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J. Differential Equations 202 (2004) 354–366
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Positive periodic solutions of functional differential equations

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Received June 13, 2003

Abstract

We consider the existence, multiplicity and nonexistence of positive ω-periodic solutions for the periodic equation $x'(t) = a(t)g(x)x(t) - \lambda b(t)f(x(t-\tau(t)))$, where $a, b \in C(\mathbb{R}, [0, \infty))$ are ω-periodic, $\int_0^{\omega} a(t)\,dt > 0$, $\int_0^{\omega} b(t)\,dt > 0$, $f, g \in C([0, \infty), [0, \infty))$, and $f(u) > 0$ for $u > 0$, $g(x)$ is bounded, $\tau(t)$ is a continuous ω-periodic function. Define $f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}$, $f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$, i_0 = number of zeros in the set $\{f_0, f_\infty\}$ and i_∞ = number of infinities in the set $\{f_0, f_\infty\}$. We show that the equation has i_0 or i_∞ positive ω-periodic solution(s) for sufficiently large or small $\lambda > 0$, respectively.

Keywords: Positive periodic solution; Existence; Multiplicity; Nonexistence; Fixed index theorem

1. Introduction

In this paper, we consider the existence, multiplicity and nonexistence of positive ω-periodic solutions for the periodic equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t-\tau(t))), \tag{1.1}$$

where $\lambda > 0$ is a positive parameter.

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doi:10.1016/j.jde.2004.02.018

Chow [2], Freedman and Wu [5], Hadeler and Tomiuk [8], Kuang [12,13], Kuang and Smith [14], Mallet-Paret and Nussbaum [16] and many others studied the existence of periodic solutions of this type or its generalized forms. This type of equation has been proposed as models for a variety of physiological processes and conditions including production of blood cells, respiration, and cardiac arrhythmias. See, for example, the above references, and [7,15,21].

In this paper, we shall show that the number of positive ω-periodic solutions of (1.1) can be determined by the asymptotic behaviors of the quotient of $\frac{f(u)}{u}$ at zero and infinity. Specifically, let

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}. \tag{1.2}$$

Then we introduce new notation

$$\begin{aligned} i_0 &= \text{number of zeros in the set } \{f_0, f_\infty\}, \\ i_\infty &= \text{number of infinities in the set } \{f_0, f_\infty\}. \end{aligned} \tag{1.3}$$

It is clear that $i_0, i_\infty = 0, 1$, or 2. Then we shall show that (1.1) has i_0 or i_∞ positive ω-periodic solution(s) for sufficiently large or small λ, respectively.

Let $\mathbb{R} = (-\infty, \infty)$. We make the assumptions:

(H1) $a, b \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions, $\int_0^\omega a(t)\,dt > 0$, $\int_0^\omega b(t)\,dt > 0$. $\tau \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic function.

(H2) $f, g : [0, \infty) \to [0, \infty)$ are continuous. $0 < l \leqslant g(u) < L < \infty$ for $u \geqslant 0$, l, L are positive constants. $f(u) > 0$ for $u > 0$.

Also, let $\sigma = e^{-\int_0^\omega a(t)\,dt}$, $M(r) = \max_{0\leqslant t\leqslant r}\{f(t)\}$ and $m(r) = \min\{f(t) : \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L} r \leqslant t \leqslant r\}$.

Our main results are

Theorem 1.1. *Assume* (H1)–(H2) *hold.*

(a) *If* $i_0 = 1$ *or* 2, *then* (1.1) *has* i_0 *positive* ω-*periodic solution*(*s*) *for* $\lambda > \frac{1-\sigma^L}{m(1)\sigma^L \int_0^\omega b(s)\,ds} > 0$.

(b) *If* $i_\infty = 1$ *or* 2, *then* (1.1) *has* i_∞ *positive* ω-*periodic solution*(*s*) *for* $0 < \lambda < \frac{1-\sigma^l}{M(1)\int_0^\omega b(s)\,ds}$.

(c) *If* $i_0 = 0$ *or* $i_\infty = 0$, *then* (1.1) *has no positive* ω-*periodic solution for sufficiently large or small* $\lambda > 0$, *respectively*.

Theorem 1.2 is a direct consequence of the proof of Theorem 1.1(c). Under the conditions of Theorem 1.2 we are able to give explicit intervals of λ such that (1.1) has no positive ω-periodic solution.

Theorem 1.2. *Assume* (H1)–(H2) *hold.*

(a) *If there is a* $c_1 > 0$ *such that* $f(u) \geqslant c_1 u$ *for* $u \in [0, \infty)$, *then there exists a* $\lambda_0 = \frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds\,c_1}$ *such that for all* $\lambda > \lambda_0$ (1.1) *has no positive* ω*-periodic solution.*

(b) *If there is a* $c_2 > 0$ *such* $f(u) \leqslant c_2 u$ *for* $u \in [0, \infty)$, *then there exists a* $\lambda_0 = \frac{1-\sigma^l}{c_2 \int_0^\omega b(s)\,ds}$ *such that for all* $0 < \lambda < \lambda_0$ (1.1) *has no positive* ω*-periodic solution.*

Theorem 1.3. *Assume* (H1)–(H2) *hold and* $i_0 = i_\infty = 0$. *If*

$$\frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}\frac{1}{\max\{f_0, f_\infty\}} < \lambda < \frac{1-\sigma^l}{\int_0^\omega b(s)\,ds}\frac{1}{\min\{f_0, f_\infty\}},$$

then (1.1) *has a positive* ω*-periodic solution.*

Our arguments are based on a well-known fixed point theorem (Lemma 2.1). In order to use the fixed point theorem, some a priori estimations of possible periodic solutions are obtained. Similar arguments have been employed in [4,18,19] to prove analogous results for the existence, multiplicity and nonexistence of positive solutions of boundary-value problems. We remark that the problem of the existence of positive periodic solutions of the equation and the problem of the existence of positive solutions of boundary-value problems of similar equations have much in common in certain sense. Once we transfer both the problems into equivalent fixed point problems, the arguments for dealing with the two problems are essentially the same.

When $g \equiv 1$, some related results for similar problems were shown in [1,9,10,17,20] based on the fixed point theorem. The construction of the function $G_u(t,s)$ in this paper, which allows us to rewrite the differential equation into an equivalent integral equation, is motivated by Jiang and Wei [9] and their other subsequent work.

In addition, we are able to obtain explicit intervals of λ such that (1.1) has no, one or two positive ω-periodic solution(s). From numerical and computational points of view, the expressions for the intervals of λ are more useful although they are not optimal. It would be interesting to further investigate the relationship between the nonlinearities and the intervals of λ such that (1.1) has no, one or two positive ω-periodic solution(s).

2. Preliminaries

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 2.1 (Deimling [3], Guo and Lakshmikantham [6] and Krasnoselskii [11]). *Let E be a Banach space and K a cone in E. For $r>0$, define $K_r = \{u \in K : ||x|| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : ||x|| = r\}$.*

(i) *If $||Tx|| \geqslant ||x||$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $||Tx|| \leqslant ||x||$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (1.1), let X be the Banach space $\{u(t) : u(t) \in C(\mathbb{R}, \mathbb{R}), u(t+\omega) = u(t)\}$ with $||u|| = \sup_{t \in [0,\omega]} |u(t)|$, $u \in X$.

Define K be a cone in X by

$$K = \left\{ u \in X : u(t) \geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L} ||u||, t \in [0, \omega] \right\},$$

Also, define, for r a positive number, Ω_r by

$$\Omega_r = \{u \in K : ||u|| < r\}.$$

Note that $\partial\Omega_r = \{u \in K : ||u|| = r\}$.

Let the map $T_\lambda : K \to X$ be defined by

$$T_\lambda u(t) = \lambda \int_t^{t+\omega} G_u(t,s) b(s) f(u(s - \tau(s)))\, ds, \tag{2.1}$$

where

$$G_u(t,s) = \frac{e^{-\int_t^s a(\theta) g(u(\theta))\, d\theta}}{1 - e^{-\int_0^\omega a(\theta) g(u(\theta))\, d\theta}}.$$

Note that

$$\frac{\sigma^L}{1-\sigma^L} \leqslant G_u(t,s) \leqslant \frac{1}{1-\sigma^l}, \quad t \leqslant s \leqslant t+\omega.$$

Lemma 2.2. *Assume* (H1)–(H2) *hold. Then $T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is compact and continuous.*

Proof. In view of the definition of K, for $u \in K$, we have

$$\begin{aligned}(T_\lambda u)(t+\omega) &= \lambda \int_{t+\omega}^{t+2\omega} G_u(t+\omega, s)b(s)f(u(s-\tau(s)))\,ds \\ &= \lambda \int_{t}^{t+\omega} G_u(t+\omega, \theta+\omega)b(\theta+\omega)f(u(\theta+\omega-\tau(\theta+\omega)))\,d\theta \\ &= \lambda \int_{t}^{t+\omega} G_u(t, s)b(s)f(u(s-\tau(s)))\,ds \\ &= (T_\lambda u)(t).\end{aligned}$$

It is easy to see that $\int_t^{t+\omega} b(s)f(u(s-\tau(s)))\,ds$ is a constant because of the periodicity of $b(t)f(u(t-\tau(t)))$. One can show that, for $u \in K$ and $t \in [0, \omega]$,

$$\begin{aligned}T_\lambda u(t) &\geqslant \frac{\sigma^L}{1-\sigma^L}\lambda \int_t^{t+\omega} b(s)f(u(s-\tau(s)))\,ds \\ &= \frac{\sigma^L}{1-\sigma^L}\lambda \int_0^{\omega} b(s)f(u(s-\tau(s)))\,ds \\ &= \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}\frac{1}{1-\sigma^l}\lambda \int_0^{\omega} b(s)f(u(s-\tau(s)))\,ds \\ &\geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||T_\lambda u||.\end{aligned}$$

Thus $T_\lambda(K) \subset K$ and it is easy to show that $T_\lambda : K \to K$ is compact and continuous. $\square$

Lemma 2.3. *Assume* (H1)–(H2) *hold. Eq.* (1.1) *is equivalent to the fixed point problem of T_λ in K.*

Proof. If $u \in K$ and $T_\lambda u = u$, then

$$\begin{aligned}u'(t) &= \frac{d}{dt}\left(\lambda \int_t^{t+\omega} G_u(t,s)b(s)f(u(s-\tau(s)))\,ds\right) \\ &= \lambda G_u(t, t+\omega)b(t+\omega)f(u(t+\omega-\tau(t+\omega))) - \lambda G_u(t,t)b(t)f(u(t-\tau(t))) \\ &\quad + a(t)g(u(t))T_\lambda u(t) \\ &= \lambda[G_u(t,t+\omega) - G_u(t,t)]b(t)f(u(t-\tau(t))) + a(t)g(u(t))T_\lambda u(t) \\ &= a(t)g(u(t))u(t) - \lambda b(t)f(u(t-\tau(t))).\end{aligned}$$

Thus u is a positive ω-periodic solution of (1.1). On the other hand, if u is a positive ω-periodic function, then $\lambda b(t)f(u(t-\tau(t))) = a(t)g(u(t))u(t) - u'(t)$ and

$$\begin{aligned}
T_\lambda u(t) &= \lambda \int_t^{t+\omega} G_u(t,s)b(s)f(u(s-\tau(s)))\,ds \\
&= \int_t^{t+\omega} G_u(t,s)(a(s)g(u(s))u(s) - u'(s))\,ds \\
&= \int_t^{t+\omega} G_u(t,s)a(s)g(u(s))u(s)\,ds - \int_t^{t+\omega} G_u(t,s)u'(s)\,ds \\
&= \int_t^{t+\omega} G_u(t,s)a(s)g(u(s))u(s)\,ds \; - G_u(t,s)u(s)|_t^{t+\omega} \\
&\quad - \int_t^{t+\omega} G_u(t,s)a(s)g(u(s))u(s)\,ds \\
&= u(t).
\end{aligned}$$

Furthermore, in view of the proof of Lemma 2.2, we also have $u(t) \geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||u||$ for $t \in [0,\omega]$. Thus u is a fixed point of T_λ in K. $\square$

Lemma 2.4. *Assume* (H1)–(H2) *hold and let* $\eta > 0$. *If* $u \in K$ *and* $f(u(t)) \geqslant u(t)\eta$ *for* $t \in [0,\omega]$, *then*

$$||T_\lambda u|| \geqslant \lambda \frac{\sigma^{2L}(1-\sigma^l)}{(1-\sigma^L)^2}\eta \int_0^\omega b(s)\,ds\,||u||.$$

Proof. Since $u \in K$ and $f(u(t)) \geqslant u(t)\eta$ for $t \in [0,\omega]$, we have

$$\begin{aligned}
(T_\lambda u)(t) &\geqslant \frac{\sigma^L}{1-\sigma^L}\lambda \int_0^\omega b(s)f(u(s-\tau(s)))\,ds \\
&\geqslant \frac{\sigma^L}{1-\sigma^L}\lambda \int_0^\omega b(s)u(s-\tau(s))\eta\,ds \\
&\geqslant \frac{\sigma^L}{1-\sigma^L}\lambda \int_0^\omega b(s)\,ds\,\eta \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||u|| \\
&= \lambda \frac{\sigma^{2L}(1-\sigma^l)}{(1-\sigma^L)^2}\eta \int_0^\omega b(s)\,ds\,||u||.
\end{aligned}$$

Thus $||T_\lambda u|| \geqslant \lambda \frac{\sigma^{2L}(1-\sigma^l)}{(1-\sigma^L)^2}\eta \int_0^\omega b(s)\,ds\,||u||$. $\square$

Lemma 2.5. *Assume* (H1)–(H2) *hold and let* $r>0$. *If* $u\in\partial\Omega_r$ *and there exists an* $\varepsilon>0$ *such that* $f(u(t))\leqslant\varepsilon u(t)$ *for* $t\in[0,\omega]$, *then*

$$||T_\lambda u||\leqslant\lambda\varepsilon||u||\frac{\int_0^\omega b(s)\,ds}{1-\sigma^l}.$$

Proof. From the definition of T, for $u\in\partial\Omega_r$, we have

$$\begin{aligned}||T_\lambda u|| &\leqslant\frac{1}{1-\sigma^l}\lambda\int_0^\omega b(s)f(u(s-\tau(s)))\,ds\\ &\leqslant\frac{1}{1-\sigma^l}\lambda\int_0^\omega b(s)\varepsilon u(s-\tau(s))\,ds\\ &\leqslant\frac{1}{1-\sigma^l}\lambda\int_0^\omega b(s)\,ds\,\varepsilon||u||\\ &=\frac{\int_0^\omega b(s)\,ds}{1-\sigma^l}\lambda\varepsilon||u||. \qquad\square\end{aligned}$$

The following two lemmas are weak forms of Lemmas 2.4 and 2.5.

Lemma 2.6. *Assume* (H1)–(H2) *hold. If* $u\in\partial\Omega_r$, $r>0$, *then*

$$||T_\lambda u||\geqslant\lambda\frac{\sigma^L\int_0^\omega b(s)\,ds}{1-\sigma^L}m(r).$$

Proof. Since $f(u(t))\geqslant m(r)$ for $t\in[0,\omega]$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.4. $\square$

Lemma 2.7. *Assume* (H1)–(H2) *hold. If* $u\in\partial\Omega_r$, $r>0$, *then*

$$||T_\lambda u||\leqslant\lambda\frac{\int_0^\omega b(s)\,ds}{1-\sigma^l}M(r).$$

Proof. Since $f(u(t))\leqslant M(r)$ for $t\in[0,\omega]$, it is easy to see that this lemma can be shown in a similar manner as in Lemma 2.5. $\square$

3. Proof of Theorem 1.1

Proof. (a) Choose a number $r_1=1$. By Lemma 2.6 we infer that there exists a $\lambda_0=\frac{1-\sigma^L}{m(r_1)\sigma^L\int_0^\omega b(s)\,ds}>0$, such that

$$||T_\lambda u||>||u|| \quad \text{for } u\in\partial\Omega_{r_1},\ \lambda>\lambda_0.$$

If $f_0 = 0$, we can choose $0 < r_2 < r_1$ so that $f(u) \leqslant \varepsilon u$ for $0 \leqslant u \leqslant r_2$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon \frac{\int_0^\omega b(s)\,ds}{1-\sigma^l} < 1.$$

Thus $f(u(t)) \leqslant \varepsilon u(t)$ for $u \in \partial\Omega_{r_2}$ and $t \in [0, \omega]$. We have by Lemma 2.5 that

$$||T_\lambda u|| \leqslant \lambda\varepsilon \frac{\int_0^\omega b(s)\,ds}{1-\sigma^l} ||u|| < ||u|| \quad \text{for } u \in \partial\Omega_{r_2}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \quad i(T_\lambda, \Omega_{r_2}, K) = 1.$$

Thus $i(T_\lambda, \Omega_{r_1} \backslash \bar{\Omega}_{r_2}, K) = -1$ and T_λ has a fixed point in $\Omega_{r_1} \backslash \bar{\Omega}_{r_2}$, which is a positive ω-periodic solution of (1.1) for $\lambda > \lambda_0$.

If $f_\infty = 0$, there is an $\hat{H} > 0$ such that $f(u) \leqslant \varepsilon u$ for $u \geqslant \hat{H}$, where the constant $\varepsilon > 0$ satisfies

$$\lambda\varepsilon \frac{\int_0^\omega b(s)\,ds}{1-\sigma^l} < 1.$$

Let $r_3 = \max\left\{2r_1, \frac{\hat{H}}{\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}}\right\}$ and it follows that $u(t) \geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||u|| \geqslant \hat{H}$ for $u \in \partial\Omega_{r_3}$ and $t \in [0, \omega]$. Thus $f(u(t)) \leqslant \varepsilon u(t)$ for $u \in \partial\Omega_{r_3}$ and $t \in [0, \omega]$. In view of Lemma 2.5, we have

$$||T_\lambda u|| \leqslant \lambda\varepsilon \frac{\int_0^\omega b(s)\,ds}{1-\sigma^l} ||u|| < ||u|| \quad \text{for } u \in \partial\Omega_{r_3}.$$

Again, it follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \quad i(T_\lambda, \Omega_{r_3}, K) = 1.$$

Thus $i(T_\lambda, \Omega_{r_3} \backslash \bar{\Omega}_{r_1}, K) = 1$, and (1.1) has a positive ω-periodic solution for $\lambda > \lambda_0$.

If $f_0 = f_\infty = 0$, it is easy to see from the above proof that T_λ has a fixed point u_1 in $\Omega_{r_1} \backslash \bar{\Omega}_{r_2}$ and a fixed point u_2 in $\Omega_{r_3} \backslash \bar{\Omega}_{r_1}$ such that

$$r_2 < ||u_1|| < r_1 < ||u_2|| < r_3.$$

Consequently, (1.1) has two positive ω-periodic solutions for $\lambda > \lambda_0$ if $f_0 = f_\infty = 0$.

(b) Choose a number $r_1 = 1$. By Lemma 2.7 we infer that there exists a $\lambda_0 = \frac{1-\sigma^l}{M(r_1)\int_0^\omega b(s)\,ds}$, such that

$$||T_\lambda u|| < ||u|| \quad \text{for } u \in \partial\Omega_{r_1}, \ 0 < \lambda < \lambda_0.$$

If $f_0 = \infty$, there is a positive number $r_2 < r_1$ such that $f(u) \geqslant \eta u$ for $0 \leqslant u \leqslant r_2$, where $\eta > 0$ is chosen so that

$$\lambda \frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2}\eta > 1.$$

Then

$$f(u(t)) \geqslant \eta u(t) \quad \text{for } u \in \partial\Omega_{r_2}, \;\; t \in [0,\omega].$$

Lemma 2.4 implies that

$$||T_\lambda u|| \geqslant \lambda \frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2}\eta ||u|| > ||u|| \quad \text{for } u \in \partial\Omega_{r_2}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \quad i(T_\lambda, \Omega_{r_2}, K) = 0.$$

Thus $i(T_\lambda, \Omega_{r_1} \backslash \bar{\Omega}_{r_2}, K) = 1$ and T_λ has a fixed point in $\Omega_{r_1} \backslash \bar{\Omega}_{r_2}$ for $0 < \lambda < \lambda_0$, which is a positive ω-periodic solution of (1.1).

If $f_\infty = \infty$, there is an $\hat{H} > 0$ such that $f(u) \geqslant \eta u$ for $u \geqslant \hat{H}$, where $\eta > 0$ is chosen so that

$$\lambda \frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2}\eta > 1.$$

Let $r_3 = \max\left\{2r_1, \frac{\hat{H}}{\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}}\right\}$. If $u \in \partial\Omega_{r_3}$, then

$$\min_{0 \leqslant t \leqslant \omega} u(t) \geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||u|| \geqslant \hat{H}$$

and hence,

$$f(u(t)) \geqslant \eta u(t) \quad \text{for } t \in [0,\omega].$$

Again, it follows from Lemma 2.4 that

$$||T_\lambda u|| \geqslant \lambda \frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2}\eta ||u|| > ||u|| \quad \text{for } u \in \partial\Omega_{r_3}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \quad i(T_\lambda, \Omega_{r_3}, K) = 0$$

and hence, $i(T_\lambda, \Omega_{r_3}\backslash\bar{\Omega}_{r_1}, K) = -1$. Thus, T_λ has a fixed point in $\Omega_{r_3}\backslash\bar{\Omega}_{r_1}$ for $0<\lambda<\lambda_0$, which is a positive ω-periodic solution of (1.1).

If $f_0 = f_\infty = \infty$, it is easy to see from the above proof that T_λ has a fixed point u_1 in $\Omega_{r_1}\backslash\bar{\Omega}_{r_2}$ and a fixed point u_2 in $\Omega_{r_3}\backslash\bar{\Omega}_{r_1}$ such that

$$r_2 < ||u_1|| < r_1 < ||u_2|| < r_3.$$

Consequently, (1.1) has two positive ω-periodic solutions for $0<\lambda<\lambda_0$ if $f_0 = f_\infty = \infty$.

(c) If $i_0 = 0$, then $f_0 > 0$ and $f_\infty > 0$. It follows that there exist positive numbers η_1, η_2, r_1 and r_2, such that $r_1 < r_2$ and

$$f(u) \geqslant \eta_1 u \quad \text{for } u \in [0, r_1],$$

$$f(u) \geqslant \eta_2 u \quad \text{for } u \in [r_2, \infty).$$

Let $c_1 = \min\left\{\eta_1, \eta_2, \min_{r_1 \leqslant u \leqslant r_2}\left\{\frac{f(u)}{u}\right\}\right\} > 0$. Thus, we have

$$f(u) \geqslant c_1 u \quad \text{for } u \in [0, \infty).$$

Assume $v(t)$ is a positive ω-periodic solution of (1.1). We will show that this leads to a contradiction for $\lambda > \lambda_0$, where $\lambda_0 = \frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds\, c_1}$. Since $T_\lambda v(t) = v(t)$ for $t \in [0, \omega]$, it follows from Lemma 2.4 that, for $\lambda > \lambda_0$,

$$\begin{aligned} ||v|| &= ||T_\lambda v|| \\ &\geqslant \lambda \frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2} c_1 ||v|| \\ &> ||v||, \end{aligned}$$

which is a contradiction.

If $i_\infty = 0$, then $f_0 < \infty$ and $f_\infty < \infty$. It follows that there exist positive numbers ε_1, ε_2, r_1 and r_2 such that $r_1 < r_2$,

$$f(u) \leqslant \varepsilon_1 u \quad \text{for } u \in [0, r_1],$$

$$f(u) \leqslant \varepsilon_2 u \quad \text{for } u \in [r_2, \infty).$$

Let $c_2 = \max\left\{\varepsilon_1, \varepsilon_2, \max_{r_1 \leqslant u \leqslant r_2}\left\{\frac{f(u)}{u}\right\}\right\} > 0$. Thus, we have

$$f(u) \leqslant c_2 u \quad \text{for } u \in [0, \infty).$$

Assume $v(t)$ is a positive ω-periodic solution of (1.1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0$, where $\lambda_0 = \frac{1-\sigma^l}{c_2\int_0^\omega b(s)\,ds}$. Since $T_\lambda v(t) = v(t)$ for $t \in [0, \omega]$, it follows from Lemma 2.5 that, for $0<\lambda<\lambda_0$,

$$\begin{aligned}||v|| &= ||T_\lambda v|| \\ &\leqslant \lambda \frac{\int_0^\omega b(s)\,ds}{1-\sigma^l} c_2 ||v|| \\ &< ||v||,\end{aligned}$$

which is a contradiction. □

4. Proof of Theorem 1.3

Proof. If $f_\infty > f_0$, then $\frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds\, f_\infty} < \lambda < \frac{1-\sigma^l}{\int_0^\omega b(s)\,ds\, f_0}$. It is easy to see that there exists an $0<\varepsilon<f_\infty$ such that

$$\frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds(f_\infty-\varepsilon)} < \lambda < \frac{1-\sigma^l}{\int_0^\omega b(s)\,ds(f_0+\varepsilon)}.$$

Now, turning to f_0 and f_∞, there is an $r_1>0$, such that $f(u) \leqslant (f_0+\varepsilon)u$ for $0\leqslant u\leqslant r_1$. Thus $f(u(t)) \leqslant (f_0+\varepsilon)u(t)$ for $u\in\partial\Omega_{r_1}$ and $t\in[0,\omega]$. We have by Lemma 2.5 that

$$||T_\lambda u|| \leqslant \lambda(f_0+\varepsilon)\frac{\int_0^\omega b(s)\,ds}{1-\sigma^l}||u|| < ||u|| \quad \text{for } u\in\partial\Omega_{r_1}.$$

On the other hand, there is an $\hat{H}>r_1$ such that $f(u) \geqslant (f_\infty-\varepsilon)u$ for $u\geqslant\hat{H}$. Let $r_2 = \max\left\{2r_1, \frac{\hat{H}}{\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}}\right\}$ and it follows that $u(t) \geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||u|| \geqslant \hat{H}$ for $u\in\partial\Omega_{r_2}$ and $t\in[0,\omega]$. Thus $f(u(t)) \geqslant (f_\infty-\varepsilon)u(t)$ for $u\in\partial\Omega_{r_2}$ and $t\in[0,\omega]$. In view of Lemma 2.4, we have

$$||T_\lambda u|| \geqslant \lambda(f_\infty-\varepsilon)\frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2}||u|| > ||u|| \quad \text{for } u\in\partial\Omega_{r_2}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 1, \quad i(T_\lambda, \Omega_{r_2}, K) = 0.$$

Thus $i(T_\lambda, \Omega_{r_2}\backslash\bar{\Omega}_{r_1}, K) = -1$. Hence, T_λ has a fixed point in $\Omega_{r_2}\backslash\bar{\Omega}_{r_1}$. Consequently, (1.1) has a positive ω-periodic solution.

If $f_\infty < f_0$, then $\frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds\,f_0} < \lambda < \frac{1-\sigma^l}{\int_0^\omega b(s)\,ds\,f_\infty}$. It is easy to see that there exists an $0 < \varepsilon < f_0$ such that

$$\frac{(1-\sigma^L)^2}{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds(f_0-\varepsilon)} < \lambda < \frac{1-\sigma^l}{\int_0^\omega b(s)\,ds(f_\infty+\varepsilon)}.$$

Now, turning to f_0 and f_∞, there is an $r_1 > 0$ such that $f(u) \geqslant (f_0 - \varepsilon)u$ for $0 \leqslant u \leqslant r_1$. Thus $f(u(t)) \geqslant (f_0 - \varepsilon)u(t)$ for $u \in \partial\Omega_{r_1}$ and $t \in [0, \omega]$. We have by Lemma 2.4 that

$$||T_\lambda u|| \geqslant \lambda(f_0 - \varepsilon)\frac{\sigma^{2L}(1-\sigma^l)\int_0^\omega b(s)\,ds}{(1-\sigma^L)^2}||u|| > ||u|| \quad \text{for } u \in \partial\Omega_{r_1}.$$

On the other hand, there is an $\hat{H} > r_1$ such that $f(u) \leqslant (f_\infty + \varepsilon)u$ for $u \geqslant \hat{H}$. Let $r_2 = \max\left\{2r_1, \frac{\hat{H}}{\frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}}\right\}$ and it follows that $u(t) \geqslant \frac{\sigma^L(1-\sigma^l)}{1-\sigma^L}||u|| \geqslant \hat{H}$ for $u \in \partial\Omega_{r_2}$ and $t \in [0, \omega]$. Thus $f(u(t)) \leqslant (f_\infty + \varepsilon)u(t)$ for $u \in \partial\Omega_{r_2}$ and $t \in [0, \omega]$. In view of Lemma 2.5, we have

$$||T_\lambda u|| \leqslant \lambda(f_\infty + \varepsilon)\frac{\int_0^\omega b(s)\,ds}{1-\sigma^l}||u|| < ||u|| \quad \text{for } u \in \partial\Omega_{r_2}.$$

It follows from Lemma 2.1 that

$$i(T_\lambda, \Omega_{r_1}, K) = 0, \quad i(T_\lambda, \Omega_{r_2}, K) = 1.$$

Thus $i(T_\lambda, \Omega_{r_2}\backslash\bar{\Omega}_{r_1}, K) = 1$. Hence, T_λ has a fixed point in $\Omega_{r_2}\backslash\bar{\Omega}_{r_1}$. Consequently, (1.1) has a positive ω-periodic solution. $\square$

Acknowledgments

Haiyan Wang thanks Y. Kuang for suggesting the notation i_0 and i_∞ for the paper and other remarks. Haiyan Wang is also grateful to the reviewers for their comments.

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