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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR ELLIPTIC EQUATIONS IN AN ANNULUS

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ABSTRACT. We study the equation $-\Delta u = \lambda g(|x|)f(u) \quad R_1 < |x| < R_2, \quad x \in \mathbb{R}^N, \quad N \geq 1$ subject to linear boundary conditions at R_1 and R_2. Under assumptions concerning sub- or superlinearity of f, we establish existence, non-existence, and multiplicity results for positive solutions.

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1. Introduction

In this paper we consider existence and nonexistence of positive radial solutions of the equation

$$-\Delta u = \lambda g(|x|)f(u), \quad R_1 < |x| < R_2 \tag{1.1}$$

where $x \in \mathbb{R}^N$, $N \geq 1$, along with linear boundary conditions at R_1 and R_2 which include

$$u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad |x| = R_2 \tag{1.2a}$$

$$u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad \frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_2 \tag{1.2b}$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_2 \tag{1.2c}$$

Here $r = |x|$ and $\frac{\partial}{\partial r}$ denotes differentiation in the radial direction, $0 < R_1 < R_2 < \infty$. Because of the radial symmetry, we seek criteria for existence (or nonexistence) of positive radial solutions of (1.1) which then must satisfy

$$-u''(r) - \frac{N-1}{r}\, u'(r) = \lambda g(r) f(u(r)), \quad R_1 < r < R_2. \tag{1.3}$$

Further, via a standard change of variable, (1.3) may be written as an ODE on $(0,1)$. Consequently, we shall consider the following BVP with linear boundary conditions

$$-u'' = \lambda h(t) f(u), \quad 0 < t < 1 \tag{I}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) &= 0 \\ \gamma u(1) + \delta u'(1) &= 0 \end{aligned} \tag{II}$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative real constants with $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$. We shall also assume that $f(u) > 0$ for $u > 0$ and $h(t) \geq 0$ for $t \geq 0$, with both f and h continuous on their respective domains. The BVP (1.1), (1.2) and (I), (II) has been the subject of numerous investigations (cf. [1-17]) under various assumptions. In [7], [8], the BVP (I), (II) (with $\lambda = 1$) was investigated and various criteria for existence of one (or several) positive solutions were obtained. We introduce the notation

$$f_0 = \lim_{u \to 0+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} \tag{1.4}$$

and we shall say that f is sub (super) linear at 0 in case $f_0 = +\infty$ $(f_0 = 0)$. Similarly, f is sub (super) linear at ∞ in case $f_0 = 0$ $(f_\infty = \infty)$. The main result of this paper may then be stated as

Theorem 1. *Let* f, g *be continuous real valued functions defined on* $[0, \infty)$ *with* $f(u) > 0$ *for* $u > 0$ *and* $g(t) \geq 0$ *for* $t \geq 0$ *and assume* $g(t) \not\equiv 0$ *in some neighborhood of* $t = 1/2$. *Then:*

a) *if* $f_0 = f_\infty = 0$ *there exists* $\lambda_1 > 0$ *such that* (I), (II) *has at least two distinct positive solutions for* $\lambda \geq \lambda_1$.
b) *if* $f_0 = f_\infty = \infty$ *there exists* $\lambda_2 > 0$ *such that* (I), (II) *has at least two distinct positive solutions for* $0 < \lambda \leq \lambda_2$.
c) *if* $f_0 = 0$ *or* $f_\infty = 0$ *there exists* $\lambda_3 > 0$ *such that* (I), (II) *has at least one positive solution for* $\lambda \geq \lambda_3$.
d) *if* $f_0 = \infty$ *or* $f_\infty = \infty$ *there exists* $\lambda_4 > 0$ *such that* (I), (II) *has at least one positive solution for* $0 < \lambda \leq \lambda_4$.
e) *if* $f(u) \geq cu$ *for* $u \geq 0$ *and some constant* $c > 0$, *there exists* $\lambda_5 > 0$ *such that* (I), (II) *has no positive solution for* $\lambda \geq \lambda_5$.
f) *if* $f(u) \leq cu$ *for* $u \geq 0$ *and some constant* $c > 0$, *there exists* $\lambda_6 > 0$ *such that* (I), (II) *has no positive solution for* $0 < \lambda \leq \lambda_6$.

The proof of Theorem 1 is based on the following Fixed Point Theorem of cone expansion/compression type. (See [6] for a proof and additional details.)

Theorem 2. *Let X be a Banach space, $K \subseteq X$ a cone and assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $F : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Fu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1$ *and*

$\|Fu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2$

or

(ii) $\|Fu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1$ *and*

$\|Fu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2$.

Then F has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. Notation and Proofs

The BVP (I), (II) is equivalent to the integral equation

$$u(t) = \lambda \int_0^1 k(t,s)h(s)f(u(s))ds \equiv Fu(t) \tag{2.1}$$

where $u \in X := C[0,1]$ and $k(t,s)$ is the Green's function for the problem $-u'' = 0$ subject to the boundary conditions (II) and is given explicitly by

$$k(t,s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1 \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

The operator $F : X \to X$ is completely continuous and if we define the cone $K \subset X$ by

$$K = \{u \in X : \ u(t) \geq 0, \ \min_J u(t) \geq \sigma \|u\|\} \tag{2.3}$$

where $J = [\frac{1}{4}, \frac{3}{4}]$, $\|u\| = \sup_{t \in [0,1]} |u(t)|$ and $\sigma = \min\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\}$, then one can verify directly that $k(t,s)/k(s,s) \geq \sigma$ for $t \in J$, $s \in [0,1]$. Hence, for $u \in X$ we have

$$\begin{aligned} \min_{t \in J} (Fu)(t) &= \min_{t \in J} \lambda \int_0^1 k(t,s)h(s)f(u(s))ds \\ &\geq \lambda\sigma \int_0^1 k(s,s)h(s)f(u(s))ds \\ &\geq \lambda\sigma \max_{t \in [0,1]} \int_0^1 k(t,s)h(s)f(u(s))ds \\ &= \sigma\|Fu\|. \end{aligned}$$

Consequently, $Fu \in K$ for any $u \in X$.

Proof of Theorem 1a. For $u \in K$, we have

$$\begin{aligned} (Fu)(\tfrac{1}{2}) &= \lambda \int_0^1 k(\tfrac{1}{2}, s)h(s)f(u(s))ds \\ &\geq \lambda \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)f(u(s))ds. \end{aligned} \tag{2.4}$$

For any $p > 0$ we define

$$m(p) = \min\{\int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)f(u(s))ds : \ u \in K, \ \|u\| = p\}. \tag{2.5}$$

Since $h(s) \not\equiv 0$ and $f(u) > 0$ it follows from $\frac{p}{4} \leq u(s) \leq p$ on J that $m(p) > 0$. Let $0 < p_1 < p_2$ be arbitrary and define

$$\lambda_1 = \max\{\frac{p_1}{m(p_1)}, \frac{p_2}{m(p_2)}\}. \tag{2.6}$$

From (2.4) – (2.6) and for $\lambda \geq \lambda_1$ it follows that $\|Fu\| \geq \|u\|$ for $\|u\| = p_1$ and $\|u\| = p_2$. Since $f_0 = 0$ for any $\lambda \geq \lambda_1$ we can choose $q_1 > 0$ and $\eta > 0$ such that $2q_1 < p_1$ and such that $f(u) \leq \eta u$ for $0 < u \leq q_1$ where

$$\eta\lambda \int_0^1 k(s,s)h(s)ds \leq 1. \tag{2.7}$$

Then for $u \in K$ and $\|u\| = q_1$ we have

$$\begin{aligned}(Fu)(t) &= \lambda \int_0^1 k(t,s)h(s)f(u(s))ds \\ &\le \lambda \int_0^1 k(s,s)h(s)f(u(s))ds \\ &\le \lambda\eta\|u\| \int_0^1 k(s,s)h(s)ds \le \|u\|.\end{aligned}$$

Thus, $\|Fu\| \le \|u\|$ for $u \in K$ and $\|u\| = q_1$. Also, since $f_\infty = 0$ we may choose $q_2 > 2p_2$ such that $f(u) \le \eta u$ for $u \ge q_2$ where η satisfies (2.7). Now if f is bounded, say $f(u) \le N$ for all $u \in (0,\infty)$, then we may also suppose that $N\lambda \int_0^1 k(s,s)h(s)ds \le q_2$. Then for $u \in K$ and $\|u\| = q_2$ we have

$$\begin{aligned}(Fu)(t) &= \lambda \int_0^1 k(t,s)h(s)f(u(s))ds \\ &\le N\lambda \int_0^1 k(s,s)h(s)ds \le q_2 = \|u\|\end{aligned}$$

so $\|Fu\| \le \|u\|$ for $u \in K$ and $\|u\| = q_2$.

If f is unbounded then $q_2 > 2p_2$ is chosen so that $f(u) \le f(q_2)$ for $0 < u \le q_2$. Then for $u \in K$ and $\|u\| = q_2$ we have

$$\begin{aligned}(Fu)(t) &= \lambda \int_0^1 k(t,s)h(s)f(u(s))ds \\ &\le \lambda f(q_2) \int_0^1 k(s,s)h(s)ds \\ &\le \lambda\eta q_2 \int_0^1 k(s,s)h(s)ds \le q_2\end{aligned}$$

and so again we have $\|Fu\| \le \|u\|$ for $u \in K$ and $\|u\| = q_2$. If we now set

$$\Omega_1 = \{u \in X : \|u\| < q_1\}$$

$$\Omega_2 = \{u \in X : \|u\| < p_1\}$$

$$\Omega_3 = \{u \in X : \|u\| < p_2\}$$

$$\Omega_4 = \{u \in X : \|u\| < q_2\}$$

then we conclude from Theorem 2 that F has a fixed point $u_1 \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and $u_2 \in K \cap (\overline{\Omega}_4 \backslash \Omega_3)$ with $q_1 \le \|u_1\| \le p_1$ and $p_2 \le \|u_2\| \le q_2$. Since $p_1 < p_2$, u_1, u_2 are distinct and are both positive for $0 < t < 1$.

Proof of Theorem 1b. For any $u \in K$ we have

$$(Fu)(t) = \lambda \int_0^1 k(t,s)h(s)f(u(s))ds \le \lambda \int_0^1 k(s,s)h(s)f(u(s))ds. \tag{2.8}$$

Let $0 < p_1 < p_2$ be given and let $M_i = \max\{f(u) : 0 \le u \le p_i\}$, $i = 1,2$. Then we have, from (2.8)

$$(Fu)(t) \le (\lambda \int_0^1 k(s,s)h(s)ds) M_1, \quad \|u\| = p_1$$

and

$$(Fu)(t) \le (\lambda \int_0^1 k(s,s)h(s)ds) M_2, \quad \|u\| = p_2.$$

Therefore, we may choose $\lambda_2 > 0$ such that for $0 < \lambda \le \lambda_2$ we have $(Fu)(t) \le p_1$ and $(Fu)(t) \le p_2$ for $\|u\| = p_1$ and $\|u\| = p_2$, respectively. If we set

$$\Omega_2 = \{u \in X : \|u\| < p_1\}$$

$$\Omega_3 = \{u \in X : \|u\| < p_2\}$$

then $\|Fu\| \le \|u\|$ for $\lambda \le \lambda_2$ and $u \in K \cap \partial\Omega_2$ and $\|Fu\| \le \|u\|$ for $\lambda \le \lambda_2$ and $u \in K \cap \partial\Omega_3$.

Now since $f_0 = \infty$, there exists $q_1 < \frac{1}{2} p_1$ such that $f(u) \ge Mu$ for $0 < u \le q_1$ where $M > 0$ satisfies

$$\lambda\sigma M \ge \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)ds \ge 1. \tag{2.9}$$

Then for $u \in K$ and $\|u\| = q_1$ we have

$$\begin{aligned}(Fu)(\tfrac{1}{2}) &= \lambda \int_0^1 k(\tfrac{1}{2}, s)h(s)f(u(s))ds \\ &\geq \lambda \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)f(u(s))ds \\ &\geq \lambda \sigma M\Big(\int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)ds\Big)\|u\| \geq \|u\|\end{aligned}$$

so letting $\Omega_1 = \{u \in X : \|u\| < q_1\}$ we have

$$\|Fu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1.$$

Similarly, since $f_\infty = \infty$, it follows that there is $q > 0$ such that $f(u) \geq Mu$ for $u \geq q$ where M satisfies (2.9). If we now put $q_2 = \max\{2p_2, \frac{q}{\sigma}\}$, then for $u \in K$ and $\|u\| = q_2$ we have $\min_J u(t) \geq \sigma\|u\| \geq q$ and

$$\begin{aligned}(Fu)(\tfrac{1}{2}) &= \lambda \int_0^1 k(\tfrac{1}{2}, s)h(s)f(u(s))ds \\ &\geq \lambda \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)f(u(s))ds \\ &\geq M\lambda \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)u(s)ds \\ &\geq M\lambda\sigma\Big(\int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)ds\Big)\|u\| \geq \|u\|.\end{aligned}$$

Therefore, if we define Ω_4

$$\Omega_4 = \{u \in X : \|u\| < q_2\}$$

then $\|Fu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_4$. Thus, if we apply the Fixed Point as in the proof of part a), we conclude the existence of $u_1 \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and $u_2 \in K \cap (\overline{\Omega}_4 \backslash \Omega_3)$ which are positive solutions of (I), (II) with $0 < q_1 \leq \|u_1\| \leq p_1 < p_2 \leq \|u_2\| \leq q_2$.

Proof of Theorem 1c and 1d. The proofs of parts c) and d) are similar to those for a) and b). If we consider Ω_1 and Ω_2 in the proof of a), then it follows that c) is true for $f_0 = 0$. Similarly, c) is true for $f_\infty = 0$ by considering Ω_3 and Ω_4. Likewise, d) follows from the proof of b).

Proof of Theorem 1e. Suppose $v(t)$ is a positive solution of (I), (II). Therefore, since $v \in K$

$$\begin{aligned}v(\tfrac{1}{2}) &= \lambda \int_0^1 k(\tfrac{1}{2}, s)h(s)f(v(s))ds \\ &\geq \lambda \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)f(v(s))ds \\ &\geq \lambda c \int_{1/4}^{3/4} k(\tfrac{1}{2}\, s)h(s)v(s)ds \\ &\geq \lambda c\sigma\Big(\int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)ds\Big)\|v\|.\end{aligned}$$

Hence, if λ is sufficiently large so that

$$\lambda c\sigma \int_{1/4}^{3/4} k(\tfrac{1}{2}, s)h(s)ds > 1$$

then we have $v(\frac{1}{2}) > \|v\|$, which is a contradiction. This proves part e).

Proof of Theorem 1f. Suppose $v(t)$ is a positive solution of (I), (II). Then we have

$$\begin{aligned}v(t) &= \lambda \int_0^1 k(t, s)h(s)f(v(s))ds \\ &\leq c\lambda \int_0^1 k(s, s)h(s)v(s)ds \\ &\leq c\lambda\Big(\int_0^1 k(s, s)h(s)ds\Big)\|v\|.\end{aligned}$$

Thus, if λ is sufficiently small so that

$$c\lambda \int_0^1 k(s,s)h(s)ds < 1$$

then we have $v(t) < \|v\|$ for $0 \le t \le 1$, which is a contradiction. This proves part f).

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