One-Loop Low Energy Effective Action in QED (worldline method)

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References:

- V.P. Gusynin and I.A. Shovkovy, Can. J. Phys. **74**, 282 (1996).

- V.P. Gusynin and I.A. Shovkovy, `hep-th/9804143`

Outline

1. Effective action in QED
    - its meaning and range of validity;
    - status of the problem.
2. Method of calculation
    - worldline formalism;
    - perturbation theory and Feynman rules.
3. General result (derivative expansion).
4. Two special cases.
5. Conclusion.

1. The effective action in QED

The original theory $\stackrel{?}{\rightarrow}$ an effective theory
(exact) (approximate)

- Reason:
  to get rid of irrelevant degrees of freedom.

- Low-energy dynamics in QED

  1. massive fermions decouple;

  2. electromagnetic field is relevant;

  3. virtual fermions $\rightarrow$ vacuum currents;

  4. Maxwell equations $\rightarrow$ nonlinear.

- In QED $\overset{\int (\text{fermions})}{\longrightarrow}$ Effective Action.

- Results ($F^{\mu\nu}$ is constant or varies slowly).

  1. The Heisenberg–Euler eff. action (1936)

$$\mathcal{L}_{HE} = \mathcal{F} + \frac{2\alpha^2 \left(4\mathcal{F}^2 + 7\mathcal{G}^2\right)}{45\pi^2 m^4} + \ldots, \quad (1)$$

where $\mathcal{F} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and $\mathcal{G} = \frac{1}{8} \epsilon^{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa}$.

  2. The Schwinger effective action (1951)

$$\mathcal{L}_S = \mathcal{F} + \int_0^\infty \frac{ds}{8\pi^2 s^3} e^{-im^2 s} \Big[esK_+ \coth(esK_+) \times esK_- \cot(esK_-) - 1 - \frac{2e^2 s^2}{3} \mathcal{F} \Big], \quad (2)$$

where $K_\pm = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} \pm \mathcal{F}}$.

- Derivative expansion

1. The Hauknes expansion in $1/m^2$ (1984)

$$\mathcal{L}_H = \mathcal{L}_{HE} + \frac{\alpha}{360\pi m^2}\left(\frac{1}{9}F^{\mu\nu}F_{\mu\nu,\lambda}{}^{\lambda} + \right. \quad (3)$$
$$\left. + \frac{7}{2}F^{\mu\nu,\lambda}F_{\mu\nu,\lambda} - F^{\lambda\mu,}{}_{\mu}F_{\lambda\nu,}{}^{\nu}\right) + O\left(\frac{\alpha^2}{m^6}\right),$$

2. Derivative expansion for a special case (Lee, Pac, Shin, 1989) in spinor QED_{3+1}:
$$\mathcal{G} = 0 \quad \text{and} \quad F^{\mu\nu} = \Phi(x)\mathbf{F}^{\mu\nu}$$

$$\mathcal{L}_{der} = \mathcal{L}_S - i\frac{\partial_\mu\Phi\partial^\mu\Phi}{(8\pi)^2 e^2\Phi^4}\int_0^\infty \frac{ds}{s}e^{-im^2 s}$$
$$\times\frac{d^3}{ds^3}\Big[es\Phi\tanh(es\Phi)\Big] \quad (4)$$

3. Arbitrary background $F_{\mu\nu} = const$?
OUR RESULT (below)

2. Method of calculation

- One-loop effective action

$$W^{(1)}(A) \equiv \int d^n x \mathcal{L}^{(1)} = -i \ln \mathrm{Det}(i\hat{\mathcal{D}} - m)$$
$$= -\frac{i}{2} \ln \mathrm{Det}\left(\mathcal{D}_\mu^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + m^2\right), \quad (5)$$

so that

$$\mathcal{L}^{(1)}(A) = \frac{i}{2}\int_0^\infty \frac{d\tau}{\tau} e^{-im^2\tau} tr\langle x| \exp(-i\tau H)|x\rangle, \quad (6)$$

where the "Hamiltonian" is given by

$$H = -\Pi_\mu \Pi^\mu + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}(x), \quad (7)$$
$$\Pi_\mu = -i\left(\partial_\mu - ieA_\mu\right).$$

- $tr\langle z|U(\tau)|y\rangle$ — a quantum mechanical (!) evolution operator.

- Quantum mechanical path integral (!):

$$tr\langle z|U(\tau)|y\rangle = \frac{1}{M}\int \mathcal{D}[x,\psi]e^{iS_{bos}+iS_{fer}} \quad (8)$$

with

$$L_{bos}(x) = -\frac{1}{4}\frac{dx_\nu}{dt}\frac{dx^\nu}{dt} - eA_\nu(x)\frac{dx^\nu}{dt}, \quad (9)$$

$$L_{fer}(\psi,x) = \frac{i}{2}\psi_\nu\frac{d\psi^\nu}{dt} - ie\psi^\nu\psi^\lambda F_{\nu\lambda}(x). \quad (10)$$

and boundary conditions

$$x(0) = y, \quad x(\tau) = z, \quad \psi(0) = -\psi(\tau). \quad (11)$$

- Fock-Schwinger gauge

$$\begin{aligned} A_\nu(x) &= \frac{1}{2}(x^\lambda - y^\lambda)F_{\lambda\nu}(y) \\ &+\frac{1}{3}(x^\lambda - y^\lambda)(x^\sigma - y^\sigma)\partial_\sigma F_{\lambda\nu}(y) \quad (12) \\ &+\frac{1}{8}(x^\lambda - y^\lambda)(x^\sigma - y^\sigma)(x^\mu - y^\mu)\partial_\sigma\partial_\mu F_{\lambda\nu}(y) + \ldots \end{aligned}$$

- $F_{\mu\nu} = const \rightarrow$ Gaussian integral

- Derivative terms $\rightarrow$ "interactions"

- Well defined perturbative expansion in N_{der}

- Feynman rules

  1. All vacuum diagrams

  2. The propagators:

$$\langle T x_\mu(t_1) x_\nu(t_2) \rangle \quad \rightarrow \quad \mu \;\text{———————}\; \nu$$
$$\langle T \psi_\mu(t_1) \psi_\nu(t_2) \rangle \quad \rightarrow \quad \mu \;--\rightarrow--\; \nu$$

  3. The interactions

$$\frac{eF_{\nu_0\nu_1,\nu_2\ldots\nu_{n+1}}}{n!(n+2)} \frac{dx^{\nu_0}}{dt} x^{\nu_1} \ldots x^{\nu_{n+1}}, \qquad \text{(Fig. 1)}$$
$$\frac{ie}{n!} F_{\lambda\mu,\nu_1\ldots\nu_n} \psi^\lambda \psi^\mu x^{\nu_1} \ldots x^{\nu_n}, \qquad \text{(Fig. 2)}$$

generate two types of VERTICES

- A few comments on the pertur. theory

  1. Infinite number of different vertices
  2. Still, a finite number of diagrams in each order, N_{der}, of the pertur. theory
  3. Disconnected diagrams are included
  4. All derivatives come from the vertices

- For example, (a 2-derivative contribution)

- Problem of propagators,
  $D_{\mu\nu}(t_1, t_2) = D(t_1, t_2, F..)_{\mu\nu}$ (???)

- Set of matrices $A_{(j)\mu\nu}$:

$$F^{\lambda\mu}A_{(j)\mu\nu} = A^{\lambda\mu}_{(j)}F_{\mu\nu} = f_j A^{\lambda}_{(j)\nu} \quad (13)$$

where

$$f_0 = 0, \quad f_{\pm 1} = \pm\sqrt{2\mathcal{F}} \quad \underline{(2+1)D} \quad (14)$$

$$f_{1,2} = \pm iK_-, \quad f_{3,4} = \pm K_+ \quad \underline{(3+1)D} \quad (15)$$

- $A^{\mu\nu}_{(j)}$ satisfy the identities

$$\sum_j A^{\mu\nu}_{(j)} = \eta^{\mu\nu}, \qquad A^{\mu}_{(j)\mu} = 1, \quad (16)$$

$$A^{\mu\nu}_{(k)}A_{(j)\nu\lambda} = A^{\mu}_{(j)\lambda}\delta_{kj} \quad (17)$$

- Therefore,

$$D(t_1, t_2, F..)_{\mu\nu} = \sum_j D(t_1, t_2, f_j)A_{(j)\mu\nu} \quad (18)$$

3. General result (derivative expansion)

$$
\begin{aligned}
&\langle x|U_{bos}(\tau)|x\rangle = \langle x|U_{bos}(\tau)|x\rangle_0 \\
\times\ &\Big\{1 - \frac{i}{8} eF_{\nu\lambda,\mu\kappa} \sum_{j,l} C^{V}_{(j,l)} (A^{\nu\lambda}_{(j)} A^{\mu\kappa}_{(l)} + 2A^{\nu\mu}_{(j)} A^{\lambda\kappa}_{(l)}\ ) \\
&+ \frac{i}{18} e^2 F_{\nu\lambda,\mu} F_{\sigma\kappa,\rho} \sum_{j,l,k} \Big[C^{VV}_{1(j,l,k)} (A^{\nu\lambda}_{(j)} A^{\kappa\sigma}_{(l)} A^{\mu\rho}_{(k)} \\
&\qquad + A^{\nu\mu}_{(j)} A^{\kappa\rho}_{(l)} A^{\lambda\sigma}_{(k)} + 2A^{\nu\lambda}_{(j)} A^{\kappa\rho}_{(l)} A^{\mu\sigma}_{(k)}) \\
&+ C^{VV}_{2(j,l,k)} (A^{\nu\sigma}_{(j)} A^{\kappa\lambda}_{(l)} A^{\mu\rho}_{(k)} + A^{\nu\rho}_{(j)} A^{\kappa\mu}_{(l)} A^{\lambda\sigma}_{(k)} \\
&\qquad + 2A^{\nu\sigma}_{(j)} A^{\kappa\mu}_{(l)} A^{\lambda\rho}_{(k)}) \\
&+ 2C^{VV}_{3(j,l,k)} (A^{\nu\lambda}_{(j)} A^{\kappa\mu}_{(l)} A^{\sigma\rho}_{(k)} + A^{\kappa\rho}_{(j)} A^{\nu\sigma}_{(l)} A^{\lambda\mu}_{(k)}) \\
&+ C^{VV}_{4(j,l,k)} A^{\nu\kappa}_{(j)} A^{\lambda\mu}_{(l)} A^{\sigma\rho}_{(k)} \\
&+ C^{VV}_{5(j,l,k)} A^{\nu\kappa}_{(j)} (A^{\lambda\sigma}_{(l)} A^{\mu\rho}_{(k)} + A^{\lambda\rho}_{(l)} A^{\mu\sigma}_{(k)}) \Big] \Big\} \qquad (19)
\end{aligned}
$$

(in scalar QED) where

$$
\langle x|U_{bos}(\tau)|x\rangle_0^{3+1} = -\frac{i}{(4\pi\tau)^2} \frac{(e\tau K_-)(e\tau K_+)}{\sin(e\tau K_-)\sinh(e\tau K_+)},
$$

and

$$
\langle x|U_{bos}(\tau)|x\rangle_0^{2+1} = -\frac{\exp[-i\pi/4]}{(4\pi\tau)^{3/2}} \frac{(e\tau\sqrt{2\mathcal{F}})}{\sinh(e\tau\sqrt{2\mathcal{F}})}
$$

- Expansion in powers of $1/m^2$ (in τ)

$$tr\langle x|U_{scal}(\tau)|x\rangle = tr\langle x|U_{scal}(\tau)|x\rangle_0 \times\Big\{1 - \frac{ie^2\tau^3}{30}F^{\nu\lambda}F_{\nu\lambda,\mu}{}^{\mu} - -\frac{ie^2\tau^3}{180}\Big(4F^{\nu\lambda,\mu}F_{\nu\lambda,\mu} + F^{\nu\lambda,}{}_{\lambda}F_{\nu\mu,}{}^{\mu}\Big) + \ldots\Big\} \tag{20}$$

in <u>scalar</u> QED, and

$$tr\langle x|U_{spin}(\tau)|x\rangle = tr\langle x|U_{spin}(\tau)|x\rangle_0 \times\Big\{1 + \frac{ie^2\tau^3}{20}F^{\nu\lambda}F_{\nu\lambda,\mu}{}^{\mu} + +\frac{ie^2\tau^3}{180}\Big(\frac{7}{2}F^{\nu\lambda,\mu}F_{\nu\lambda,\mu} - F^{\nu\lambda,}{}_{\lambda}F_{\nu\mu,}{}^{\mu}\Big) + \ldots\Big\} \tag{21}$$

in <u>spinor</u> QED.

- Range of validity

$$\frac{|eF|}{m^2} \ll 1 \quad \text{and} \quad \frac{(\partial|eF|)^2}{m^2(eF)^2} \ll 1$$

- Eq. (21) $\rightarrow$ Hauknes result.

4. Two special cases

- Pure magnetic background

Scalar QED

$$\mathcal{L}^{(D)}_{der}(B) = \frac{e^2\,(\partial_\perp B)^2}{(4\sqrt{\pi})^D|eB|^{\frac{6-D}{2}}}\int\limits_0^\infty \frac{d\omega}{\omega^{\frac{D-2}{2}}}e^{-\frac{m^2}{|eB|}\omega} \times\left(\frac{d^3}{d\omega^3}+\frac{d}{d\omega}\right)\left(\frac{\omega}{\sinh\omega}\right), \qquad (22)$$

— Spatial derivatives $\to$ energy decrease (if $|eB|/m^2 < Const$).

Spinor QED

$$\mathcal{L}^{(D)}_{der}(B) = -\frac{e^2\,(\partial_\perp B)^2}{4(2\sqrt{\pi})^D|eB|^{\frac{6-D}{2}}}\int\limits_0^\infty \frac{d\omega}{\omega^{\frac{D-2}{2}}}e^{-\frac{m^2}{|eB|}\omega} \times\frac{d^3}{d\omega^3}\Big(\omega\coth(\omega)\Big), \qquad (23)$$

— Spatial derivatives $\to$ energy decrease.

- An instability?

- Pure electric background

Spinor QED

$$\mathcal{L}^{(D)}_{der}(E) = \frac{e^2\left(\partial_{\|}E\right)^2 i^{\frac{2-D}{2}}}{4(2\sqrt{\pi})^D|eE|^{\frac{6-D}{2}}}\int_0^\infty \frac{d\omega}{\omega^{\frac{D-2}{2}}}e^{-i\frac{m^2}{|eE|}\omega} \times \frac{d^3}{d\omega^3}\left(\omega\coth\omega\right). \quad (24)$$

where $\left(\partial_{\|}E\right)^2 \equiv \left(\partial_0 E\partial_0 E - \partial_1 E\partial_1 E\right)$.

- Correction to $\mathcal{I}m\mathcal{L}(E)$ (by definition, $\sigma \equiv \frac{m^2\pi}{|eE|}$)

$$\mathcal{I}m\mathcal{L}^{(2+1)}_{der} = \frac{e^2\left(\partial_{\|}E\right)^2}{2^8\pi^3|eE|^{3/2}}\sum_{n=1}^\infty \frac{\exp(-\sigma n)}{n^{5/2}} \times \times[15+18\sigma n+12(\sigma n)^2+8(\sigma n)^3], \quad (25)$$

$$\mathcal{I}m\mathcal{L}^{(3+1)}_{der} = \frac{e^2\left(\partial_{\|}E\right)^2}{2^6\pi^4|eE|}\sum_{n=1}^\infty \frac{\exp(-\sigma n)}{n^{3}} \times \times[6+6\sigma n+3(\sigma n)^2+(\sigma n)^3], \quad (26)$$

- The corrections are finite as $m \to 0$

5. Conclusion

1. Derivative expansion of the one-loop low-energy effective action in QED_{2+1} and QED_{3+1} (scalar and spinor).

2. All known limits.

3. Explicit expressions for the probability of the particle pairs creation by an electric field due to derivatives.

4. A direct test of breaking the derivative expansion as $m \to 0$.

5. Looking for applications...