Lecture on QCD at asymptotic densities

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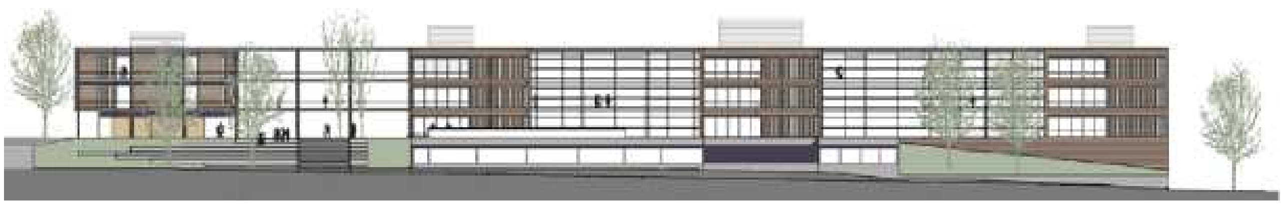

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What color superconductivity is about?

Local (Global) color symmetry is broken (in a fixed gauge) by a diquark condensate,

$$\phi \equiv \langle \psi^T O \psi \rangle \neq 0 \qquad \text{(not a gauge invariant quantity)}$$

In the case of a local field ϕ,

$$O \equiv O_{\rm Dirac} \otimes O_{\rm color} \otimes O_{\rm flavor} : \qquad O = -O^T \qquad \text{(Pauli principle)}$$

(i) Spin-0 channel: $O^T_{\rm Dirac} = -O_{\rm Dirac}$

(ii) One-gluon exchange interaction:

$$\sum_{A=1}^{N_c^2-1} T^A_{a'a} T^A_{b'b} = \overbrace{\underbrace{-\frac{N_c+1}{4N_c}\left(\delta_{aa'}\delta_{b'b} - \delta_{ab'}\delta_{a'b}\right)}_{\text{color-antisymmetric } \bar{3}\text{-plet}}}^{\text{attraction}} \overbrace{\underbrace{+\frac{N_c-1}{4N_c}\left(\delta_{aa'}\delta_{b'b} + \delta_{ab'}\delta_{a'b}\right)}_{\text{color-symmetric 6-plet}}}^{\text{repulsion}}$$

(iii) Thus, $O^T_{\rm flavor} = -O_{\rm flavor}$

Notation

Nambu-Gorkov spinor field:

$$\Psi = \begin{pmatrix} \psi \\ \psi^C \end{pmatrix} \quad \text{and} \quad \bar{\Psi} = \begin{pmatrix} \bar{\psi} & \bar{\psi}^C \end{pmatrix}$$

where $\bar{\psi} = \psi^\dagger \gamma_0$,

$$\begin{aligned} \psi^C &\equiv C\bar{\psi}^T = -\gamma_0 C \psi^* \quad \text{and} \quad C \quad \text{is a charge conjugation matrix} \\ \bar{\psi}^C &= \left(\psi^C\right)^\dagger \gamma_0 = -\psi^T C^{-1} \end{aligned}$$

By definition, $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ and $C = -C^T$.

One may introduce quadratic terms (in quark fields)

$$\begin{aligned} \text{diquark term:} \quad \bar{\psi}^C \tilde{\Delta} \psi &= -\psi^T C^{-1} \tilde{\Delta} \psi \\ \oplus \text{ h.c. term:} \quad \bar{\psi} \Delta \psi^C &= -\psi^\dagger \gamma_0 \Delta \gamma_0 C \psi^* \end{aligned}$$

i.e., $\tilde{\Delta} = \gamma_0 \Delta^\dagger \gamma_0$

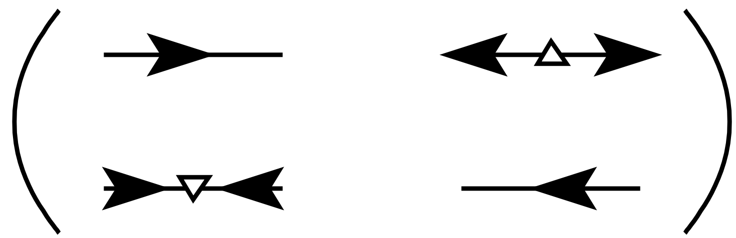
Inverse quark propagator

"Normal" quadratic term,

$$\begin{aligned}\bar\psi\left(\not p+\mu\gamma_0+m\right)\psi &= -\psi^T\left(\not p^T+\mu\gamma_0^T+m\right)\gamma_0^T\psi^* = \bar\psi^C\left(\not p-\mu\gamma_0+m\right)\psi^C\\ &= \frac{1}{2}\bar\psi\left(\not p+\mu\gamma_0+m\right)\psi+\frac{1}{2}\bar\psi^C\left(\not p-\mu\gamma_0+m\right)\psi^C\end{aligned}$$

Thus,

$$S^{-1}=\left(\begin{array}{cc}\left(S_0^+\right)^{-1} & \Delta\\ \tilde\Delta & \left(S_0^-\right)^{-1}\end{array}\right)\equiv$$



with $\left(S_0^\pm\right)^{-1}=\not p\pm\mu\gamma_0+m$.

Then,

$$S=\left(\begin{array}{cc}S_0^+\left(1-\Delta S_0^-\tilde\Delta S_0^+\right)^{-1} & -S_0^+\Delta S_0^-\left(1-\tilde\Delta S_0^+\Delta S_0^-\right)^{-1}\\ -S_0^-\tilde\Delta S_0^+\left(1-\Delta S_0^-\tilde\Delta S_0^+\right)^{-1} & S_0^-\left(1-\tilde\Delta S_0^+\Delta S_0^-\right)^{-1}\end{array}\right)$$

Dirac, color and flavor structure of Δ

$$\Delta \equiv \Delta_{\rm Dirac} \otimes \Delta_{\rm color} \otimes \Delta_{\rm flavor}$$

where

$$\begin{aligned} \Delta^{ij}_{\rm flavor} &\sim \varepsilon^{ij} \quad (N_f = 2) \quad \text{or} \quad \sim \varepsilon^{ijk} \quad (N_f = 3) \\ (\Delta_{\rm color})_{ab} &\sim \varepsilon_{abc} \\ \Delta_{\rm Dirac} &\sim \begin{cases} C & \to \text{ parity odd } \phi \\ C\gamma^5 & \to \text{ parity even } \phi \end{cases} \end{aligned}$$

Example. 2SC phase:

$$\Delta^{ij}_{ab} = \varepsilon^{ij}\varepsilon_{ab3} C\gamma^5 \left(\Delta^+\Lambda^+_p + \Delta^-\Lambda^-_p\right) \quad \text{with} \quad \Lambda^\pm_p = \frac{1}{2}\left(1 \pm \gamma_0 \frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)$$

where $\Delta_\pm$ are complex-valued functions.

Thus, "order parameter" is $\phi_3 = \left\langle \left(\bar{\psi}^C\right)^a_i \varepsilon^{ij}\varepsilon_{ab3} C\gamma^5 \psi^b_j \right\rangle$

Schwinger-Dyson (gap) equation

Consistency equation for dynamically generated self-energy of quarks,

$$S_p^{-1} - S_{p,0}^{-1} = \left(\begin{array}{cc} \Sigma^+ & \Delta \\ \tilde{\Delta} & \Sigma^- \end{array} \right) = 4\pi\alpha_s \int \frac{d^4k}{(2\pi)^4} \Gamma^{A,\mu} \, S_k \, \Gamma^{B,\nu} \, \mathcal{D}_{\mu\nu}^{AB}(k-p)$$

Quark-gluon vertex:

$$\bar{\psi} T^A \gamma^\mu \psi = -\bar{\psi}^C (T^A)^T \gamma^\mu \psi^C = \frac{1}{2} \bar{\Psi} \left(\begin{array}{cc} T^A \gamma^\mu & 0 \\ 0 & -(T^A)^T \gamma^\mu \end{array} \right) \Psi \equiv \frac{1}{2} \bar{\Psi} \Gamma^{A,\mu} \Psi$$

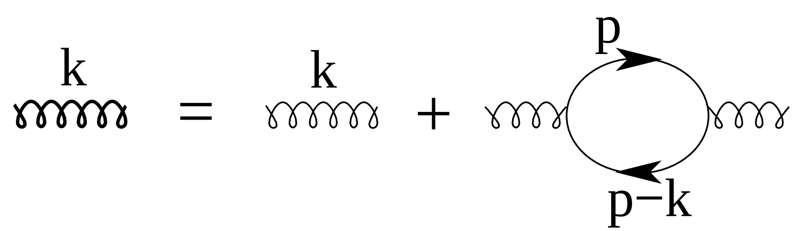
What is a good approximation for $\mathcal{D}_{\mu\nu}^{AB}(q)$ in dense quark matter?

Is the interaction screened? And, if so, how strong is screening?

Screening of gluon-exchange interaction

Gluon propagator in dense medium:

$$\mathcal{D}^{-1}_{\mu\nu}(k) = D^{-1}_{0,\mu\nu}(k) - \Pi_{\mu\nu}(k),$$ i.e.,



Electrical Debye screening and magnetic dynamical screening [Son,hep-ph/9812287]:

$$i\mathcal{D}_{\mu\nu}(k_4,|\vec{k}|) \simeq -\frac{O^{(el)}_{\mu\nu}}{k_4^2+|\vec{k}|^2+2M_D^2} - \frac{|\vec{k}|O^{(mag)}_{\mu\nu}}{|\vec{k}|^3+\pi M_D^2|k_4|/2},$$

where $M_D^2 = \alpha_s N_f \mu^2/\pi$ is the Debye mass

$$r_{el} \sim \frac{1}{M_D} \ll \xi_0 \text{ (short-range) vs. } \frac{r_{mag}}{r_{el}} \sim \left(\frac{\xi_0}{r_{el}}\right)^{1/3} \gg 1 \text{ (long-range)}$$

Magnetic gluon interaction is long-range in the static limit, $k_4 \to 0$

Gap equation in $N_f = 2$ superconductor

$$\Delta_p^{\pm} = \frac{4\pi\alpha_s}{3}\int\frac{d^4k}{(2\pi)^4}\left(\frac{\Delta_k^- \,\mathrm{Tr}\left(\gamma_\mu\Lambda_k^+\gamma_\nu\Lambda_p^{\pm}\right)}{k_0^2-(|\mathbf{k}|-\mu)^2-|\Delta_k^-|^2}+\frac{\Delta_k^+ \,\mathrm{Tr}\left(\gamma_\mu\Lambda_k^-\gamma_\nu\Lambda_p^{\pm}\right)}{k_0^2-(|\mathbf{k}|+\mu)^2-|\Delta_k^+|^2}\right)\mathcal{D}_{k-p}^{\mu\nu}$$

Low-energy and high-energy quasiparticles:

$$k_0 \simeq \sqrt{(|\mathbf{k}|-\mu)^2+|\Delta_k^-|^2}$$

$$k_0 \simeq \sqrt{(|\mathbf{k}|+\mu)^2+|\Delta_k^+|^2} \simeq |\mathbf{k}|+\mu$$

i.e., the actual "gap" parameter is $\Delta_0 = |\Delta_k^-|_{k\simeq\mu}$

Approximaion in the (Euclidean) gap equation,

$$\int\frac{k^2 dk d\Omega}{k_4^2+(|\mathbf{k}|-\mu)^2+\Delta_0^2}\mathrm{Tr}\left(\gamma_\mu\Lambda_k^+\gamma_\nu\Lambda_p^{\pm}\right)\mathcal{D}_{k-p}^{\mu\nu} \simeq \underbrace{\frac{4i\pi^2}{3\sqrt{k_4^2+\Delta_0^2}}\ln\frac{\lambda(\alpha_s)\mu}{|k_4-p_4|}}_{\text{long-range magnetic gluons}}$$

Approximate equation

$$\Delta(p_4) \simeq \frac{\alpha_s}{9\pi} \int \frac{dk_4 \Delta(k_4)}{\sqrt{k_4^2 + \Delta_0^2}} \underbrace{\ln \frac{\lambda(\alpha_s)\mu}{|k_4 - p_4|}}_{\text{extra log}}$$

Approximate kernel:

$$\ln \frac{\lambda(\alpha_s)\mu}{|k_4 - p_4|} \simeq \theta(|k_4| - |p_4|) \ln \frac{\lambda(\alpha_s)\mu}{|k_4|} + \theta(|p_4| - |k_4|) \ln \frac{\lambda(\alpha_s)\mu}{|p_4|}$$

Then, the gap equation is equivalent to

$$p_4 \Delta(p_4)'' + \Delta(p_4)' + \frac{2\alpha_s}{9\pi} \frac{\Delta(p_4)}{\sqrt{p_4^2 + \Delta_0^2}} = 0$$

with the boundary conditions

$$\begin{aligned} p_4 \Delta(p_4)'|_{p_4=0} &= 0 \quad (IR) \\ \Delta(\lambda\mu) &= 0 \quad (UV) \end{aligned}$$

Approximate solution

Solution in IR $(p_4 \ll \Delta_0)$: $\quad \Delta(p_4) = \Delta_0 J_0\left(\sqrt{\frac{8\alpha_s p_4}{9\pi\Delta_0}}\right)$

Solution in UV $(p_4 \gg \Delta_0)$: $\quad \Delta(p_4) = Const \times \sin\left(\sqrt{\frac{2\alpha_s}{9\pi}} \ln\frac{\lambda\mu}{p_4}\right)$

After matching $\Delta(p_4)$ and $\Delta(p_4)'$ at $p_4 = \Delta_0$,

$$\Delta_0 \simeq \lambda\mu \quad \underbrace{\exp\left(-\frac{3\pi^{3/2}}{2^{3/2}\sqrt{\alpha_s}}\right)}_{\text{long-range mag. gluons}}$$

where $\quad \lambda = \underbrace{\frac{2(4\pi)^{3/2}}{\alpha_s^{5/2}}}_{\text{elec. gluons}} \times \underbrace{\exp\left(-\frac{4+\pi^2}{8}\right)}_{\text{quark self-energy corr.}} \times$(higher order corrections)

Symmetries of the ground state

$$\phi_3 = \left\langle \left(\bar{\psi}^C\right)_i^a \varepsilon^{ij} \varepsilon_{ab3} C \gamma^5 \psi_j^b \right\rangle \neq 0$$

Original symmetry:

$$\underbrace{SU(2)_{\rm L} \times SU(2)_{\rm R}}_{\text{chiral symmetry}} \times \underbrace{U(1)_{\rm B}}_{\text{baryon number}} \times \underbrace{[U(1)_{\rm em} \times SU(3)_{\rm c}]}_{\text{gauge symmetry}} \times \underbrace{U(1)_{\rm A}}_{\text{approx. axial}}$$

Residual symmetry in the ground state:

$$SU(2)_{\rm L} \times SU(2)_{\rm R} \times \tilde{U}(1)_{\rm B} \times [\tilde{U}(1)_{\rm em} \times SU(2)_{\rm c}]$$

where

$$\tilde{U}(1)_{\rm B}: \qquad \tilde{B} = \mathbf{1}_{\rm color} - \frac{2}{\sqrt{3}} T_{8,\rm color} = \mathrm{diag}_{\rm color}(0,0,1)$$

$$\tilde{U}(1)_{\rm em}: \qquad \tilde{Q} = Q_{\rm flavor} \otimes \mathbf{1}_{\rm color} - \mathbf{1}_{\rm flavor} \otimes \frac{1}{\sqrt{3}} T_{8,\rm color}$$

$N_f = 3$ color superconductivity

Order parameter:

$$\left\langle \left(\bar{\psi}^C\right)^a_i \varepsilon^{ijk} \varepsilon_{abc} C \gamma^5 \psi^b_j \right\rangle \sim \delta^c_k$$

or, in terms of chiral fields,

$$\left\langle \psi^{a,\alpha}_{L,i} \varepsilon^{ijk} \varepsilon_{abc} \varepsilon_{\alpha\beta} \psi^{b,\beta}_{L,j} \right\rangle = -\left\langle \psi^{a,\dot{\alpha}}_{R,i} \varepsilon^{ijk} \varepsilon_{abc} \varepsilon_{\dot{\alpha}\dot{\beta}} \psi^{b,\dot{\beta}}_{R,j} \right\rangle \sim \delta^c_k$$

Color-flavor locking:

$\langle \mathrm{LL} \rangle$	is invariant under	$g_{\mathrm{L,flavor}} \otimes g_{\mathrm{color}}$	if	$g_{\mathrm{color}} \equiv g^{-1}_{\mathrm{L,flavor}}$
$\langle \mathrm{RR} \rangle$	is invariant under	$g_{\mathrm{R,flavor}} \otimes g_{\mathrm{color}}$	if	$g_{\mathrm{color}} \equiv g^{-1}_{\mathrm{R,flavor}}$

i.e., residual global symmetry is $SU(3)_{\mathrm{L+R}}$

As in vacuum, there appear 8 Nambu-Goldstone bosons:

$$\pi^0,\ \pi^{\pm},\ K^0,\ \bar{K}^0,\ K^{\pm},\ \eta$$

Symmetries of CFL phase

$$\phi\delta_k^c = \left\langle \left(\bar{\psi}^C\right)_i^a \varepsilon^{ijk}\varepsilon_{abc} C\gamma^5\psi_j^b \right\rangle \neq 0$$

Original symmetry:

$$\underbrace{SU(3)_{\rm L}\times SU(3)_{\rm R}}_{\substack{\text{chiral symmetry}\\ U(1)_{\rm em}\in SU(3)_{\rm L}\times SU(3)_{\rm R}}} \times \underbrace{U(1)_{\rm B}}_{\text{baryon number}} \times \underbrace{[SU(3)_{\rm c}]}_{\text{color gauge symmetry}} \times \underbrace{U(1)_{\rm A}}_{\text{approx. axial}}$$

Residual symmetry in the ground state:

$$SU(3)_{\rm L+R}\times \mathbf{Z}_2 \times [\tilde{U}(1)_{\rm em}]$$

where

$$\tilde{U}(1)_{\rm em}: \qquad \tilde{Q} = Q_{\rm flavor}\otimes \mathbf{1}_{\rm color} - \mathbf{1}_{\rm flavor}\otimes\left(T_3 + \frac{1}{\sqrt{3}}T_8\right)_{\rm color}$$

Low-energy action in the CFL phase

$$\mathcal{L}_{\rm eff} = \frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_0\Sigma\partial_0\Sigma^\dagger - v^2\partial_i\Sigma\partial_i\Sigma^\dagger\right) + \frac{1}{2}\left[(\partial_0\phi)^2 - v^2(\partial_i\phi)^2\right] + \frac{1}{2}\left[(\partial_0\eta')^2 - v^2(\partial_i\eta')^2\right]$$

where

$$\Sigma = \exp\left(i\lambda^A\pi^A/f_\pi\right)$$

and

$$f_\pi^2 = \frac{21-8\ln 2}{36\pi^2}\mu^2 \quad \text{and} \quad v^2 = \frac{1}{2} \quad (\text{in QCD at } \mu\to\infty)$$

Effect of strage quark mass on mesons

Denote $M = \text{diag}(m_u, m_d, m_s)$. Then, $\mathcal{L}_{\text{eff}}$ has several new terms,

$$\frac{3\Delta^2}{4\pi^2}\left(\text{Tr}(M\Sigma^\dagger)\text{Tr}(M\Sigma^\dagger) - \text{Tr}(M\Sigma^\dagger M\Sigma^\dagger)\right)$$

in addition to

$$\partial_0\Sigma \to \nabla_0\Sigma = \partial_0\Sigma + i\frac{MM^\dagger}{2\mu}\Sigma - i\Sigma\frac{M^\dagger M}{2\mu}$$

Note that $\mu^s_{\text{eff}} = m_s^2/(2\mu)$.
When μ^s_{eff} is larger than the mass of K, $m_K^{(0)} \simeq \Delta/\mu\sqrt{m_q m_s}$, Bose-Einstein condensation should occur, i.e.,

$$\langle\Sigma\rangle \simeq \exp\left(i\theta\lambda_4\right)$$

where

$$\cos\theta \simeq \left(\frac{m_K^{(0)}}{\mu^s_{\text{eff}}}\right)^2 \quad \text{for} \quad \mu^s_{\text{eff}} > m_K^{(0)}$$

Abnormal number of Goldstone bosons

Toy model of kaon condensation in CFL phase

$$\mathcal{L} = (\partial + i\mu)\Phi^\dagger(\partial - i\mu)\Phi - v^2\partial_i\Phi^\dagger\partial_i\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$$

where Φ is a complex doublet field (i.e., K^0, $\bar{K}^0$, K^-, K^+)

(i) $m > \mu$, normal phase with $\begin{cases} m_{K^-,\bar{K}^0} = m + \mu, \\ m_{K^+,K^0} = m - \mu \end{cases}$

(ii) $\mu > m$, broken phase with $\langle \Phi^T \rangle = (0, \phi_0) \neq 0$, i.e.,

$$SU(2) \times U(1) \to U(1) \qquad \text{THREE broken generators}$$

However, there are only TWO NG bosons:

$$\omega_2 \simeq \frac{v^2 q^2}{2\mu}, \quad (!) \qquad \tilde{\omega}_2 \simeq \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}}\, vq$$

No superfluidity of matter in this phase!

Effect of strage quark mass on baryons

$$\mathcal{L} = \mathrm{Tr}\left(N^\dagger i v^\mu D_\mu N\right) - D\mathrm{Tr}\left(N^\dagger v^\mu \gamma_5\{A_\mu, N\}\right) - F\mathrm{Tr}\left(N^\dagger v^\mu \gamma_5[A_\mu, N]\right)$$
$$+ \frac{\Delta}{2}\left[\left(\mathrm{Tr}\left(N_L N_L\right) - [\mathrm{Tr}(N_L)]^2\right) - (L \to R) + h.c.\right]$$