Arizona State University
Polytechnic Campus

Huada School on QCD
MAY 29, 2017
CCNU, Wuhan

Lecture #1
Magnetic Catalysis: basics

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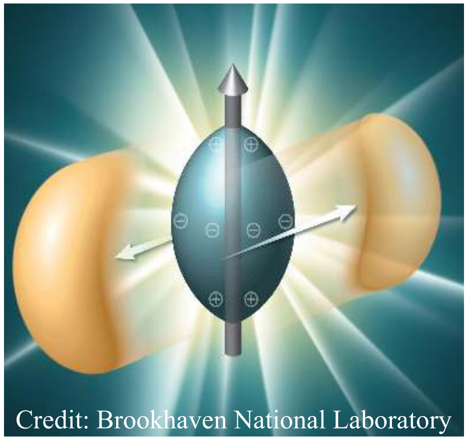
Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports **576** (2015)

MAGNETIC CATALYSIS: PLAN OF LECTURES

- Dirac fermions in magnetic field
- Dimensional reduction
- Magnetic catalysis: basics
- Magnetic catalysis in toy model
- Magnetic catalysis in QED
- Magnetic catalysis in QCD
- Anisotropic confinement
- Inverse catalysis
- Phase diagram

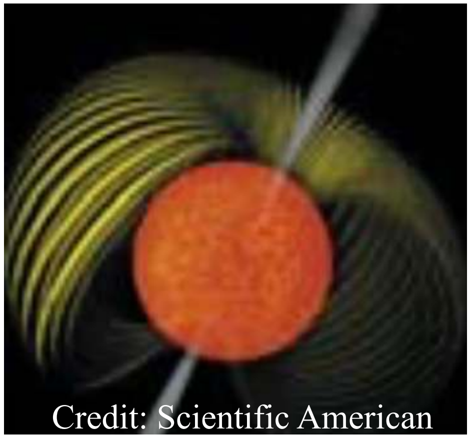
QCD IN MAGNETIC FIELDS

- Relativistic collisions of *heavy ions* produce quark-gluon plasma & strong magnetic fields



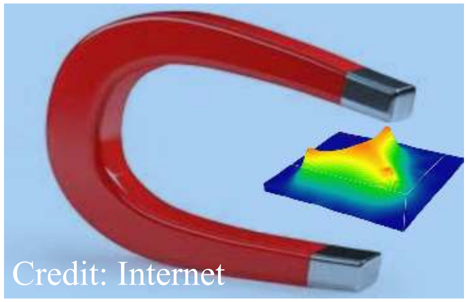
10^{18} - 10^{19} Gauss ($\sqrt{|eB|} \sim$ 100 MeV)

- Quark matter may form inside *magnetars*



10^{14} - 10^{16} Gauss ($\sqrt{|eB|} \sim$1 MeV to 10 MeV)

- Strong magnetic field is an instructive *theoretical tool* to study confined gauge theories such as QCD



$\gtrsim 10^{19}$ Gauss ($\sqrt{|eB|} \gtrsim$100 MeV to 10 MeV)

Dirac fermions

- Lagrangian density for charged Dirac fermions (units with $c = 1$):

$$\mathcal{L} = \bar{\psi}\,(i\gamma^{\mu} D_{\mu} - m)\psi$$

where $D_{\mu} = \partial_{\mu} + ieA_{\mu}$, $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$ and $g^{\mu\nu} = (1,-1,-1,-1)$

- Consider the following two types of global transformations:

Electric charge conservation: $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\alpha\gamma^{5}}\psi$:Chiral charge conservation

where $\gamma^{5} = i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$

The corresponding Noether's currents are

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi \quad \text{and} \quad j_{5}^{\mu} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi$$

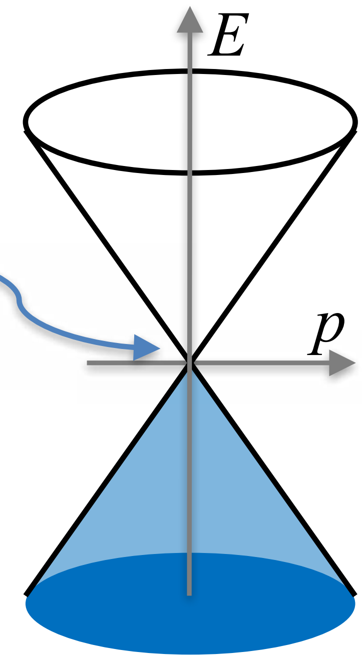
They satisfy the relations:

$$\partial_{\mu} j^{\mu} = 0 \quad \text{and} \quad \partial_{\mu} j_{5}^{\mu} = 2i\, m\, \bar{\psi}\gamma^{5}\psi$$

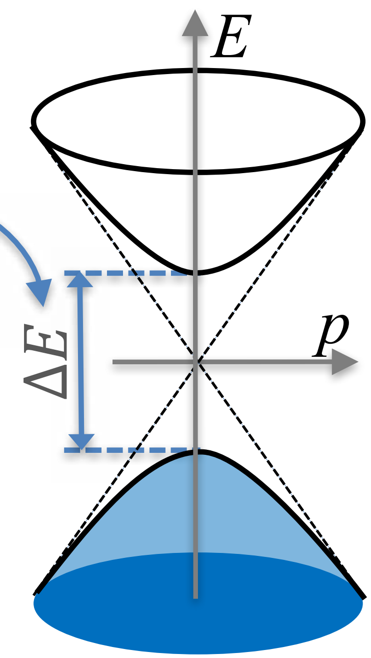
Both transformations are **symmetries** when $m = 0$, but chiral symmetry is broken when $m \neq 0$. *[The chiral anomaly may complicate the situation]*

DIRAC VACUUM

- $m = 0$: Dirac vacuum is a semimetal
  - No energy gap between the filled Dirac sea states and the empty positive-energy states ($E = \pm p$)
  - However, the density of states *vanishes* at E=0
  - A nonzero electric current could be produced by an arbitrarily small electric field



- $m \neq 0$: Dirac vacuum is an insulator
  - Energy gap $\Delta E = 2m$ between the antiparticle and particle states $\left(E = \pm\sqrt{p^2 + m^2}\right)$
  - the density of states @ E=0 *vanishes* (no states)
  - electric current is exponentially small, i.e., $e^{-\pi\, m^2/|eE|}$ (due to Schwinger pair creation)



DIRAC FERMIONS AT B≠0

- Dirac equation for charged fermions:

$$\left(i\gamma^\mu D_\mu - m\right)\psi = 0$$

where $A_\mu = \left(A_0, -\vec{\boldsymbol{A}}\right)$ and the Landau gauge $\vec{\boldsymbol{A}} = (-By, 0{,}0)$ is used.

- Look for a solution in the form: $\psi = \left(i\gamma^\mu D_\mu + m\right)\phi$. Then,

$$\left[-\partial_0^2 + (\partial_x + ieBy)^2 + \partial_y^2 + \partial_z^2 + i\gamma^1\gamma^2 eB - m^2\right]\phi = 0$$

- Normalized solutions for ϕ have the form

$$\phi_{k,\pm} \propto \frac{1 \pm i\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}\varphi_k(y)e^{-i\omega t + ip_x x + ip_z z}$$

where φ_k are harmonic oscillator wave functions, i.e.,

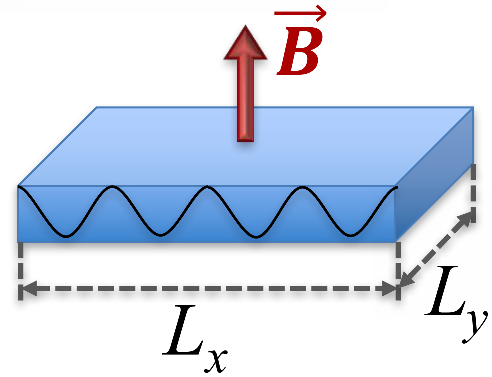
$$\varphi_k \propto H_k(\xi)e^{-\frac{\xi^2}{2}}, \quad \xi = \frac{y}{l} + p_x l\,\mathrm{sgn}(eB) \quad \text{and} \quad l = \frac{1}{\sqrt{|eB|}}$$

- The dispersion relation is given by

$$\omega = E_n^\pm = \pm\sqrt{2n|eB| + p_z^2 + m^2}$$

where $n = \underbrace{k + \frac{1}{2}}_{\text{orbital}} + \underbrace{\mathrm{sgn}(eB)s_z}_{\text{spin}}$ and $s_z = \pm\frac{1}{2}$ is an eigenvalue of $\frac{i}{2}\gamma^1\gamma^2$

DEGENERACY OF LANDAU LEVELS



- The Landau level energies are independent of p_x

$$E_n^{\pm} = \pm\sqrt{2n|eB| + p_z^2 + m^2}$$

- This means that each level is highly degenerate
- Let's calculate the degeneracy by confining the system in a finite box of size $L_x \times L_y$ with periodic boundary conditions
- The wave function is a plane wave in the x direction: $\psi(x) \propto e^{ip_x x}$

$$\psi(0) = \psi(L_x) \quad \Longrightarrow \quad e^{ip_x L_x} = 1 \quad \Longrightarrow \quad p_x = \frac{2\pi n}{L_x}, \quad n = 1,2,\ldots,N_{\max}$$

- The value of p_x sets the center of the Landau orbit in y-direction:

$$y_c \approx p_x l^2 \quad \Longrightarrow \quad p_{x,\max} l^2 \lesssim L_y \quad \Longrightarrow \quad \frac{2\pi N_{\max}}{L_x}\frac{1}{|eB|} \approx L_y \quad \Longrightarrow \quad \frac{N_{\max}}{L_x L_y} \approx \frac{|eB|}{2\pi}$$

- The degeneracy is proportional to the field strength and the size (area) of the system in the spatial directions perpendicular to $\vec{\boldsymbol{B}}$

$$N_{\max} \approx \frac{|eB|}{2\pi} L_x L_y$$

LANDAU ENERGY SPECTRUM

- Landau energy levels at $m = 0$

$$E_n^{\pm} = \pm\sqrt{2n|eB| + p_z^2}$$

where $n = \underbrace{k + \frac{1}{2}}_{\text{orbital}} + \underbrace{\text{sgn}(eB)s_z}_{\text{spin}}$

- Lowest Landau level is spin polarized

$$E_0^{\pm} = \pm p_z \qquad (k = 0,\ s_z = -\tfrac{1}{2})$$

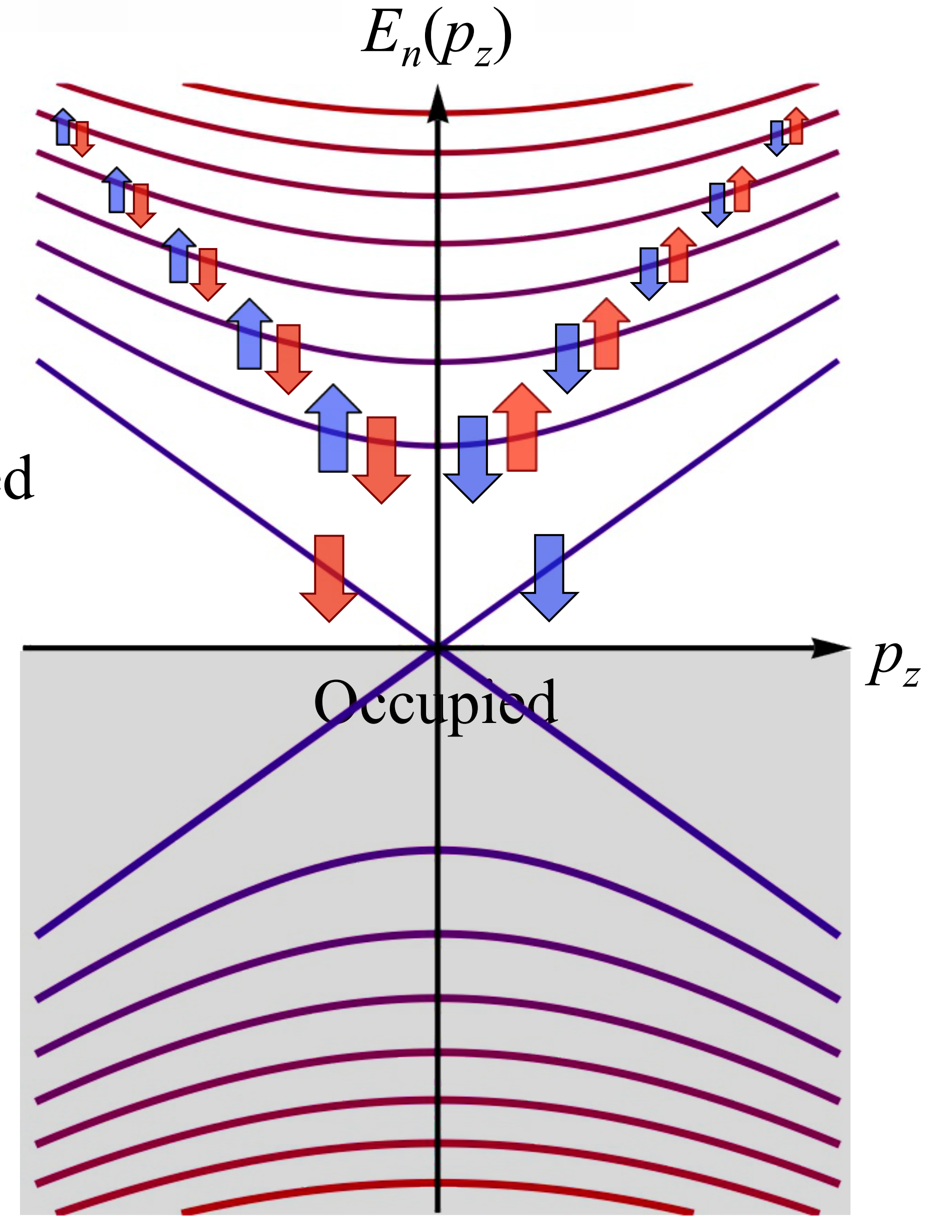
- Density of states at E=0:

$$\left.\frac{dn}{dE}\right|_{E=0} = \frac{|eB|}{2\pi}\frac{1}{2\pi} = \frac{|eB|}{4\pi^2}$$

- Higher Landau levels ($n \geq 1$) are twice as degenerate:

(i) $k = n$ & $s = -\frac{1}{2}$

(ii) $k = n - 1$ & $s = +\frac{1}{2}$



DIRAC PROPAGATOR AT B≠0

- By definition,

$$G(r,r') = i\left\langle r\middle|(i\gamma^\mu D_\mu - m)^{-1}\middle|r'\right\rangle$$

$$= i(i\gamma^\mu D_\mu + m)_r \left\langle r\middle|\left[(i\gamma^\mu D_\mu - m)(i\gamma^\nu D_\nu + m)\right]^{-1}\middle|r'\right\rangle$$

$$= i(i\gamma^\mu D_\mu + m)_r \left\langle r\middle|\left[-D^\mu D_\mu + i\gamma^1\gamma^2 eB - m^2\right]^{-1}\middle|r'\right\rangle$$

$$= i(i\gamma^\mu D_\mu + m)_r \sum \langle r|k,p_z,s_z\rangle\,(\omega^2 - E_n^2)^{-1}\langle k,p_z,s_z|r'\rangle$$

$$\mathbb{I} = \sum |k,p_z,s_z\rangle\langle k,p_z,s_z|$$

- Note that the explicit form of the wave functions is the same as before

$$\psi_{k,p_z,s_z}(r) = \langle r|k,p_z,s_z\rangle \propto H_k(\xi)e^{-\frac{\xi^2}{2}}e^{-i\omega t+ip_z z}\,U_{s_z}, \quad \text{where } \xi = \frac{y}{l} + p_x l$$

- The final expression for the propagator has the form

$$G(\omega,p_z;\vec{r}_\perp,\vec{r}_\perp') = e^{i\Phi(\vec{r}_\perp,\vec{r}_\perp')}\tilde{G}(\omega,p_z;\vec{r}_\perp - \vec{r}_\perp')$$

where $\Phi(\vec{r}_\perp,\vec{r}_\perp') = -\mathrm{e}\int_{\vec{r}_\perp'}^{\vec{r}_\perp} A_\nu dr^\nu$ is the Schwinger phase (!), and

$$\tilde{G}(\omega,p_z;\vec{r}_\perp - \vec{r}_\perp') = \int \frac{d^2\vec{p}_\perp}{(2\pi)^2} e^{i\vec{p}_\perp\cdot(\vec{r}_\perp - \vec{r}_\perp')}\,\tilde{G}(\omega,\vec{p})$$

DIRAC PROPAGATOR AT B≠0

- The Fourier transform of the translation invariant part reads

$$\tilde{G}(\omega,\vec{p}) = i e^{-\vec{p}_\perp^2 l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega,\vec{p})}{\omega^2 - E_n^2}$$

Laguerre polynomials

where

$$D_n(\omega,\vec{p}) = 2(\omega\gamma^0 - p_z\gamma^3 + m)[\mathcal{P}_+ L_n(2\vec{p}_\perp^2 l^2) - \mathcal{P}_- L_{n-1}(2\vec{p}_\perp^2 l^2)]$$
$$+4(\vec{p}_\perp \cdot \vec{\gamma}_\perp) L_{n-1}^1(2\vec{p}_\perp^2 l^2)$$

and the following notation for the spin projectors is used

$$\mathcal{P}_\pm = \frac{1 \pm i\,\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}$$

- Similarly, in momentum-coordinate space representation:

$$\tilde{G}(\omega,p_z;\vec{r}_\perp) = i \frac{e^{-\vec{r}_\perp^2/(4l^2)}}{2\pi l^2} \sum_{n=0}^{\infty} \frac{F_n(\omega,p_z;\vec{r}_\perp)}{\omega^2 - E_n^2}$$

Laguerre polynomials

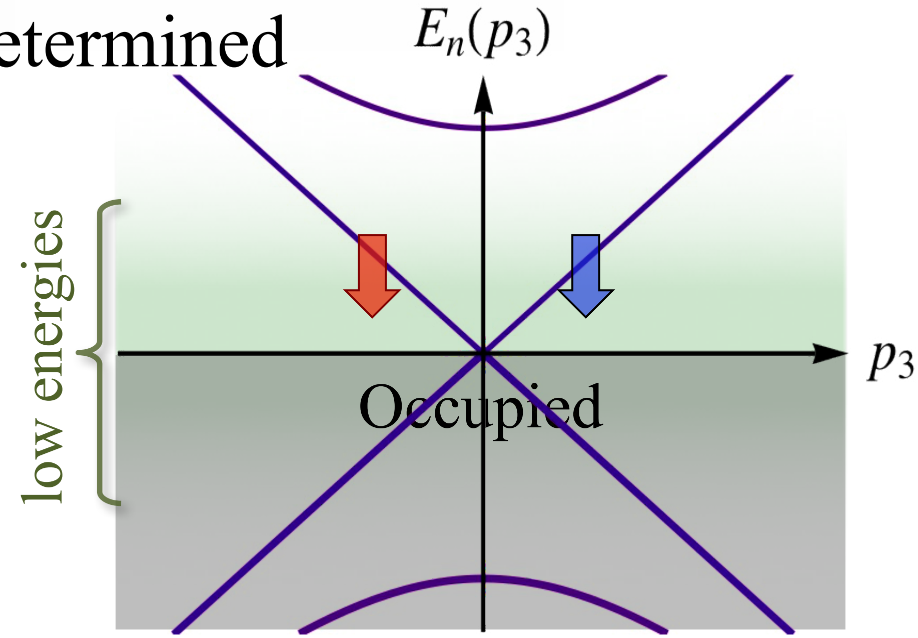
where $F_n(\omega,p_z;\vec{r}_\perp) = 2(\omega\gamma^0 - p_z\gamma^3 + m)\left[\mathcal{P}_+ L_n\left(\frac{\vec{r}_\perp^2}{2l^2}\right) - \mathcal{P}_- L_{n-1}\left(\frac{\vec{r}_\perp^2}{2l^2}\right)\right]$

$$-\frac{i}{l^2}(\vec{r}_\perp \cdot \vec{\gamma}_\perp) L_{n-1}^1\left(\frac{\vec{r}_\perp^2}{2l^2}\right)$$

Dimensional reduction

- The low-energy dynamics is determined by the lowest Landau level (n=0)

$$E_0^{\pm} = \pm p_z$$



- This is a (1+1)D spectrum!

- Propagator is also (1+1)D:

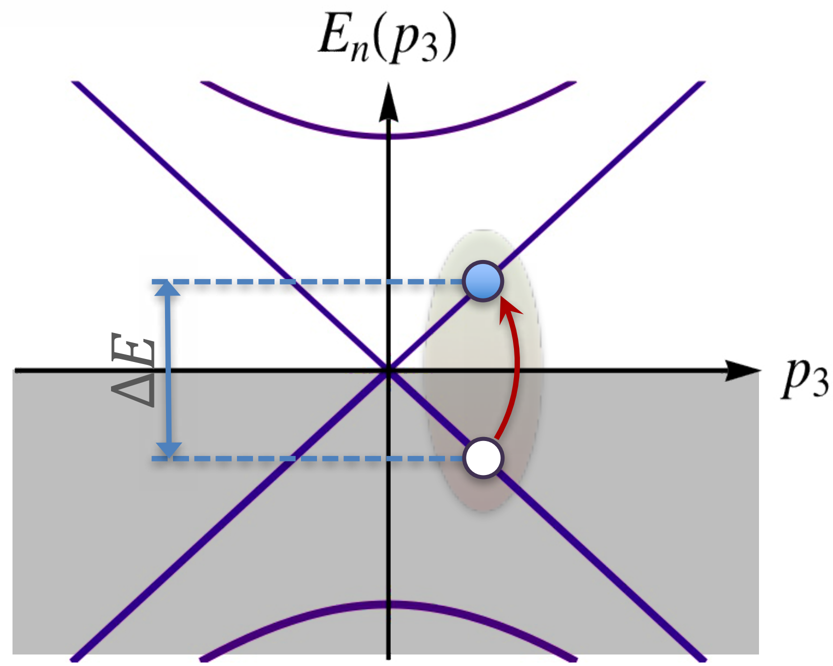
$$\tilde{G}_{LLL}(\omega, \vec{p}) = 2ie^{-\vec{p}_\perp^2 l^2} \frac{\omega\gamma^0 - p_z\gamma^3}{\omega^2 - p_z^2} \frac{1 + i\,\mathrm{sgn}(eB)\gamma^1\gamma^2}{2}$$

- In addition, there is a nonzero density of states at E=0:

$$\left.\frac{dn}{dE}\right|_{E=0} = \frac{1}{\delta E}\left(\frac{N_{\max}}{L_x L_y}\right)\left(\int_0^{\delta E} \frac{dp_z}{2\pi}\right) = \frac{|eB|}{4\pi^2}$$

PAIRING INSTABILITY

- Thought experiment:
  - Create a particle-antiparticle pair (energy price: ΔE)
  - The pair can form a bosonic bound state (energy gain: $-\epsilon_b$)
  - If $\epsilon_b > \Delta E$, copious formation of bound states is beneficial
  - Note, ΔE can be arbitrarily small when $m = 0$ (!)
  - The bound states of fermions are bosons
  - Bosons can (and will) occupy the lowest energy state ($\vec{P} = 0$), and thus form a Bose condensate $\langle \bar{\psi}\psi \rangle \neq 0$
  - Ground state (vacuum) changes its properties (e.g., chiral symmetry breaks down, an energy gap opens in spectrum)



Do bound states always form in 3D?

- Consider a 3D potential well in quantum mechanics
  [Landau-Lifshitz, Quantum Mechanics]

$$U(r) = \begin{cases} -g\dfrac{\pi^2\hbar^2}{8m_* a^2} & \text{for} \quad r \le a \\ 0 & \text{for} \quad r > a \end{cases}$$

- Bound states form only when the well is deep enough (namely, $g > 1$):

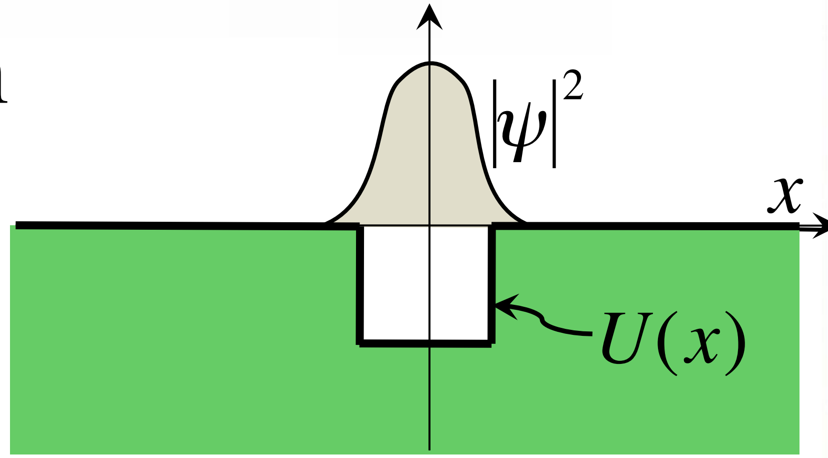
$$|E_{3D}| \approx \frac{\pi^4\hbar^2}{2^7 a^2 m_*}(g-1)^2 , \quad \text{assuming} \quad 0 < g-1 << 1$$

- There are no bound states when $g < 1$, i.e., when the well is not deep enough (in other words, when the coupling constant is not strong enough)

Compare: Bound states in 1D

- Bound states always form

$$\left|E_{1D}\right| \approx \frac{m_*}{2\hbar^2}\left(-\int_{-\infty}^{+\infty} U(x)\,dx\right)^2$$



- This is a perturbative result (!)

$$\left|E_{1D}\right| \propto g^2\,, \quad \text{when} \quad U(x) \rightarrow gU(x)$$

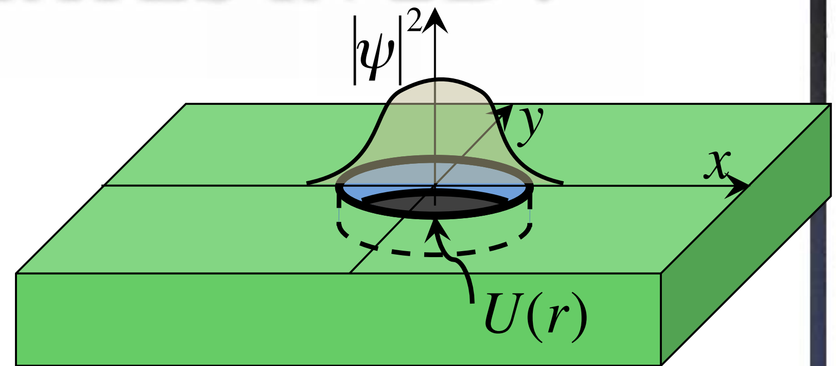
- Rigorous statement: at least one bound state exists if

$$\int \left(1+|x|\right)\left|U(x)\right| dx < \infty \quad \& \quad \int U(x)\,dx \le 0$$

[B. Simon, Annals Phys. 97 (1976) 279]

How about bound states in 2D?

- Bound states always form



$$|E_{2D}| \approx \frac{\hbar^2}{a^2 m_*} \exp\left(-\frac{\hbar^2}{m_*} \left| \int_0^\infty r U(r) dr \right|^{-1} \right)$$

- This is a non-perturbative result

$$|E_{2D}| \propto \exp\left(-\frac{C}{g} \right), \quad \text{when} \quad U(x) \to g U(x)$$

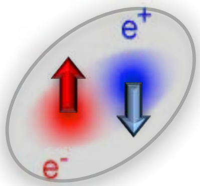
- Rigorous statement: at least one bound state exists if

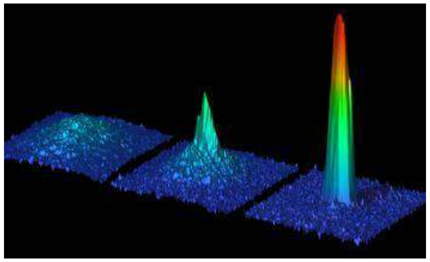
$$\int |U(x)|^{1+\varepsilon} \, d^2x < \infty, \quad \int (1+x^2)^{\varepsilon} |U(x)| \, d^2x < \infty \quad \& \quad \int U(x) \, d^2x \leq 0$$

[B. Simon, Annals Phys. 97 (1976) 279]

Universal magnetic catalysis

- Quantum field theory of charged fermions (m=0) at $\vec{\boldsymbol{B}} \neq \mathbf{0}$
  - Dimensional reduction (caused by a nonzero $\vec{\boldsymbol{B}}$)
  - Nonzero density of states ($\propto |eB|$) at E=0
  - Attraction between particles and antiparticles
- Universal outcome:
  - Copious particle-antiparticle pairing at low energies
  - Condensation of boson pairs that destabilizes the trivial Dirac vacuum
  - Spontaneous rearrangement of the ground state
  - Breakdown of chiral symmetry
  - Opening a nonzero gap in the Dirac spectrum





[Gusynin, Miransky, Shovkovy, Phys. Rev. Lett. **73**, 3499 (1994)]
[Shovkovy, Lect. Notes Phys. **871**, 13 (2013)]

- The mechanism is similar to superconductivity in metals due to Cooper pairing of electrons

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Lecture #2 Magnetic Catalysis in a Toy Model

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Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports **576** (2015)

TOY MODEL

- Let us consider a Nambu-Jona-Lasino model ($m = 0$) with four-fermion contact interaction

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu} D_{\mu}\right)\psi + \frac{G}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma^5\psi)^2\right]$$

- After the Hubbard–Stratonovich transformation, this is equivalent to

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu} D_{\mu} - \sigma - i\gamma^5\pi\right)\psi - \frac{\sigma^2 + \pi^2}{2G}$$

where the following composite fields were introduced

$$\sigma = -G\,\bar{\psi}\psi \quad \text{and} \quad \pi = -G\,\bar{\psi} i\gamma^5\psi$$

- The effective action for the composite fields reads

$$\Gamma(\sigma,\pi) = -\frac{1}{2G}\int d^4x(\sigma^2 + \pi^2) - i\,\mathrm{Tr}\ln\left[i\gamma^{\mu} D_{\mu} - \sigma - i\gamma^5\pi\right]$$

Symmetry of the Model

- $\mathrm{U_L}(1)$ symmetry transformations, $\psi \to e^{i\alpha_L(1-\gamma^5)/2}\psi$

$$\bar{\psi}\psi \to \cos\alpha_L\,\bar{\psi}\psi - \sin\alpha_L\,\bar{\psi}i\gamma^5\psi$$
$$\bar{\psi}i\gamma^5\psi \to \sin\alpha_L\,\bar{\psi}\psi + \cos\alpha_L\,\bar{\psi}i\gamma^5\psi$$

- $\mathrm{U_R}(1)$ symmetry transformations, $\psi \to e^{i\alpha_R(1+\gamma^5)/2}\psi$

$$\bar{\psi}\psi \to \cos\alpha_R\,\bar{\psi}\psi + \sin\alpha_R\,\bar{\psi}i\gamma^5\psi$$
$$\bar{\psi}i\gamma^5\psi \to -\sin\alpha_R\,\bar{\psi}\psi + \cos\alpha_R\,\bar{\psi}i\gamma^5\psi$$

- In terms of the composite fields, $\mathrm{U_L}(1)$ / $\mathrm{U_R}(1)$ transformations:

$$\sigma \to \cos\alpha_L\,\sigma - \sin\alpha_L\,\pi$$
$$\pi \to \sin\alpha_L\,\pi + \cos\alpha_L\,\sigma$$

(Note that $\rho^2 = \sigma^2 + \pi^2$ remains an invariant.)

- Just like the original action $\int \mathcal{L}\, d^4x$, the effective action $\Gamma(\sigma,\pi)$ should be invariant under the symmetry transformations, i.e.,

$$\Gamma(\sigma,\pi) = \Gamma(\rho) + \frac{1}{2} f_1^{\mu\nu}\left(\partial_\mu\sigma\partial_\nu\sigma + \partial_\mu\pi\partial_\nu\pi\right) + \cdots$$

EFFECTIVE POTENTIAL: DERIVATION

- Let us consider a homogeneous ground state with a *uniform* σ

$$\sigma = -G\langle\bar{\psi}\psi\rangle \neq 0$$

(Because of the chiral symmetry, we can always set $\pi = 0$.)

- In this case, $\Gamma(\sigma) = -\int V(\sigma) d^4x$, where the effective action is

$$V(\sigma) = \frac{\sigma^2}{2G} - \frac{i}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{tr}\left\langle x\middle| e^{-is(D^\mu D_\mu - i\gamma^1\gamma^2 eB + \sigma^2)}\middle| x\right\rangle - (\infty)$$

- By using the Schwinger result [Phys. Rev. 82, 664 (1951)]

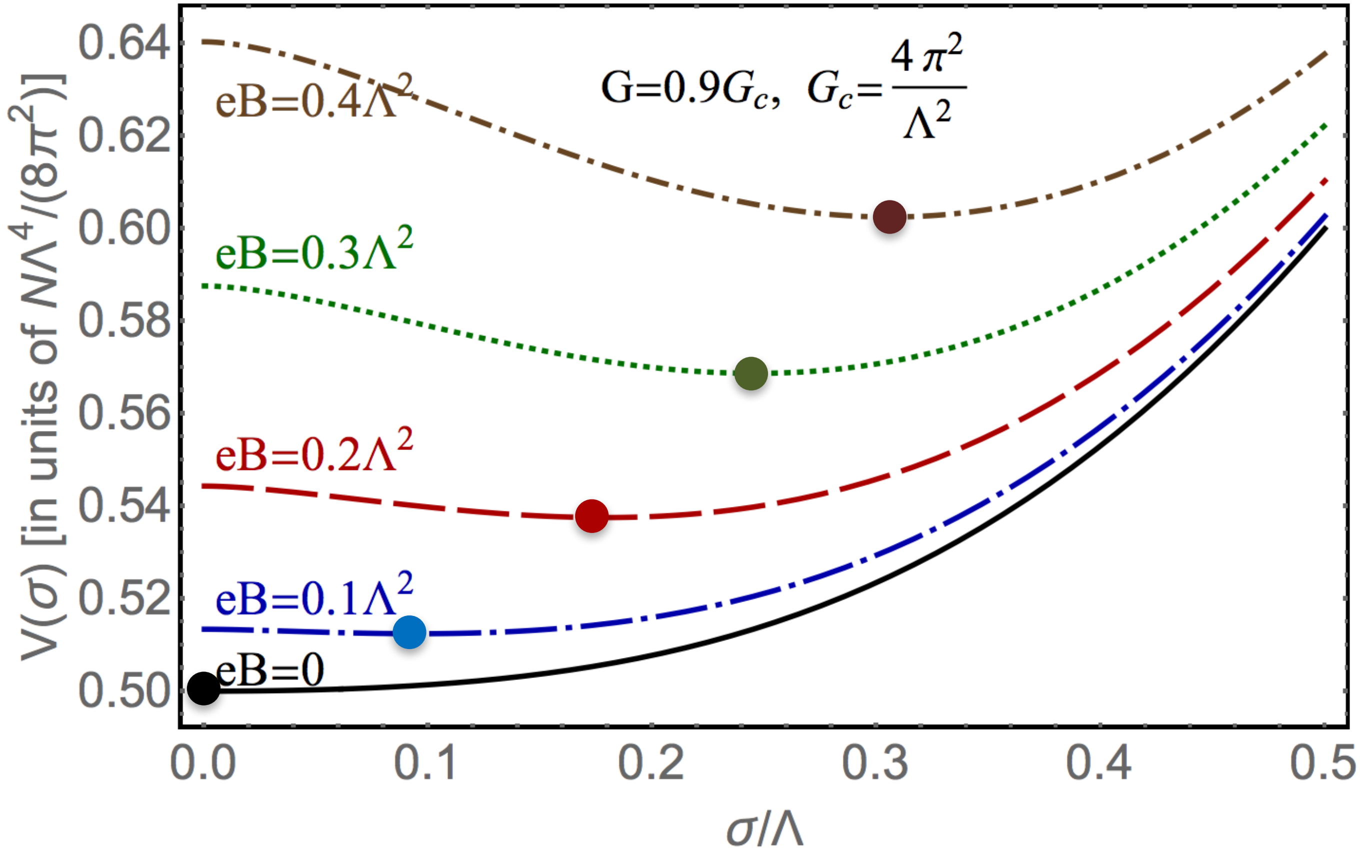
$$\left\langle x\middle| e^{-is(D^\mu D_\mu - i\gamma^1\gamma^2 eB + \sigma^2)}\middle| x\right\rangle = \frac{e^{-is\sigma^2 - i\pi/4}}{8(\pi s)^{3/2}} eBs[\cot eBs + \gamma^1\gamma^2]$$

- We derive the effective potential (after $s \to -is$):

$$V(\sigma) = \frac{\sigma^2}{2G} + \frac{eB}{8\pi^2}\int_{1/\Lambda^2}^\infty \frac{ds}{s^2} e^{-s\sigma^2} \coth eBs - (\infty)$$

EFFECTIVE POTENTIAL: RESULTS

Lowest energy ground state is defined by: $\frac{dV(\sigma)}{d\sigma} = 0$ (gap equation)

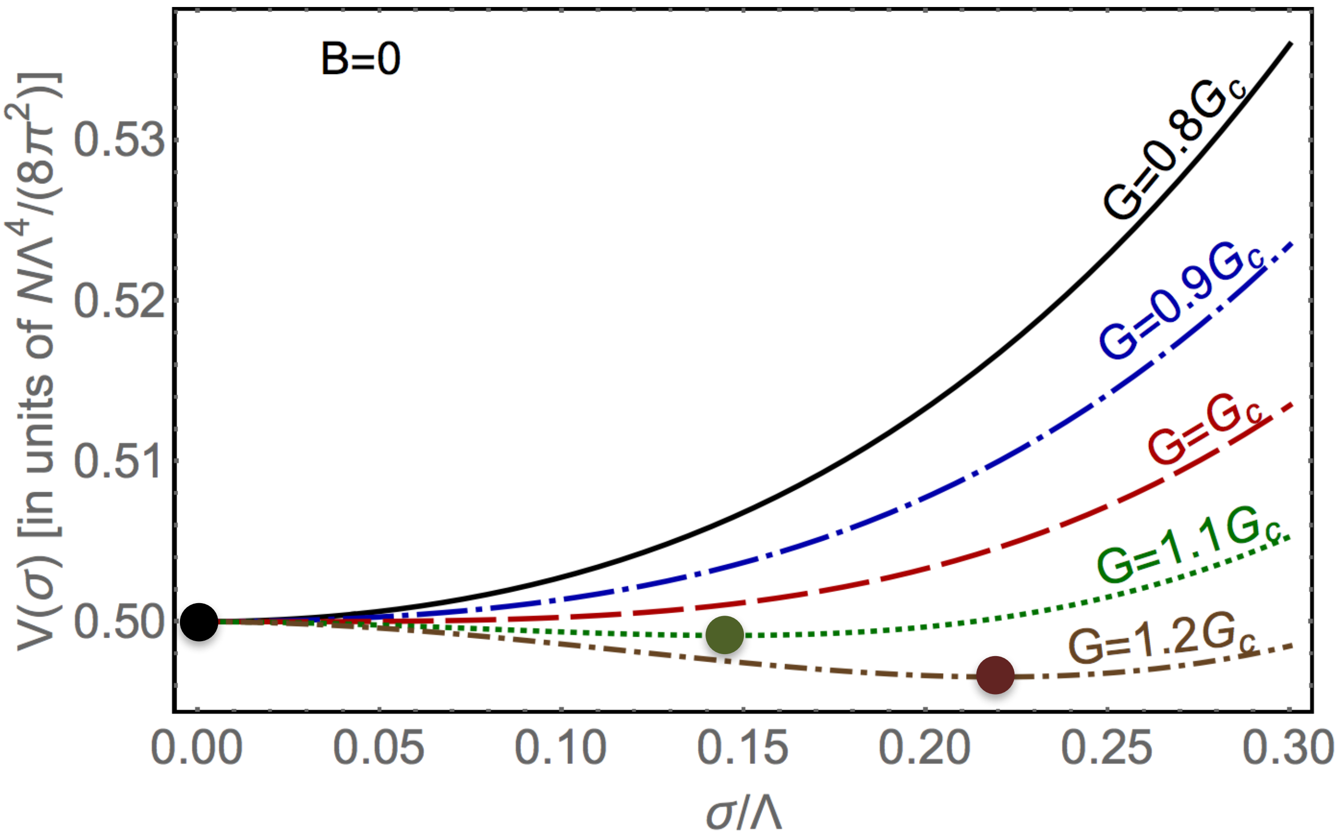


At weak coupling ($G \to 0$), the analytical solution for the minimum

$$\sigma_{\min} \simeq \frac{eB}{\pi} \exp\left(\frac{\Lambda^2}{|eB|}\right) \exp\left(-\frac{4\pi^2}{|eB|G}\right)$$

COMPARE WITH B=0 CASE

- Effective potentials for different coupling constants

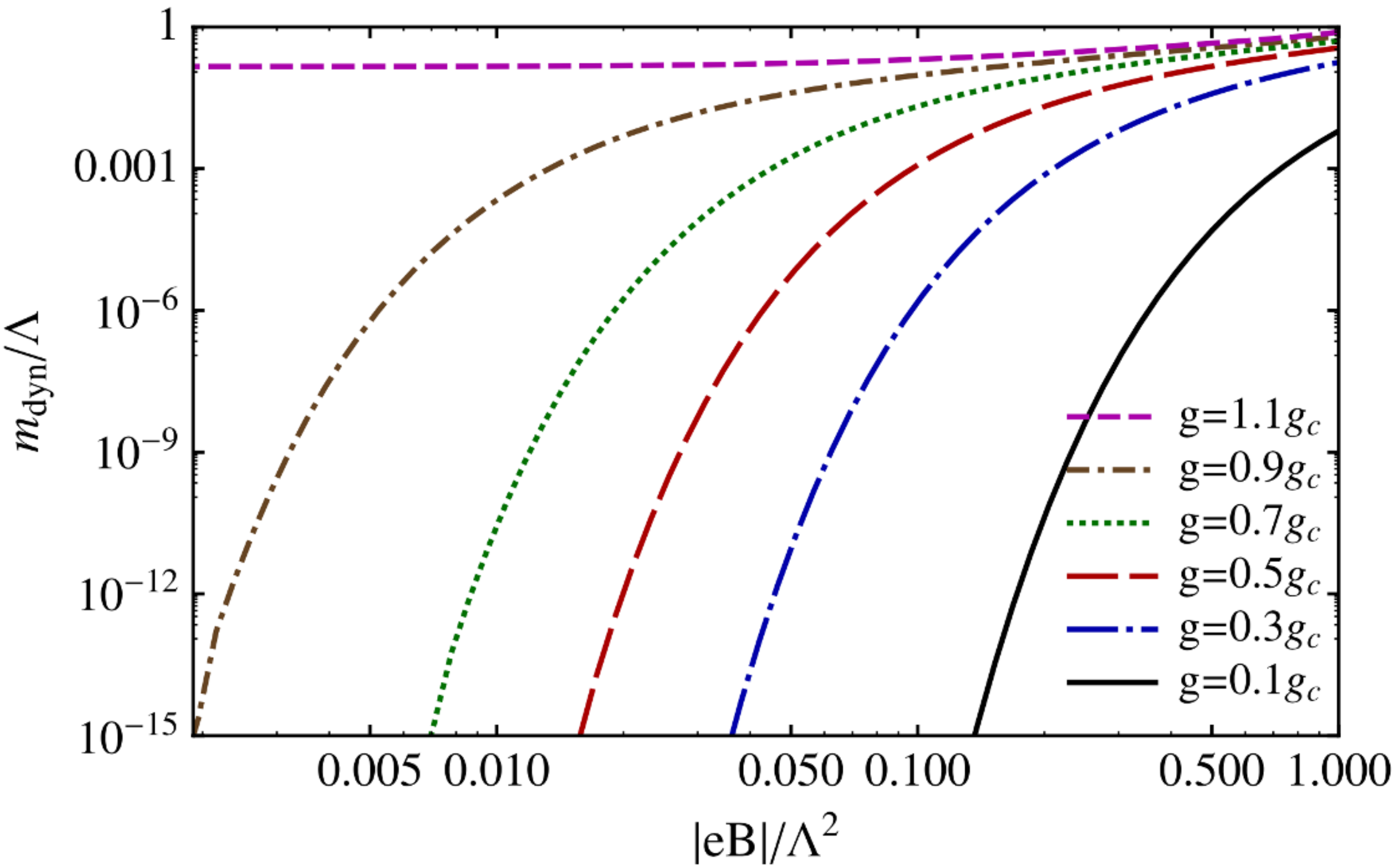


In fact, the gap equation at B=0 reads $\frac{G\Lambda^2-4\pi^2}{G} = \sigma^2 \ln\frac{\Lambda^2}{\sigma^2}$

It has a nontrivial solution $\sigma_{\min} \neq 0$ only when the coupling strength is sufficiently strong, i.e., $G > G_c = 4\pi^2/\Lambda^2$

DYNAMICAL MASS

- Recall: $\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - \sigma - i\gamma^{5}\pi\right)\psi - \frac{\sigma^2+\pi^2}{2G}$
- The ground state expectation value $\langle\sigma\rangle = \sigma_{\min}$ determines the dynamical mass of fermions m_{dyn} in the new Dirac vacuum

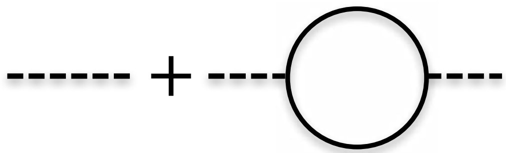


- Also, the chiral symmetry is broken in a state with $\langle\sigma\rangle \neq 0$

NAMBU-GOLDSTONE BOSONS

- When a continuous global symmetry breaks down, massless Nambu-Goldstone bosons appear in the particle spectrum

$$(D_\pi)^{-1} =$$



$$= \frac{\delta^4(x)}{G} + i\,\mathrm{tr}[G(x,0) i\gamma^5 G(0,x) i\gamma^5]$$

- The dispersion relation of NG bosons at $\vec{p} \to 0$

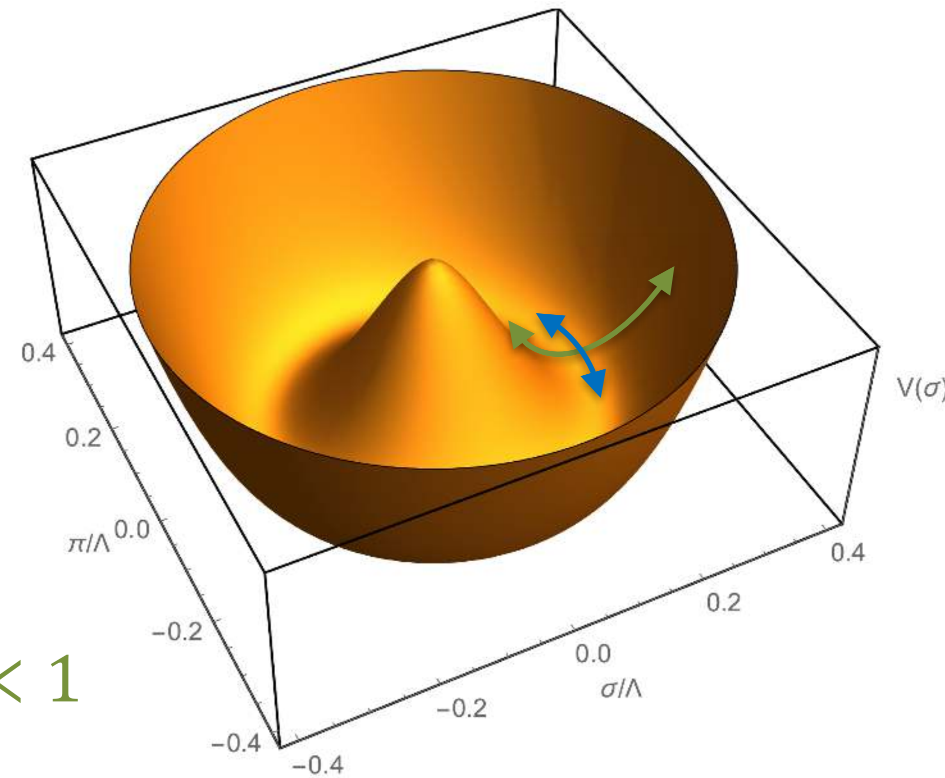
$$E_\pi = \sqrt{v_{\pi,\perp}^2 \vec{p}_\perp^2 + p_z^2}$$

where $v_{\pi,\perp} \ll 1$ at *weak coupling*

- The relation for the σ-boson

$$E_\sigma = \sqrt{M_\sigma^2 + v_{\sigma,\perp}^2 \vec{p}_\perp^2 + p_z^2}$$

where $M_\sigma = 2\sqrt{3} m_{dyn}$ & $v_{\pi,\perp} \ll 1$



NONZERO TEMPERATURE

- Partition function:

$$Z_{T,\mu} = \mathrm{Tr}\left[\exp\left(-\frac{H-\mu N}{T}\right)\right]$$

$$= \int [d\psi d\bar{\psi} d\sigma d\pi] \exp\left(i\int_0^{-i/T} dt \int d^3x \left[\bar{\psi}\left(i\gamma^\mu D_\mu - \sigma - i\gamma^5\pi\right)\psi - \frac{\sigma^2+\pi^2}{2G}\right]\right)$$

where the fermion/boson fields satisfy (anti)periodic boundary conditions in imaginary time, e.g., $\psi(0) = -\psi(-i/T)$

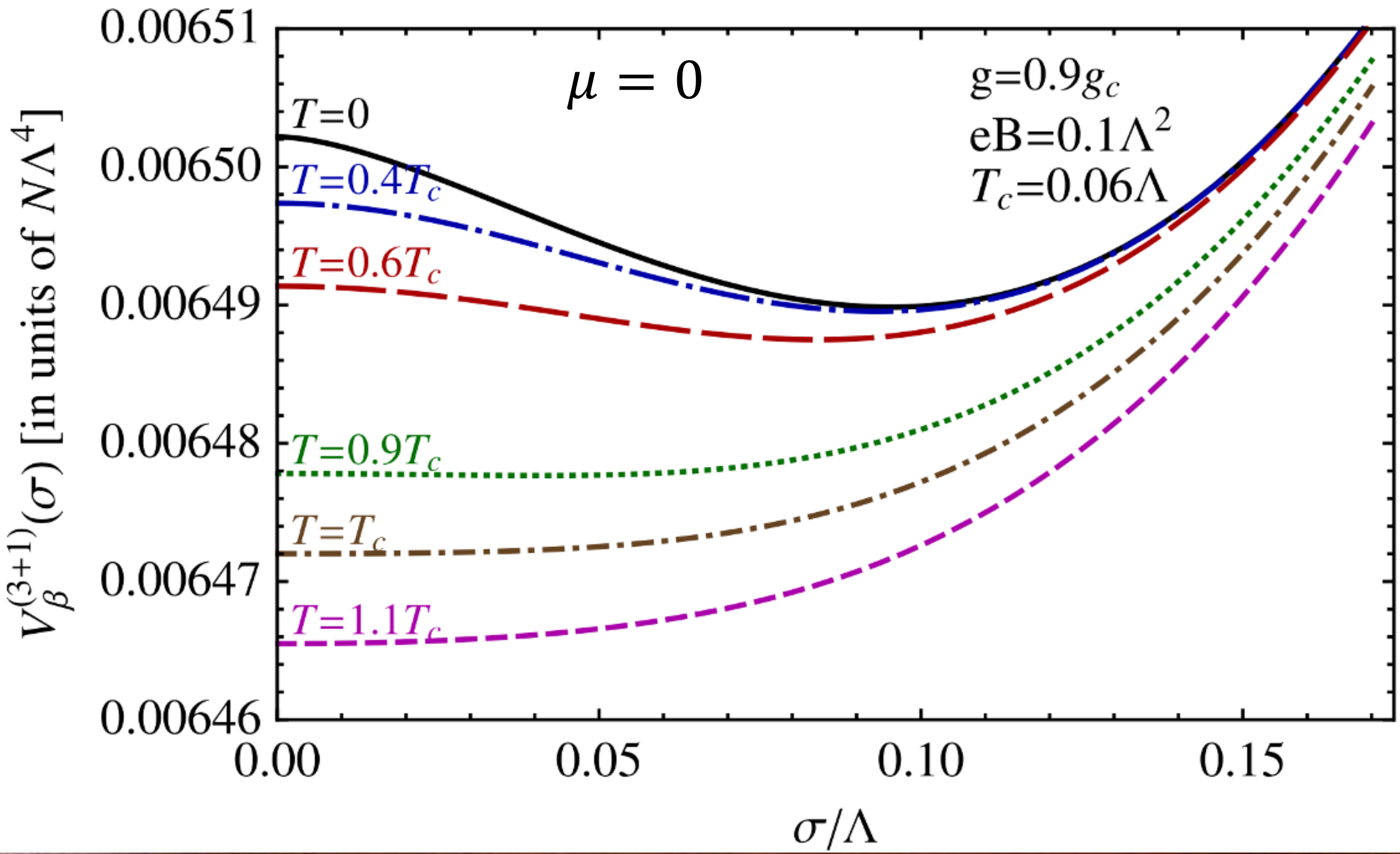
- **Note #1:** $Z_{T,\mu}$ is similar to the generating functional at *T*=0
- **Note #2:** Hubbard–Stratonovich trick ⟺ Gaussian integral
- The effective potential is similar to that at *T*=0, but with the energy integration replaced by the Matsubara sum:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{is\omega^2}(\ldots) \to iT \sum_{n=-\infty}^{\infty} e^{is(i\omega_n)^2}(\ldots)$$

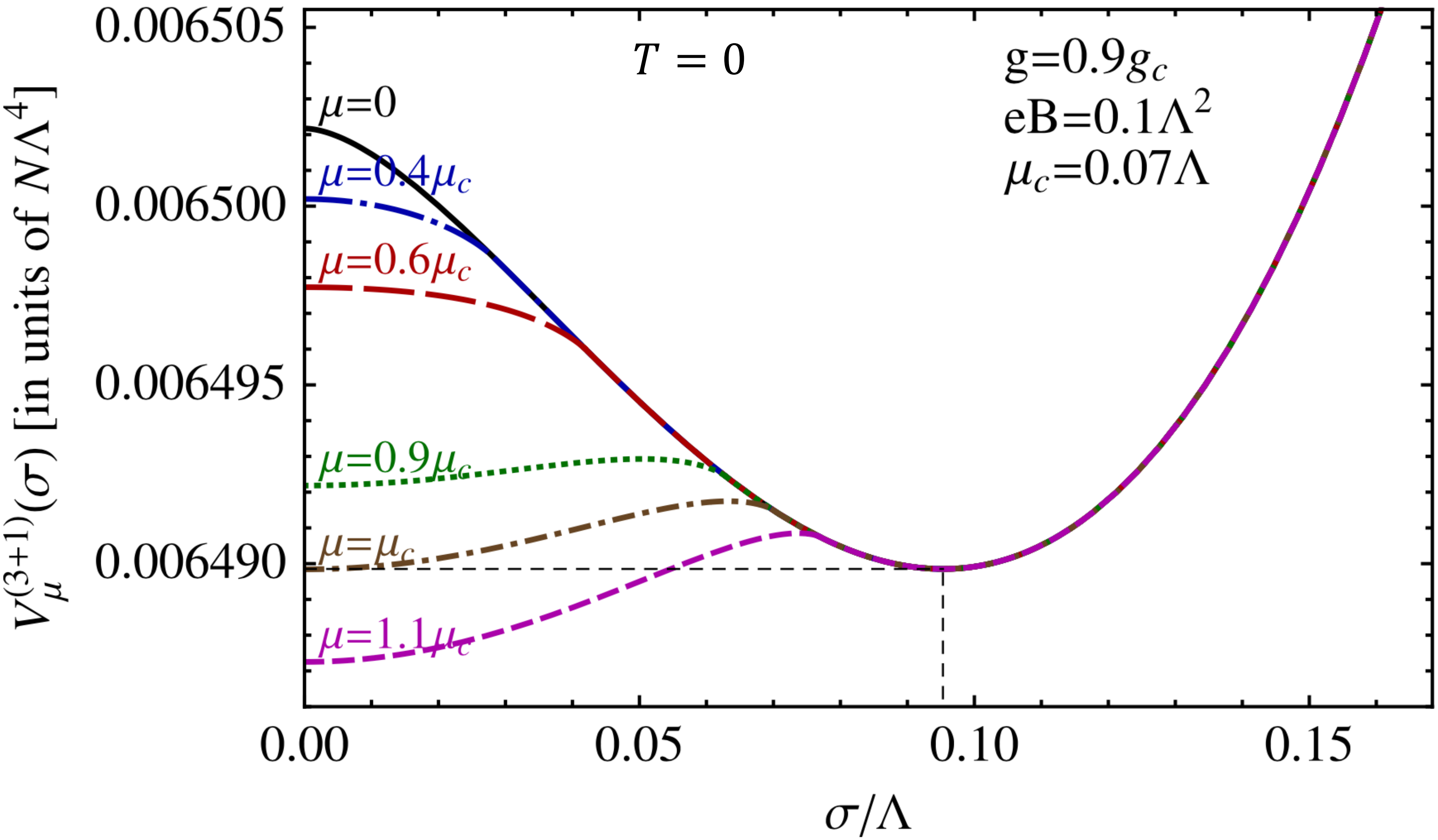
where $\omega \to i\omega_n = i\pi T(2n+1)$

Effects of nonzero temperature

$$V_{\beta,\mu}(\rho) = V(\rho) - \frac{N}{2\beta\pi^2 l^2}\int_0^\infty dk_3 \left\{ \ln\left[1 + e^{-\beta\left(\sqrt{\rho^2+k_3^2}-\mu\right)}\right] + 2\sum_{n=1}^{\infty}\ln\left[1+e^{-\beta\left(\sqrt{\rho^2+k_3^2+2n/l^2}-\mu\right)}\right] + (\mu\to-\mu)\right\}$$



EFFECTS OF NONZERO CHEMICAL POTENTIAL



Notice that at $T = 0$ the chemical potential μ has no effect on the effective potential when $\sigma > \mu$ (This is not true at $T \neq 0$)

Symmetry breaking: Methods used

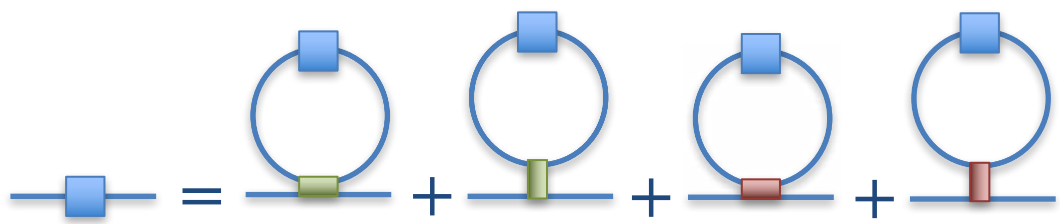
- Effective potential for the composite field, e.g., $\sigma = -G\,\bar{\psi}\psi$

$$\frac{dV(\sigma)}{d\sigma} = 0 \quad \text{(gap equation)}$$

- In NJL, e.g., $V_{NJL}(\sigma) = \frac{\sigma^2}{2G} + i\,\mathrm{tr}\ln\left[i\gamma^\mu D_\mu - \sigma\right]$, giving

$$\frac{\sigma}{G} - i\,\mathrm{tr}\left[\frac{1}{i\gamma^\mu D_\mu - \sigma}\right] = 0 \qquad \Longrightarrow \qquad \sigma = G\,\mathrm{tr}[G(x,x)]$$

- The same gap equation can be obtained from the Schwinger-Dyson equation for the fermion self-energy/propagator

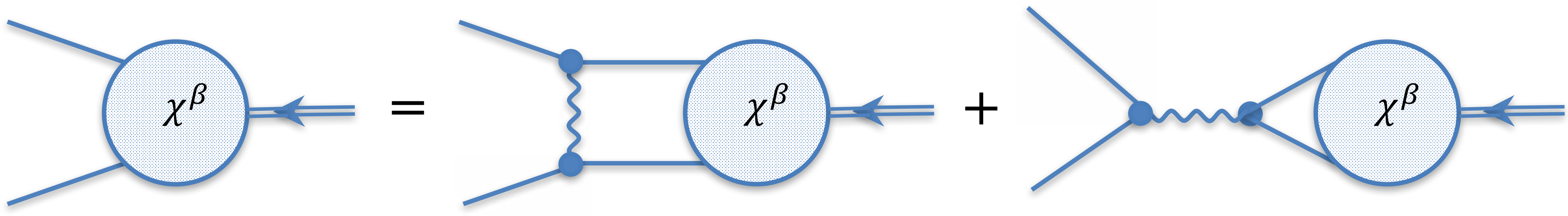


$$G^{-1}(x,x') - G_0^{-1}(x,x') = -iG\sum_i \Gamma_i[G(x,x)\Gamma_i - \mathrm{tr}\{G(x,x)\Gamma_i\}]\,\Gamma_i\delta^4(x-x')$$

where ansatz $G^{-1}(x,x') = -i\left(i\gamma^\mu D_\mu - m_{dyn}\right)\delta^4(x-x')$ is used

ANOTHER WAY: PION AS A BOUND STATE

- Homogeneous Bethe-Salpeter equation for a *massless* bound state with quantum numbers of the NG boson



- As we'll see, in NJL model in the strong-field limit, the pion's wave function in momentum space should have the structure:

$$\chi(p;P\to 0) = A(p_{\|})e^{-p_{\perp}^2 l^2}\frac{\omega\gamma^0 - p_z\gamma^3 - m}{\omega^2 - p_z^2 - m^2}\gamma^5\mathcal{P}_+\frac{\omega\gamma^0 - p_z\gamma^3 - m}{\omega^2 - p_z^2 - m^2}$$

where $A(p_{\|})$ with $p_{\|} = (\omega, p_z)$ satisfies a simple integral equation

$$A(p_{\|,E}) = \frac{G|eB|}{4\pi^3}\int\frac{A(k_{\|,E})d^2k_{\|,E}}{k_{\|,E}^2 + m^2}$$

(here mass parameter *m* is treated as a variational parameter)

AUXILIARY SCHRODINGER PROBLEM

- It is instructive to recast the problem in terms of

$$\Psi(r_{\parallel}) = \int \frac{d^2k_{\parallel}}{(2\pi)^2} \frac{e^{-ir_{\parallel}\cdot k_{\parallel}}}{k_{\parallel}^2 + m^2} A(k_{\parallel})$$

- Function $\Psi(r_{\parallel})$ satisfies the following 2D Schrodinger equation:

$$\left[-\nabla_{r_{\parallel}}^2 + m^2 + V(r_{\parallel})\right] \Psi(r_{\parallel}) = 0$$

where $-m^2$ plays the role of energy ϵ, and $V(r_{\parallel})$ is a model-dependent potential (as we will see later)

- In the NJL model, $V(r_{\parallel})$ is proportional to a δ-function

$$V(r_{\parallel}) = -\frac{G|eB|}{\pi} \delta_{\Lambda}^2(r_{\parallel}) = -\frac{G|eB|}{\pi} \int_0^{\Lambda} \frac{d^2k_{\parallel}}{(2\pi)^2} e^{-ir_{\parallel}\cdot k_{\parallel}}$$

- There exists a bound state solution ($\epsilon_b < 0$) in this Schrodinger problem and, thus, also a real solution for m, i.e.,

$$m^2 = -\epsilon_b \simeq \Lambda^2 \exp\left(-\frac{4\pi^2}{|eB|G}\right)$$ (LLL & weak coupling) ✓

ARIZONA STATE UNIVERSITY

POLYTECHNIC CAMPUS

Huada School on QCD
MAY 31, 2017
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LECTURE #3
MAGNETIC CATALYSIS IN QED

Igor Shovkovy
Arizona State University

Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports **576** (2015)

Magnetic catalysis in QED

- Lagrangian density invariant under $SU_L(N_f) \times SU_R(N_f) \times U(1)$

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}_f\left(i\gamma^\mu D_\mu\right)\psi_f$$

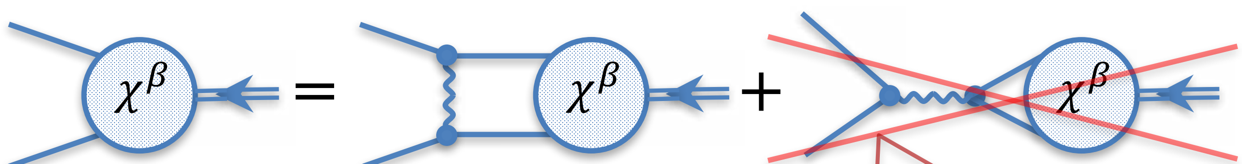
where $D_\mu = \partial_\mu + ie\left(A_\mu + a_\mu\right)$ and $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

- The Bethe–Salpeter equation for NG states $(\beta = 1, \ldots, N_f^2 - 1)$:

$$\chi_{AB}^\beta(u, u'; P) = -i\int d^4u_1 d^4u_1' d^4u_2 d^4u_2' G_{AA_1}(u, u_1) K_{A_1B_1;A_2B_2}(u_1u_1', u_2u_2') \chi_{A_2B_2}^\beta(u_2, u_2'; P) G_{B_1B}(u_2', u')$$

where the wave function is defined by $\chi_{AB}^\beta = \langle 0|T\psi_A(u)\bar{\psi}_B(u')|P;\beta\rangle$

Diagrammatically



Hartree term plays no role for NG bound states

where the kernel (in the ladder approximation) is

$$K_{A_1B_1;A_2B_2}(u_1u_1', u_2, u_2') = -4\pi i\alpha\delta_{a_1a_2}\delta_{b_2b_1}\gamma^\mu_{n_1n_2}\gamma^\nu_{m_2m_1}\mathcal{D}_{\mu\nu}(u_2' - u_2)\delta(u_1 - u_2)\delta(u_1' - u_2')$$

$$+ 4\pi i\alpha\delta_{a_1b_1}\delta_{b_2a_2}\gamma^\mu_{n_1m_1}\gamma^\nu_{m_2n_2}\mathcal{D}_{\mu\nu}(u_1 - u_2)\delta(u_1 - u_1')\delta(u_2 - u_2')$$

Solution in strong field limit

- Structure of the NG-boson wave function ($r_\mu = u_\mu - u'_\mu$):

$$\chi^{\beta}_{AB}(u,u';P) = \lambda^{\beta}_{ab} e^{-iPR} \exp\left[-ier^{\mu}A^{\text{ext}}_{\mu}(R)\right] \tilde{\chi}_{nm}(R,r;P)$$

- In the LLL approximation, the equation reduces to

$$\varphi(p_\parallel) = \frac{\pi\alpha}{(2\pi)^4}\int d^2k_\parallel \left(1 - i\gamma^1\gamma^2\right)\gamma^{\mu}\frac{\hat{k}_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}^2}\varphi(k_\parallel)\frac{\hat{k}_\parallel + m_{\text{dyn}}}{k_\parallel^2 - m_{\text{dyn}}^2}\gamma^{\nu}\left(1 - i\gamma^1\gamma^2\right) D^{\parallel}_{\mu\nu}(k_\parallel - p_\parallel)$$

where we introduced $(\hat{p}_\parallel - m_{\text{dyn}})\tilde{\chi}(p)(\hat{p}_\parallel - m_{\text{dyn}}) = \exp(-l^2\mathbf{p}_\perp^2)\varphi(p_\parallel)$

and

$$D^{\parallel}_{\mu\nu}(k_\parallel - p_\parallel) = i\pi\delta_{\mu\nu}\int_0^{\infty}\frac{dx\,\exp(-l^2x/2)}{(k_\parallel - p_\parallel)^2 + x}$$

- The solution should have the following Dirac structure

$$\varphi(p_\parallel) = A\gamma_5\left(1 - i\gamma_1\gamma_2\right)$$

Compare with the NJL model

- Finally, the equation for $A(p_\parallel)$ reads

$$A(p_\parallel) = \frac{\alpha}{2\pi^2}\int \frac{A(k_\parallel)d^2k_\parallel}{k_\parallel^2 + m_{dyn}^2}\int_0^{\infty} dx\,\frac{e^{-xl^2/2}}{x + (k_\parallel - p_\parallel)^2}$$

REDUCE TO A SCHRODINGER PROBLEM

- Rewrite the problem in terms of

$$\Psi(\mathbf{r}) = \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{e^{i\mathbf{r}\cdot k_\parallel}}{k_\parallel^2 + m_{dyn}^2} A(k_\parallel)$$

- Function $\Psi(\mathbf{r})$ satisfies the following 2D Schrodinger equation:

$$\left[-\nabla_{\mathbf{r}}^2 + m_{dyn}^2 + V(\mathbf{r})\right] \Psi(\mathbf{r}) = 0$$

where

$$V(\mathbf{r}) = -\frac{\alpha}{2\pi^2} \int d^2 p e^{i\mathbf{p}\cdot\mathbf{r}} \int_0^\infty \frac{dx \exp(-x/2)}{l^2 p^2 + x} = \frac{\alpha}{\pi l^2} \exp\left(\frac{r^2}{2l^2}\right) \mathrm{Ei}\left(-\frac{r^2}{2l^2}\right)$$

- The potential is long-ranged with the following asymptote

$$V(\mathbf{r}) \simeq -\frac{2\alpha}{\pi}\frac{1}{r^2}, \quad r \to \infty$$

- The lowest energy bound state gives

$$m_{dyn} \simeq C\sqrt{|eB|} \exp\left[-\frac{\pi}{2}\left(\frac{\pi}{2\alpha}\right)^{1/2}\right]$$

(LLL & weak coupling) ✓

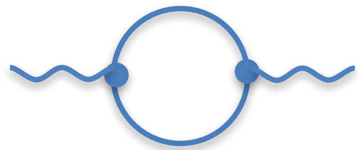
$\exp(-C/\sqrt{\alpha})$ is the result of a long-range interaction

NO SCREENING – NOT GOOD

- Photon exchange interaction is screened in a strong B-field

$$\mathcal{D}^{-1}_{\mu\nu}(u,u') = D^{-1}_{\mu\nu}(u-u') + \Pi_{\mu\nu}(u,u')$$

strong-B limit

where $\Pi_{\mu\nu} \equiv$  $\simeq (q^{\parallel}_{\mu} q^{\parallel}_{\nu} - q^2_{\parallel} g^{\parallel}_{\mu\nu}) e^{-q^2_{\perp} l^2} \Pi(q^2_{\parallel})$

- Then, the screened photon propagator reads

$$\mathcal{D}_{\mu\nu}(q) = -i\left[\frac{1}{q^2} g^{\perp}_{\mu\nu} + \frac{q^{\parallel}_{\mu} q^{\parallel}_{\nu}}{q^2 q^2_{\parallel}} + \frac{1}{q^2 + q^2_{\parallel} \Pi(q^2_{\perp}, q^2_{\parallel})}\left(g^{\parallel}_{\mu\nu} - \frac{q^{\parallel}_{\mu} q^{\parallel}_{\nu}}{q^2_{\parallel}}\right) - \frac{\lambda}{q^2}\frac{q_{\mu} q_{\nu}}{q^2}\right]$$

where the polarization function has the asymptotes

$$\Pi(q^2_{\parallel}) \simeq \frac{\bar{\alpha}}{3\pi}\frac{|eB|}{m^2_{\text{dyn}}}, \quad \text{as } |q^2_{\parallel}| \ll m^2_{\text{dyn}}$$ (extremely narrow range in $q^2_{\parallel}$)

$$\Pi(q^2_{\parallel}) \simeq -\frac{2\bar{\alpha}}{\pi}\frac{|eB|}{q^2_{\parallel}} \quad \text{as } |q^2_{\parallel}| \gg m^2_{\text{dyn}} \quad \Longrightarrow \quad \frac{1}{q^2 + q^2_{\parallel}\Pi(q^2_{\perp}, q^2_{\parallel})} \simeq \frac{1}{q^2 - M^2_{\gamma}}$$

where the effective photon screening mass is $M^2_{\gamma} = \dfrac{2\bar{\alpha}}{\pi}|eB|$

IMPROVED LADDER APPROXIMATION

- Let us re-analyze the problem with screening

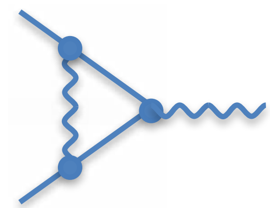
$$A(p_\parallel) = \frac{\alpha}{4\pi^2}\int \frac{A(k_\parallel)d^2k_\parallel}{k_\parallel^2+m^2}\int_0^\infty dx\left(\frac{e^{-xl^2/2}}{x+(k_\parallel-p_\parallel)^2} + \cancel{\frac{e^{-xl^2/2}}{x+(k_\parallel-p_\parallel)^2+M_\gamma^2}}\right)$$

- Improved vs. simple ladder approximations: $\alpha \to \alpha/2$
- Note, the dynamical mass is very sensitive to small α (or $\alpha/2$):

$$m_{dyn} \simeq C\sqrt{|eB|}\exp\left[-\frac{\pi}{2}\left(\frac{\pi}{2\alpha}\right)^{1/2}\right] \quad \text{(ladder approximation)}$$

and, thus, changes drastically with inclusion of screening

- The bigger problem is that the improved ladder approximation is *not* reliable either
  - The vertex corrections will change the result too
  - Singularities $\sim \ln\left(|eB|/m_{dyn}^2\right) \sim 1/\sqrt{\alpha}$ in higher-order diagrams



- Re-summation of infinitely many diagrams is needed (!)

TOWARD EXACT RESULT

- QED in a strong field looks almost like (1+1)D
- Lesson from exactly solvable (1+1)D Schwinger model: find a gauge in which all (singular) vertex corrections vanish!
- Such a (non-local) gauge exists

$$D_{\mu\nu}(q) = -i\frac{1}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - id(q_\perp^2, q_\parallel^2)\frac{q_\mu^\parallel q_\nu^\parallel}{q^2 q_\parallel^2}$$

where

$$d = -q_\parallel^2 \Pi/[q^2 + q_\parallel^2 \Pi] + q_\parallel^2/q^2$$

- The corresponding full photon propagator reads

$$\mathcal{D}_{\mu\nu}(q) = -i\frac{g_{\mu\nu}^\parallel}{q^2 + q_\parallel^2 \Pi(q_\perp^2, q_\parallel^2)} - i\frac{g_{\mu\nu}^\perp}{q^2} + i\frac{q_\mu^\perp q_\nu^\perp + q_\mu^\perp q_\nu^\parallel + q_\mu^\parallel q_\nu^\perp}{(q^2)^2}$$

- All potentially dangerous infrared singularities vanish because

$$\mathcal{P}_+\gamma_\mu\mathcal{P}_+ = \gamma_{\parallel,\mu} \quad \text{and} \quad \gamma_{\parallel,\alpha}\,\gamma_{\parallel,\mu_1}\,\gamma_{\parallel,\mu_2}\cdots\gamma_{\parallel,\mu_{2n+1}}\,\gamma_\parallel^\alpha = 0$$

RELIABLE STRONG-B LIMIT IN QED

- Let us use the method of Schwinger-Dyson equation this time:

$$\tilde{G}(x) = \tilde{G}_0(x) - 4\pi\alpha \int d^4y d^4z\, e^{-i\Phi(x,y)-i\Phi(y,z)} \tilde{G}_0(x-y)\gamma^\mu \tilde{G}(y-z)\gamma^\nu \tilde{G}(z)\mathcal{D}_{\mu\nu}(y-z)$$

where all Schwinger phases were carefully accounted for, and the nonlocal gauge is assumed in the photon propagator

$$\mathcal{D}^{-1}_{\mu\nu}(x-y) = D^{-1}_{\mu\nu}(x-y) - 4\pi\alpha\, \mathrm{tr}\left[\gamma_\mu \tilde{G}(x-y)\gamma_\nu \tilde{G}(y-x)\right]$$

- Perform Fourier transform and use LLL approximation,

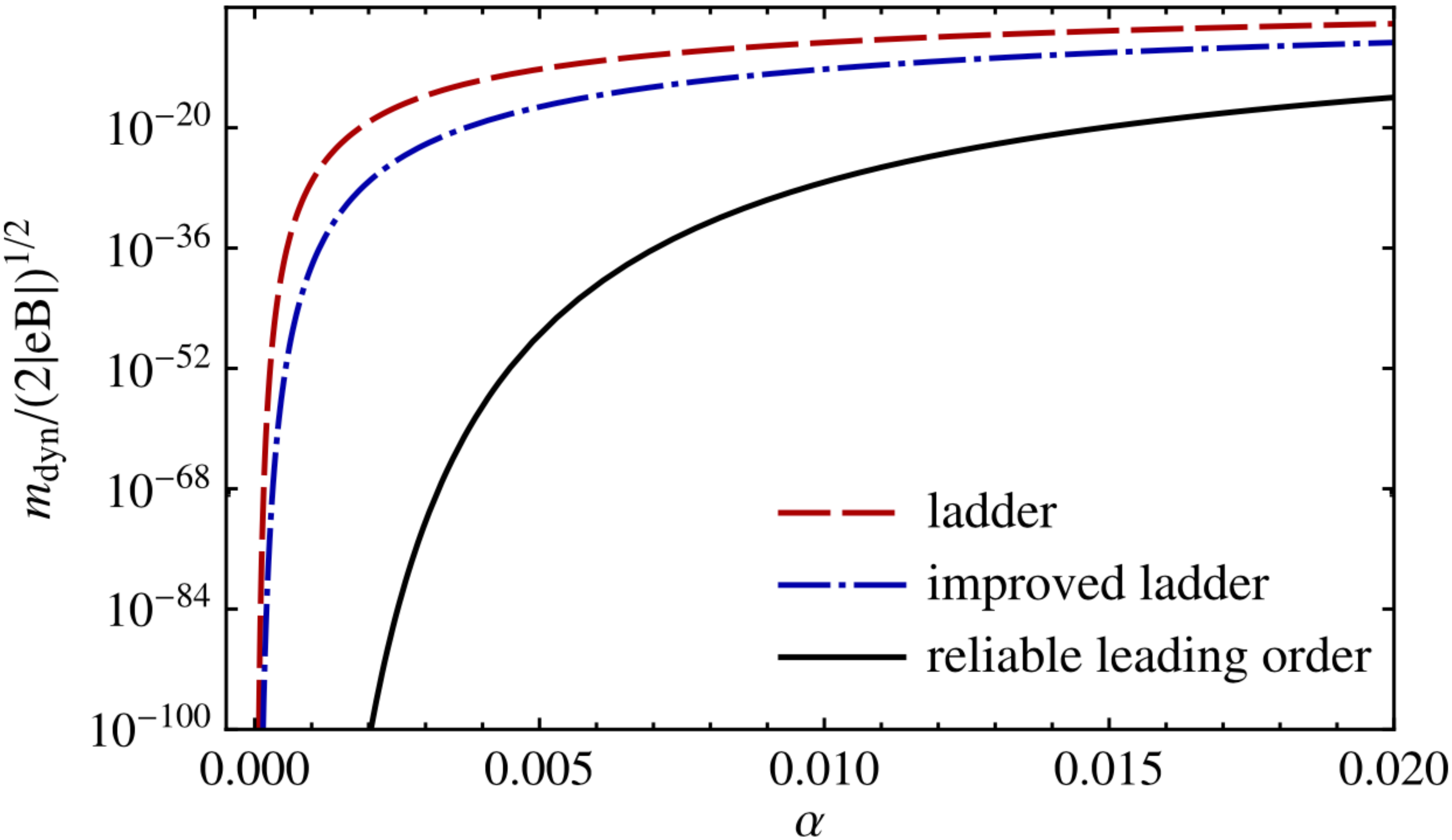
$$\tilde{G}_0(p_\parallel) = 2ie^{-\vec{p}_\perp^2 l^2} \frac{\hat{p}_\parallel}{p_\parallel^2}\mathcal{P}_+ \quad \text{and} \quad \tilde{G}(p_\parallel) = 2ie^{-\vec{p}_\perp^2 l^2} \frac{\hat{p}_\parallel + A(p_\parallel)}{p_\parallel^2 - A^2(p_\parallel)}\mathcal{P}_+$$

- Derive the following gap equation:

$$A(p_\parallel) = \frac{\alpha}{2\pi^2}\int \frac{d^2k_\parallel A(k_\parallel)}{k_\parallel^2 + A^2(p_\parallel)} \int_0^\infty dx \frac{e^{-xl^2/2}}{x + (k_\parallel - p_\parallel)^2 + M_\gamma^2 e^{-xl^2/2}}$$

- Compare with the gap equations in the (improved) ladder QED, obtained with Bethe-Salpeter method

DYNAMICAL MASS IN QED



- The numerical result is fitted well by

$$m_{dyn} \simeq \sqrt{2|eB|}\,(\alpha N_f)^{1/3} \exp\left[-\frac{\pi}{\alpha \ln\frac{C_1}{\alpha N_f}}\right], \quad C_1 \approx 1.82 \pm 0.06$$

QCD IN MAGNETIC FIELD

- QCD is strongly coupled & nonperturbative
- There are theoretical tools that provide insight
  - High-energy (weak-coupling) expansion
  - Large N_c expansion
  - High temperature limit ($T \gg \Lambda_{QCD}$)
  - High density limit ($\mu \gg \Lambda_{QCD}$)
  - Lattice QCD
- Strong magnetic field B is yet another tool
  - it probes physics at short distances $\ell \sim 1/\sqrt{|eB|}$

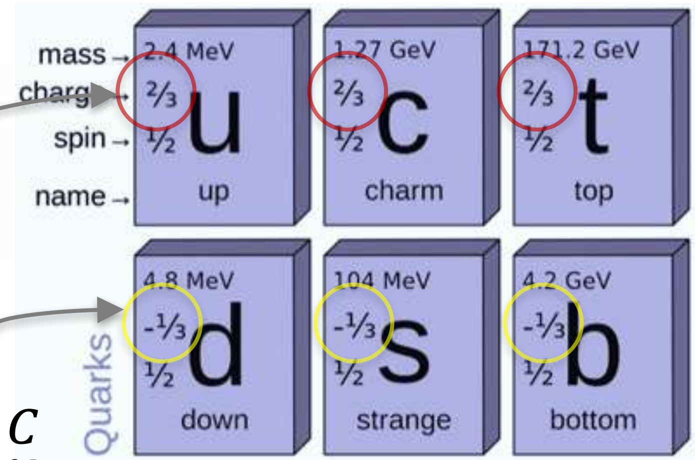
Set the stage

- Lagrangian density of QCD in an external magnetic field

$$\mathcal{L} = -\frac{1}{2} F_A^{\mu\nu} F_{\mu\nu}^A + \bar{\psi}_f (i\gamma^\mu D_\mu) \psi_f$$

where $D_\mu = \partial_\mu + igA_\mu^A \lambda^A/2 + ie_f A_\mu^{\text{ext}}$

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + gf^{ABC} A_\mu^B A_\nu^C$$



- The global chiral symmetry of the model

$$SU_L(N_u) \times SU_R(N_u) \times SU_L(N_d) \times SU_R(N_d) \times U_A^{(-)}(1)$$

chiral symmetry of up-flavors

chiral symmetry of down-flavors

anomaly-free combination of $U_A^{(u)}(1)$ and $U_A^{(d)}(1)$

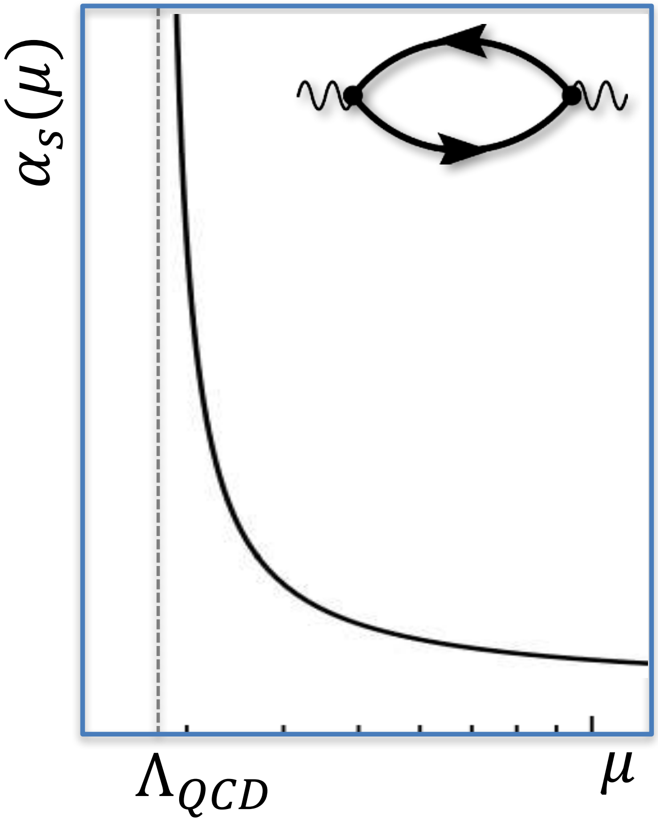
- Quark masses $m_u \neq m_d \neq 0$ break the symmetry down to

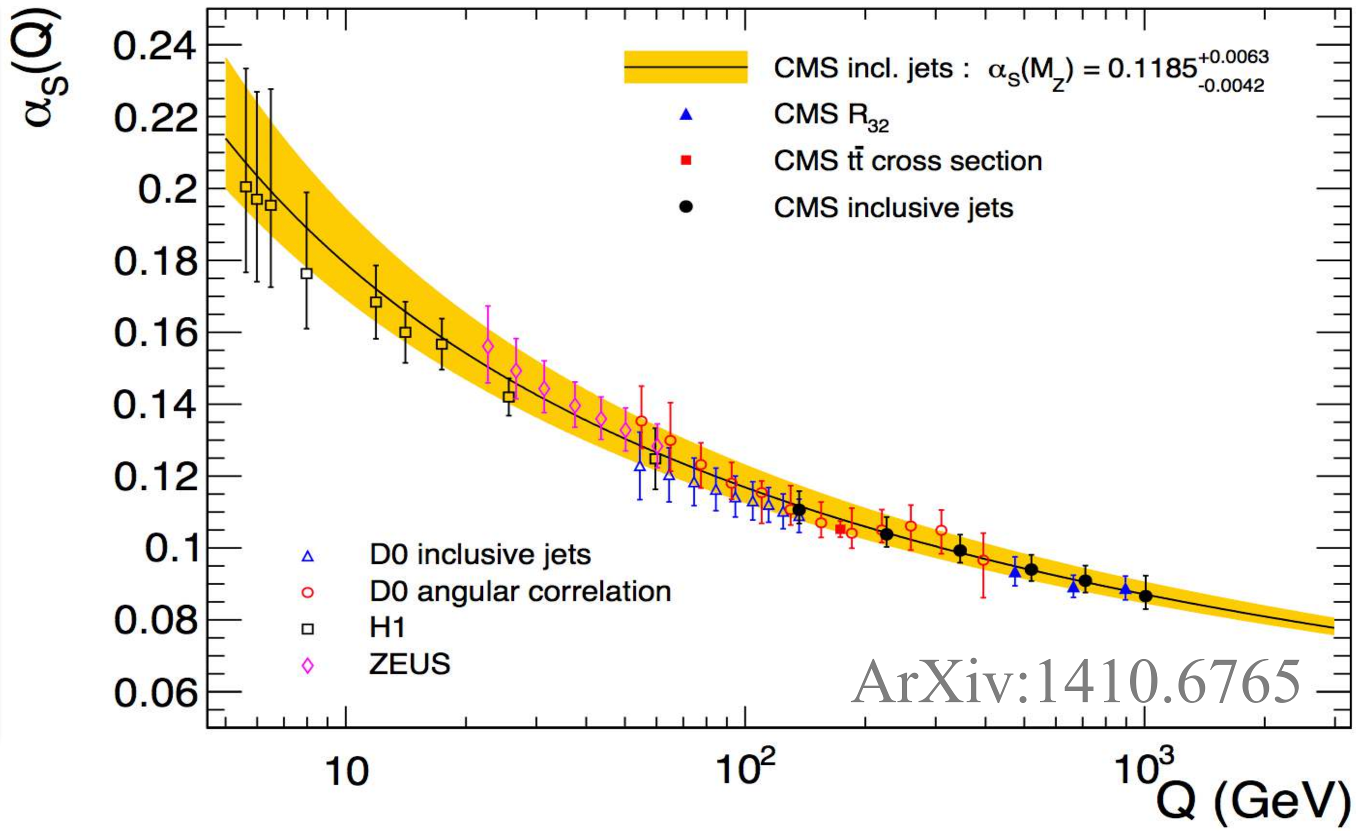
$$SU_V(N_u) \times SU_V(N_d)$$

RUNNING COUPLING & CONFINMENT

- Coupling constant in QCD runs with the energy scale,

$$\frac{1}{\alpha_s(\mu)} \simeq b \ln \frac{\mu^2}{\Lambda_{QCD}^2}, \quad \text{where} \quad b = \frac{11\, N_c - 2N_f}{12\pi}$$

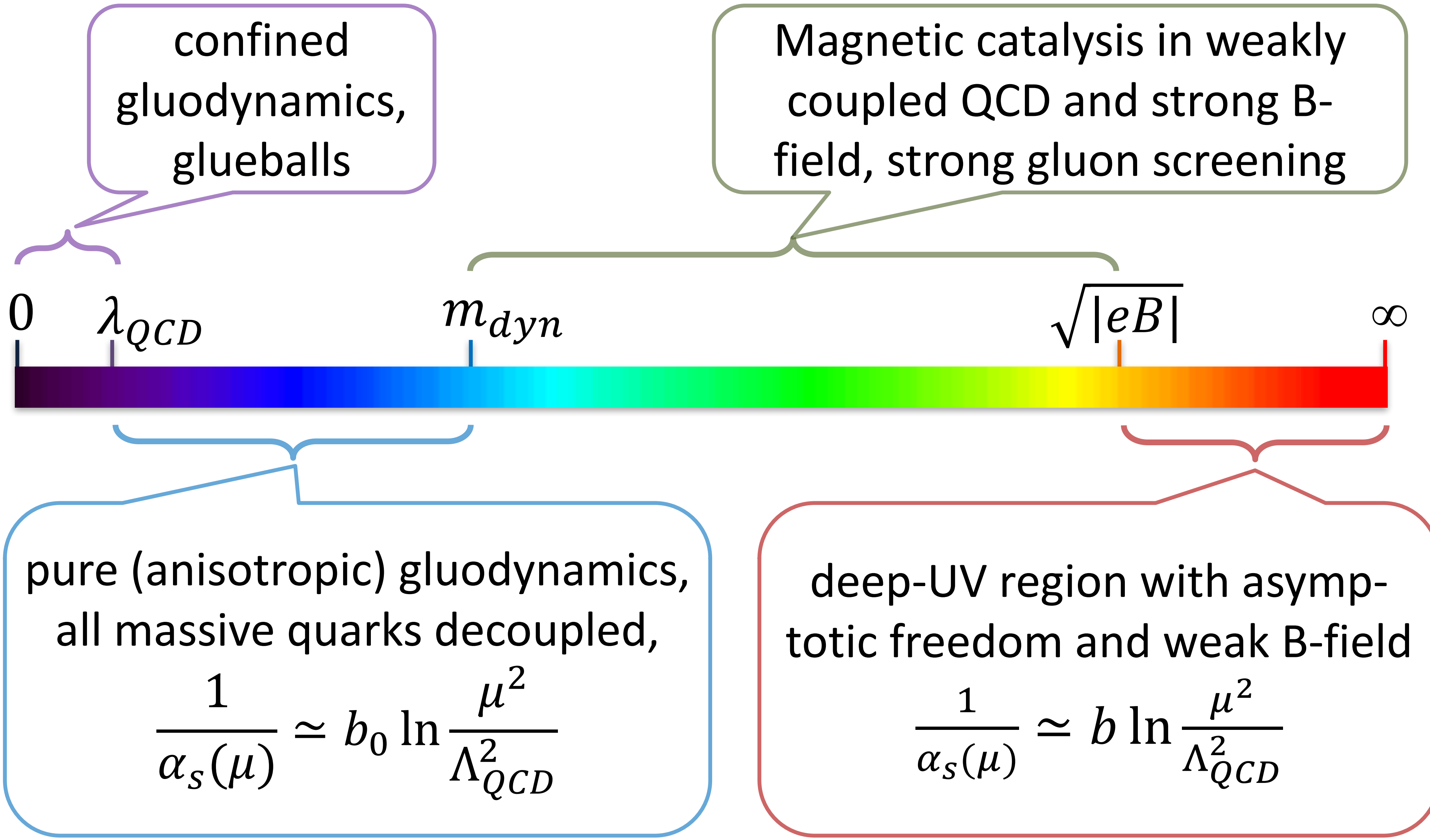




- The question is: What happens in a strong magnetic field?

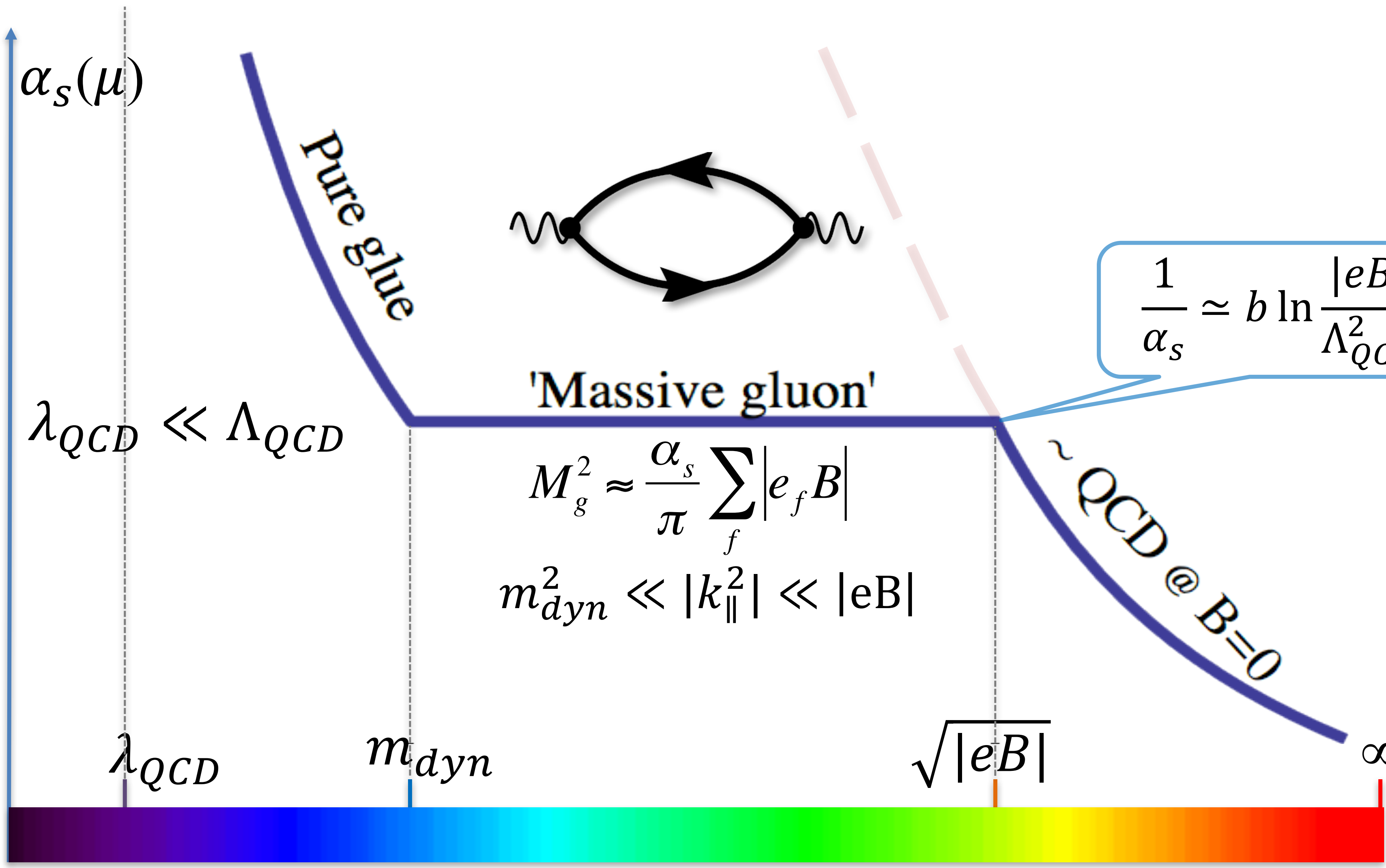
QCD IN STRONG B-FIELD

- Energy scales in the problem at hand



RUNNING α_s IN QCD AT STRONG B

- In deep UV region α_s is not affected by B-field



Schwinger-Dyson equation

- The general form of the equation is similar to that in QED

$$G^{-1}(x,y) = G_0^{-1}(x,y) + 4\pi\alpha_s\,\gamma^\mu T^A\,G(x,y)\,\gamma^\nu T^B\,\mathcal{D}_{\mu\nu}^{AB}(y-x)$$

Note that the inverse propagator $G^{-1}(x,y)$ has the same (!) Schwinger phase as $G(x,y)$

- Non-Abelian structure of the theory ($T^A T^A = C_2$): $\alpha \to \frac{N_c^2-1}{2N_c}\alpha_s$
- Screening effects are included via the polarization function

$$\mathcal{P}_{\mu\nu}^{AB}(x-y) = 4\pi\alpha_s\,\mathrm{tr}\left[\gamma_\mu T^A \tilde{G}(x-y)\gamma_\nu\lambda T^A \tilde{G}(y-x)\right]$$

- Similar to QED, in the strong field limit ($\sqrt{|eB|} \gg \Lambda_{QCD}$)

$$\mathcal{P}^{AB,\mu\nu} \simeq \frac{\alpha_s}{6\pi}\delta^{AB}\left(k_\parallel^\mu k_\parallel^\nu - k_\parallel^2 g_\parallel^{\mu\nu}\right)\sum_{q=1}^{N_f}\frac{|e_q B|}{m_q^2}, \quad \text{for } |k_\parallel^2| \ll m_q^2$$

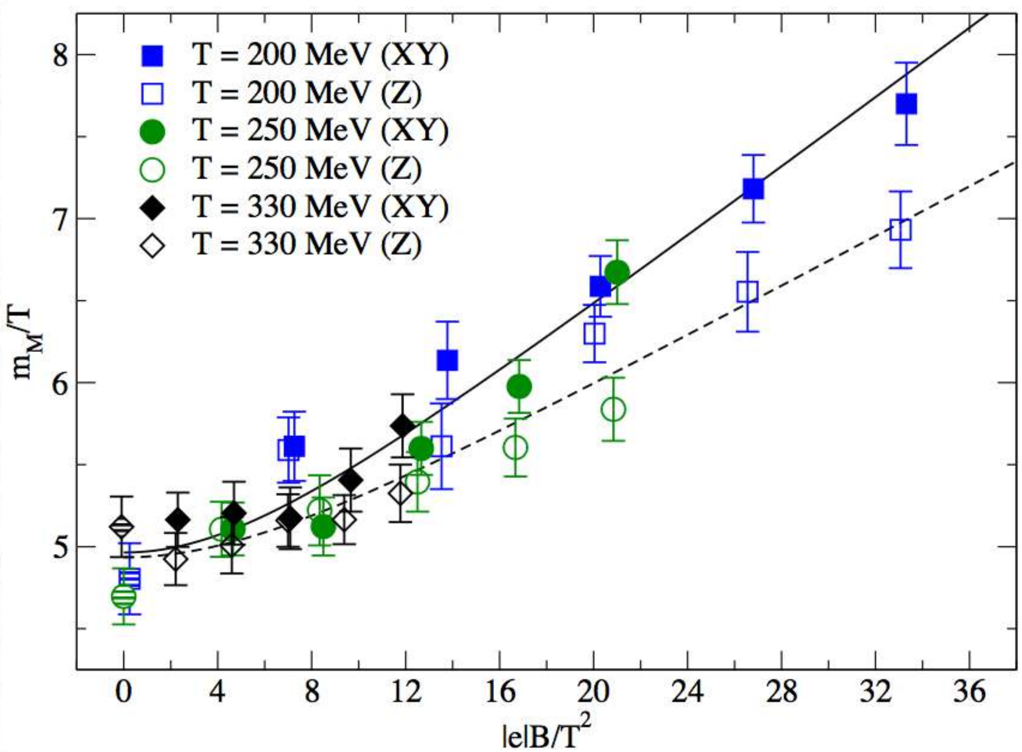
$$\mathcal{P}^{AB,\mu\nu} \simeq -\frac{\alpha_s}{\pi}\delta^{AB}\left(k_\parallel^\mu k_\parallel^\nu - k_\parallel^2 g_\parallel^{\mu\nu}\right)\sum_{q=1}^{N_f}\frac{|e_q B|}{k_\parallel^2}, \quad \text{for } m_q^2 \ll |k_\parallel^2| \ll |eB|$$

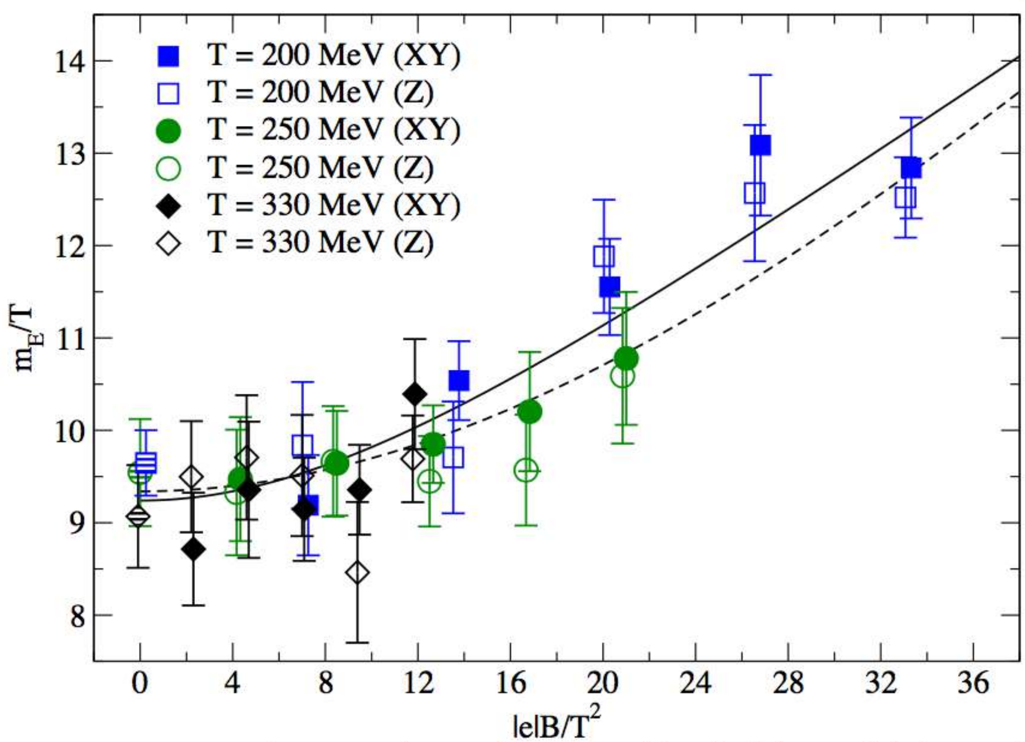
Screening masses: lattice

- Electric and magnetic screening masses on the lattice are fitted well by [Bonati et al., Phys. Rev. D 95, 074515 (2017)]

$$\frac{m_E^d}{T} = a_E^d \left[1 + c_{1;E}^d \frac{|e|B}{T^2} \text{atan}\left(\frac{c_{2;E}^d}{c_{1;E}^d} \frac{|e|B}{T^2} \right) \right]$$

(and similar for the magnetic one)





EXPRESSION FOR DYNAMICAL MASS

- In the region $m_{dyn}^2 \ll |k_\parallel^2| \ll |eB|$, which is most relevant for the fermion-pairing dynamics, the gluon has a "mass"

$$M_g^2 \simeq \frac{\alpha_s}{\pi} \sum_f |e_f B| = \frac{\alpha_s}{3\pi} (2N_u + N_d)\, |eB|$$

- As in QED, in order to tame singular infrared corrections in higher-order diagrams, a special non-local gauge is assumed for the gluon propagator

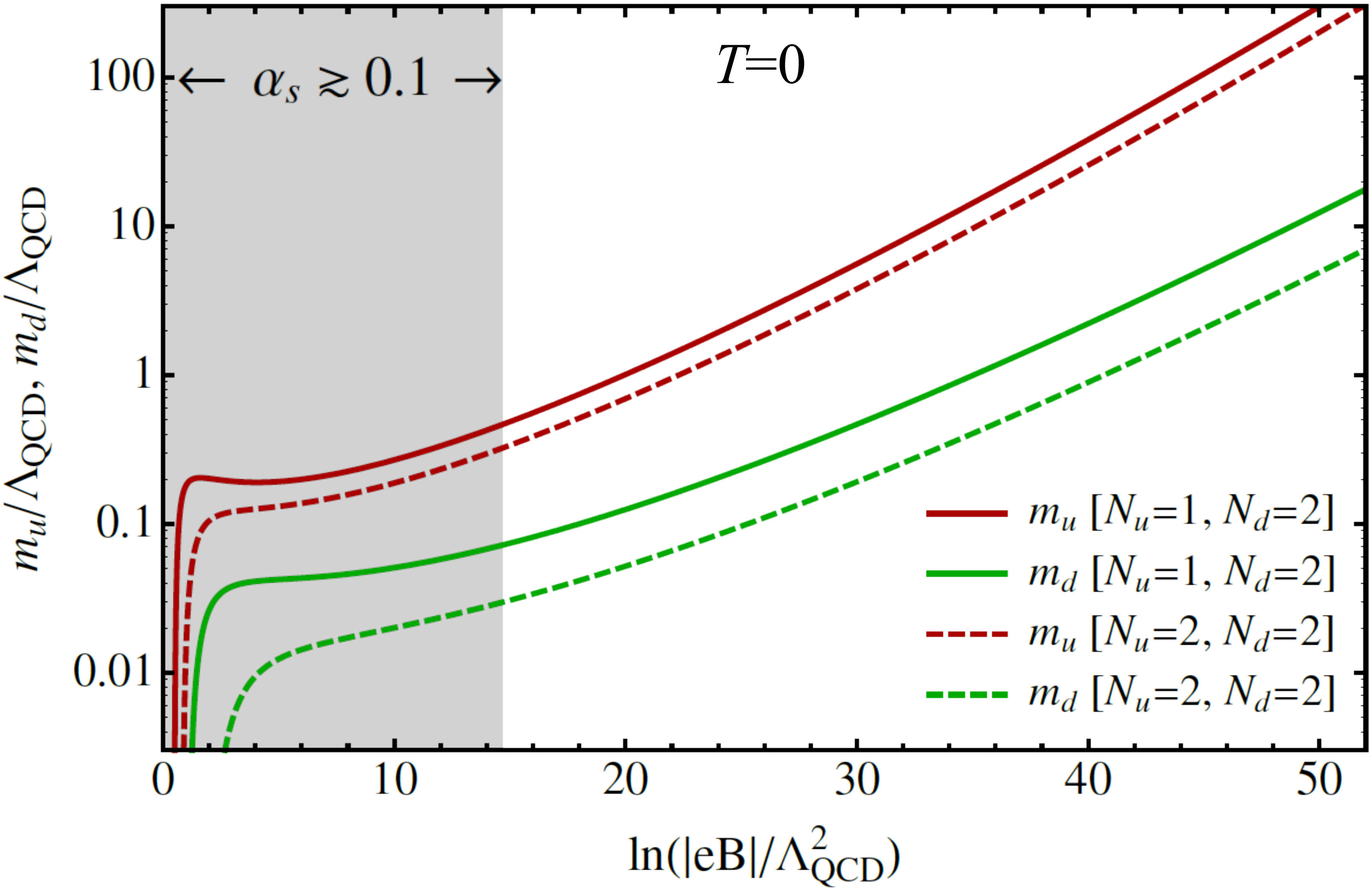
- Up to replacements $\alpha \to \frac{N_c^2 - 1}{2N_c} \alpha_s$ and $M_\gamma^2 \to M_g^2$, the gap equation looks as in QED. Thus,

$$m_q^2 \simeq 2C_1 |e_q B| \left(c_q \alpha_s\right)^{2/3} \exp\left[-\frac{4N_c \pi}{\alpha_s (N_c^2 - 1) \ln(C_2 / c_q \alpha_s)} \right]$$

where $C_1 \simeq C_2 \simeq 1$ and $c_q \simeq (2N_u + N_d)|e|/(6\pi|e_q|)$

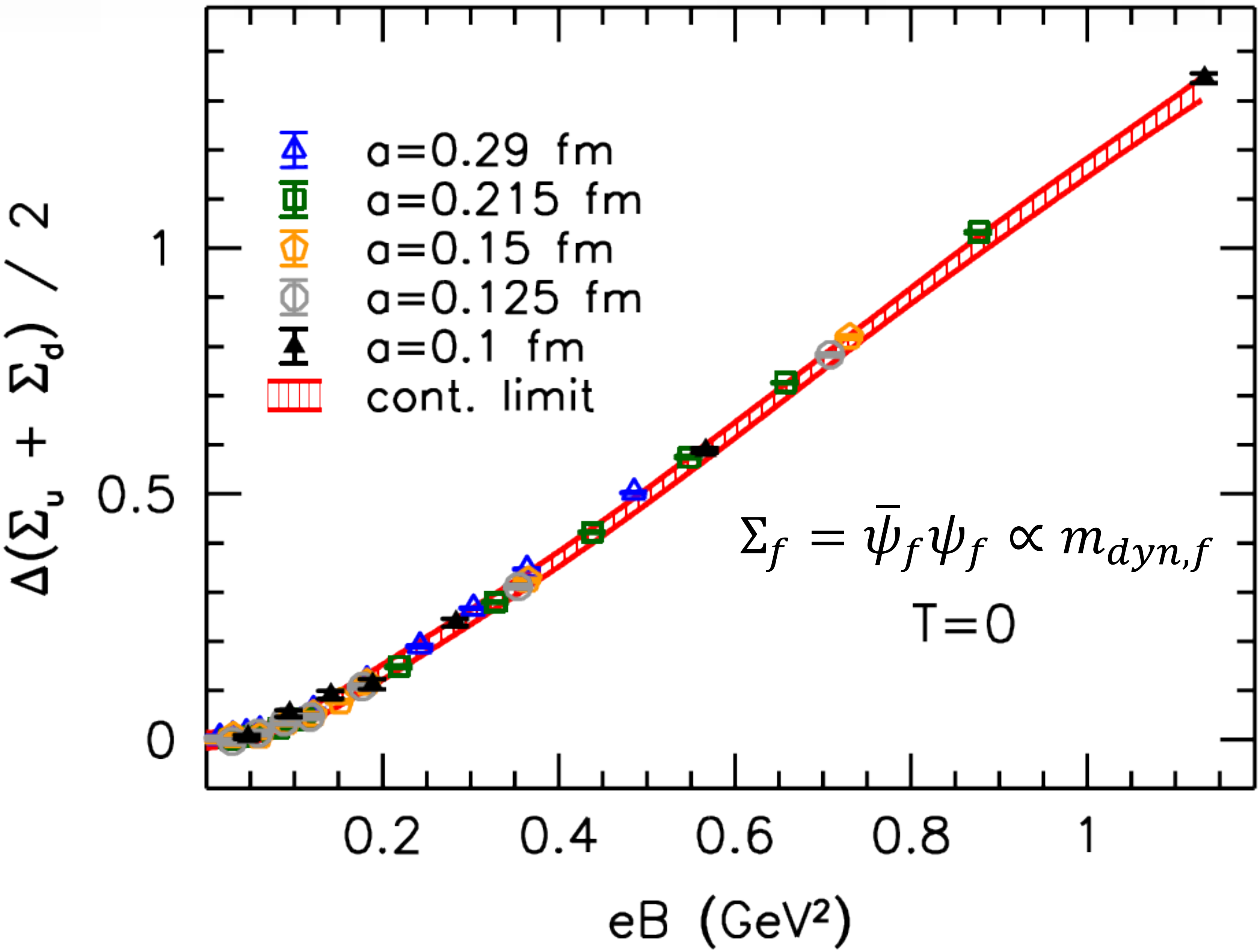
QUARK MASS VS. B

- Quantitatively, dynamical masses are $(\sqrt{|eB|} \gg \Lambda_{\mathrm{QCD}})$



[Miransky & Shovkovy, Phys. Rev. D **66** (2002) 045006]

Chiral condensate in lattice QCD



[Bali et al., Phys. Rev. D**86**, 071502 (2012)]

ARIZONA STATE UNIVERSITY
POLYTECHNIC CAMPUS

Huada School on QCD
MAY 31, 2017
CCNU, Wuhan

LECTURE #4
MAGNETIC CATALYSIS IN QCD

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Reading material: V.A. Miransky & I.A. Shovkovy, Physics Reports **576** (2015)

NAMBU-GOLDSTONE BOSONS (PIONS)

- Original global chiral symmetry

$$SU_L(N_u)\times SU_R(N_u)\times SU_L(N_d)\times SU_R(N_d)\times U_A^{(-)}(1)$$

breaks down to

$$SU_V(N_u)\times SU_V(N_d)$$

- A total number of broken-symmetry generators: $N_u^2 + N_d^2 - 1$
- Thus, there should be $(N_u^2 + N_d^2 - 1)$ massless NG bosons
- The unitary pion fields can be written in terms of the coset space generators

$$\Sigma_u \equiv \exp\left(i\sum_{A=1}^{N_u^2-1} \lambda^A \pi_u^A/f_u\right),\ \Sigma_d \equiv \exp\left(i\sum_{A=1}^{N_d^2-1} \lambda^A \pi_d^A/f_d\right)$$

and $\tilde{\Sigma} \equiv \exp\left(i\sqrt{2}\tilde{\pi}/\tilde{f}\right)$

- In a very strong magnetic field another light pseudo-NG boson, associated with anomalous $U_A(1)$, may appear

NAMBU-GOLDSTONE BOSONS (PIONS)

- The low-energy effective action should have the form

$$\mathcal{L}_{NG} \simeq \frac{f_u^2}{4} \mathrm{tr}\left(\mathrm{g}_{\parallel}^{\mu\nu} \partial_\mu \Sigma_u \partial_\nu \Sigma_u^\dagger + v_u^2 \mathrm{g}_{\perp}^{\mu\nu} \partial_\mu \Sigma_u \partial_\nu \Sigma_u^\dagger \right) + \cdots$$

- The pion decay constants are defined by

$$i \left\langle 0 \left| \bar{\psi} \gamma^\mu \gamma^5 \frac{\lambda^A}{2} \psi \right| \pi^B(P) \right\rangle = P^\mu f_\pi \delta^{AB} = -i \int \frac{d^4 k}{(2\pi)^4} \mathrm{tr}\left(\gamma^\mu \gamma^5 \frac{\lambda^A}{2} \chi_q^B(k, P) \right)$$

where $P^\mu = \left(P^0, v_\perp^2 \vec{P}_\perp, P^3\right)$

- The Bethe-Salpeter wave function looks like in QED in LLL approximation. So, we find that $v_\perp^2 \approx 0$, and

$$f_q^2 = 4N_c \int \frac{d^2 k_\perp d^2 k_\parallel}{(2\pi)^4} \exp\left(-\frac{k_\perp^2}{|e_q B|} \right) \frac{m_q^2}{(k_\parallel^2 + m_q^2)^2}$$

which can be easily calculated, giving

$$f_u^2 = \frac{N_c |eB|}{6\pi^2} \quad \text{and} \quad f_d^2 = \frac{N_c |eB|}{12\pi^2}$$

Low-energy region, $|k_\parallel^2| \lesssim m_{dyn}^2$

- Massive quarks decouple from the low-energy dynamics

- Gluons are the only "light" degrees of freedom
- Assuming that $\Lambda_{QCD}^2 \ll m_{dyn}^2$, the gluodynamics has a semi-perturbative region, $|k_\parallel^2| \lesssim m_{dyn}^2$, where

$$\frac{1}{\tilde{\alpha}_s(\mu)} - \frac{1}{\alpha_s} \simeq b_0 \ln \frac{\mu^2}{m_{dyn}^2}$$

here $b_0 = \frac{11\, N_c}{12\pi}$ and $\frac{1}{\alpha_s} \simeq b \ln \frac{|eB|}{\Lambda_{QCD}^2}$ (Recall: $b = \frac{11\, N_c - 2N_f}{12\pi}$)

- Then, we find that the new confinement scale where $\tilde{\alpha}_s = \infty$:

$$-b \ln \frac{|eB|}{\Lambda_{QCD}^2} \simeq b_0 \ln \frac{\lambda_{QCD}^2}{m_{dyn}^2} \quad \Rightarrow \quad \lambda_{QCD} = m_{dyn} \left(\frac{\Lambda_{QCD}}{\sqrt{|eB|}} \right)^{b/b_0}$$

LOW-ENERGY GLUODYNAMICS

- Quadratic part of low-energy effective action for gluons

$$\mathcal{L}^{(2)}_{\text{glue,eff}} = -\frac{1}{2}\sum_{A=1}^{N_c^2-1} A^A_\mu(-k)\left[g^{\mu\nu}k^2 - k^\mu k^\nu + \kappa\left(g^{\mu\nu}_\parallel k^2_\parallel - k^\mu_\parallel k^\nu_\parallel\right)\right] A^A_\nu(k)$$

where the susceptibility κ is extracted from the polarization tensor $\mathcal{P}^{AB}_{\mu\nu}$ in the region $\left|k_\parallel^2\right| \ll m^2_{dyn}$, i.e.,

$$\kappa = \frac{\alpha_s}{6\pi}\sum_{q=1}^{N_f}\frac{|e_q B|}{m_q^2} = \frac{1}{12C_1\pi}\sum_{q=1}^{N_f}\left(\frac{\alpha_s}{c_q^2}\right)^{1/3}\exp\left(\frac{4N_c\pi}{\alpha_s(N_c^2-1)\ln(C_2/c_q\alpha_s)}\right) \gg 1$$

- The requirement of gauge invariance allows to write down the complete expression for the gluon action

$$\mathcal{L}_{\text{glue,eff}} \simeq \frac{1}{2}\sum_{A=1}^{N_c^2-1}\left(\mathbf{E}^A_\perp\cdot\mathbf{E}^A_\perp + \epsilon E^A_3 E^A_3 - \mathbf{B}^A_\perp\cdot\mathbf{B}^A_\perp - B^A_3 B^A_3\right)$$

where $\epsilon = 1+\kappa$ is a chromo-dielectric constant (note $\epsilon \gg 1$), $E^A_i = F^A_{0i}$ and $B^A_i = {}^1\!/_2\,\varepsilon_{ijk}F^A_{jk}$ are chromo-fields

Effective potential

- By using the guidance from an analogous *anisotropic* QED, the static potential between a pair of quarks should be given by

$$V(x,y,z) \simeq \frac{g_s^2}{4\pi\sqrt{z^2+\epsilon(x^2+y^2)}}$$

which is valid for a range of distance scales $m_{dyn}^{-1} \lesssim r \lesssim \lambda_{QCD}^{-1}$

- Note that the effective coupling constants

$$\alpha_s^{\parallel} = \frac{g_s^2}{4\pi v_g^{\parallel}} \approx \frac{g_s^2}{4\pi}, \qquad \text{where} \qquad v_g^{\parallel} \approx 1$$

$$\alpha_s^{\perp} = \frac{g_s^2}{4\pi\sqrt{\epsilon}\, v_g^{\perp}} \approx \frac{g_s^2}{4\pi}, \qquad \text{where} \qquad v_g^{\perp} \approx {}^{1}/_{\sqrt{\epsilon}}$$

are approximately the same in all directions

- A posteriori, this naïve “isotropy” may justifies the use of running behavior as in isotropic gluodynamics (not rigorous)

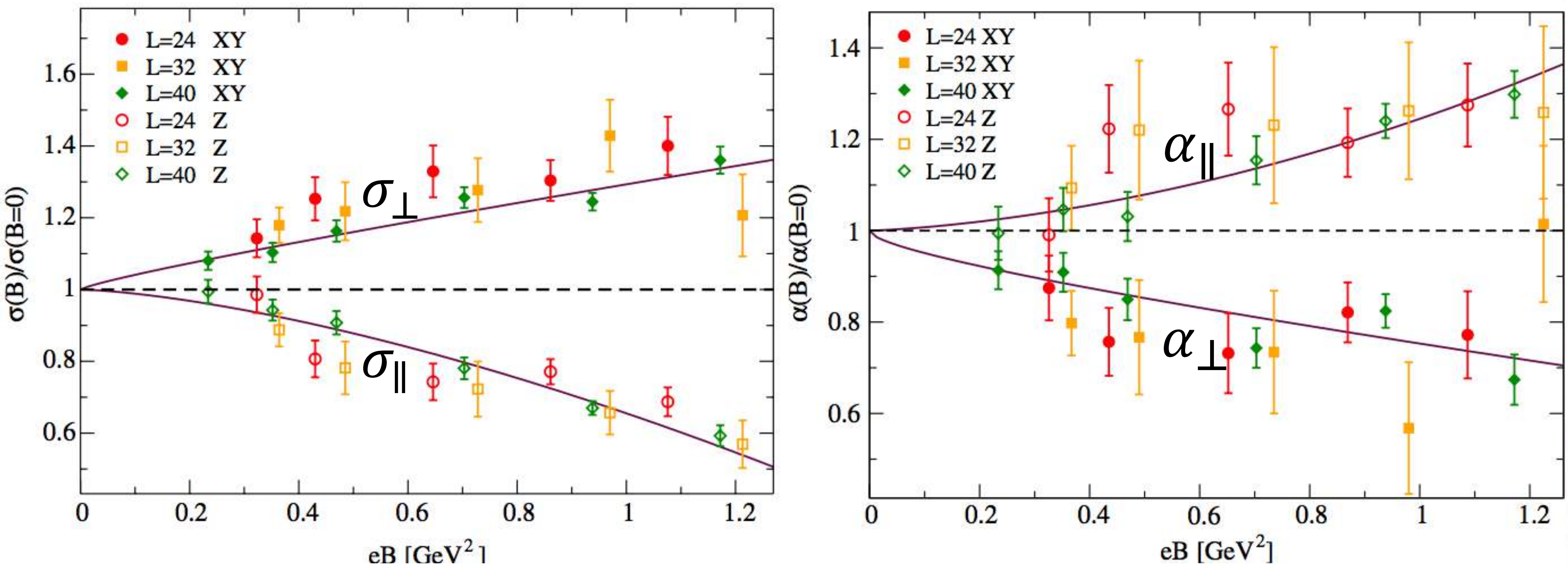
Potential on lattice

- Quark-antiquark potential was fitted by Cornell potential,

$$V(r) = -\frac{\alpha}{r} + \sigma r + V_0$$

- where σ is the string tension and α is the Coulombic coefficient

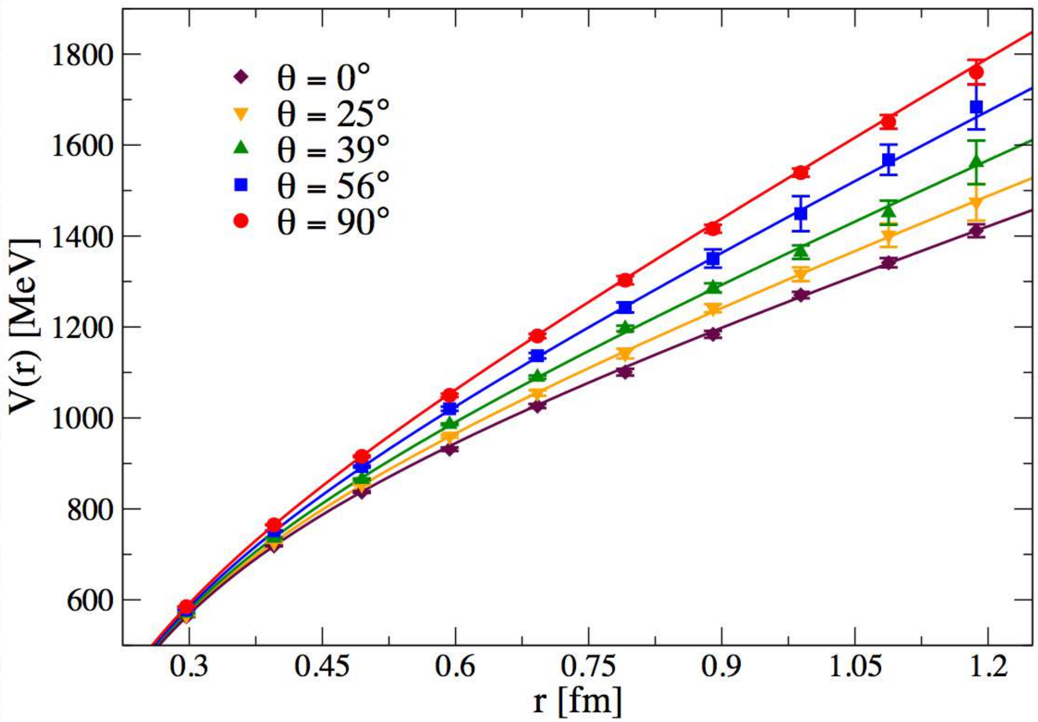
[Bonati et al., Phys. Rev. D 89, 114502 (2014)]

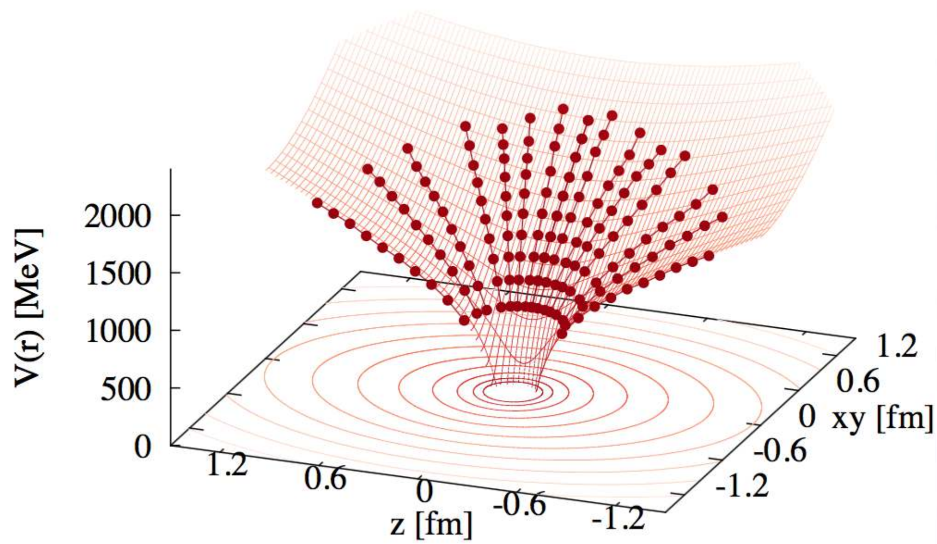


ANISOTROPY IN DETAIL

- The dependence of the potential as a function of angle θ between $\vec{B}$ and $q\bar{q}$ orientation [Bonati et al., Phys. Rev. D 94, 094007 (2016)]

$$V(r,\theta;B) = -\frac{\alpha(\theta;B)}{r} + \sigma(\theta;B)r + V_0(\theta;B)$$

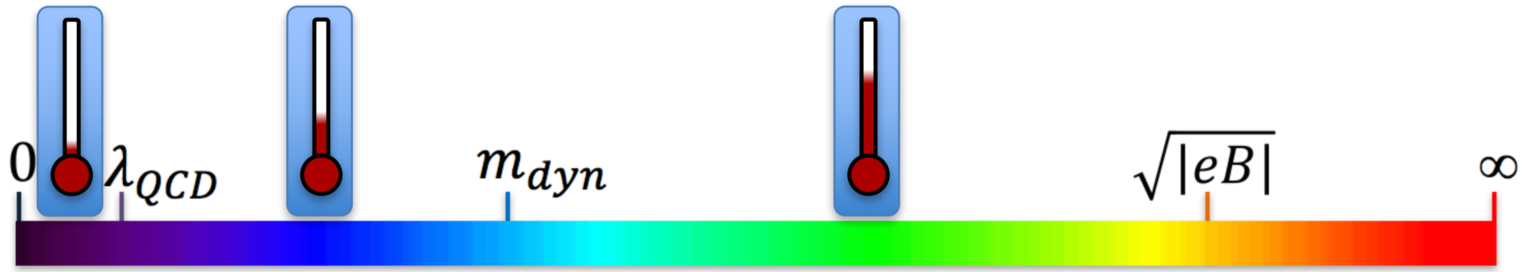




- With increasing angle θ, the string tension increases

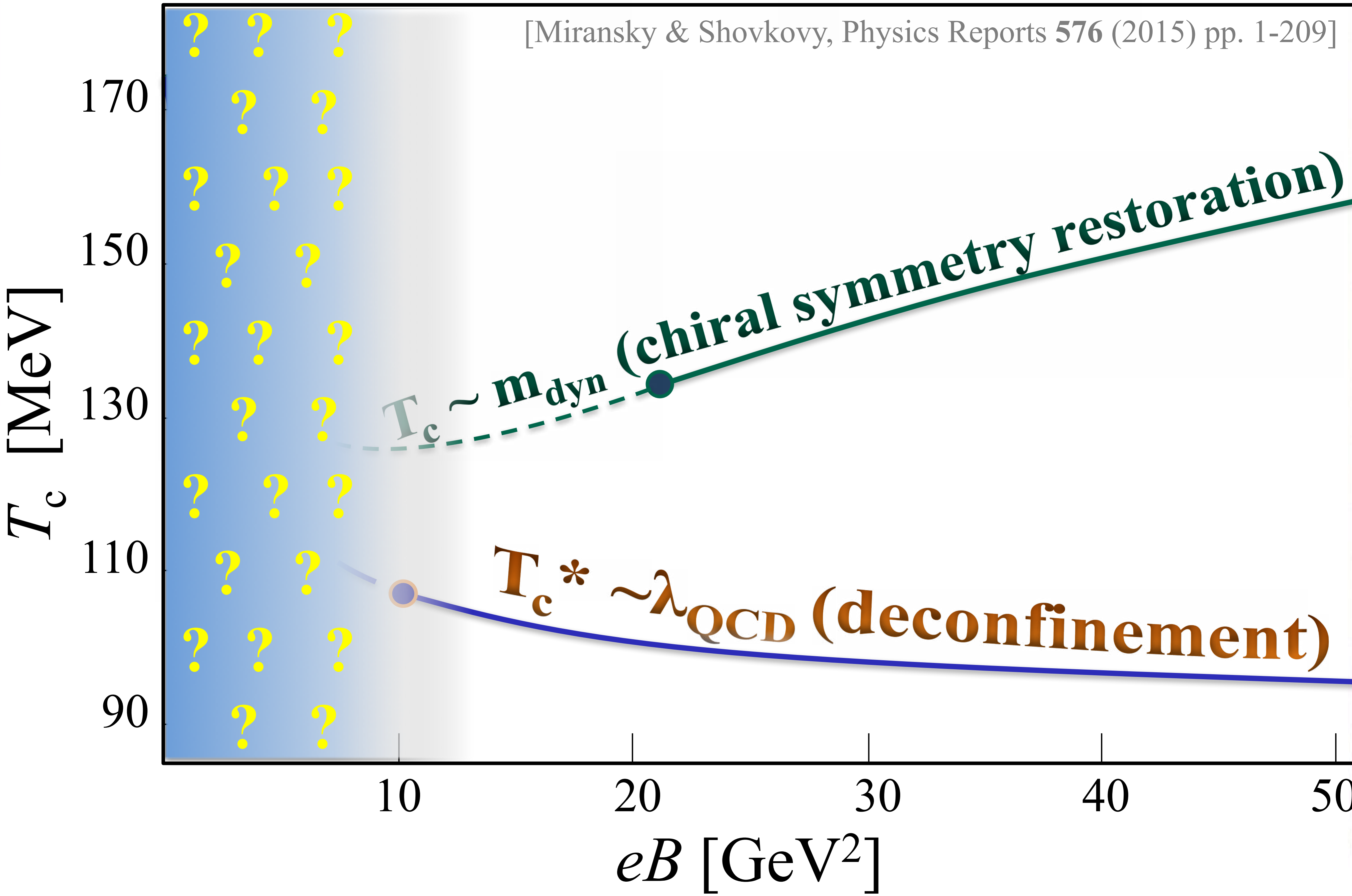
NONZERO TEMPERATURE

- What to expect at nonzero temperature (in strong B limit)?

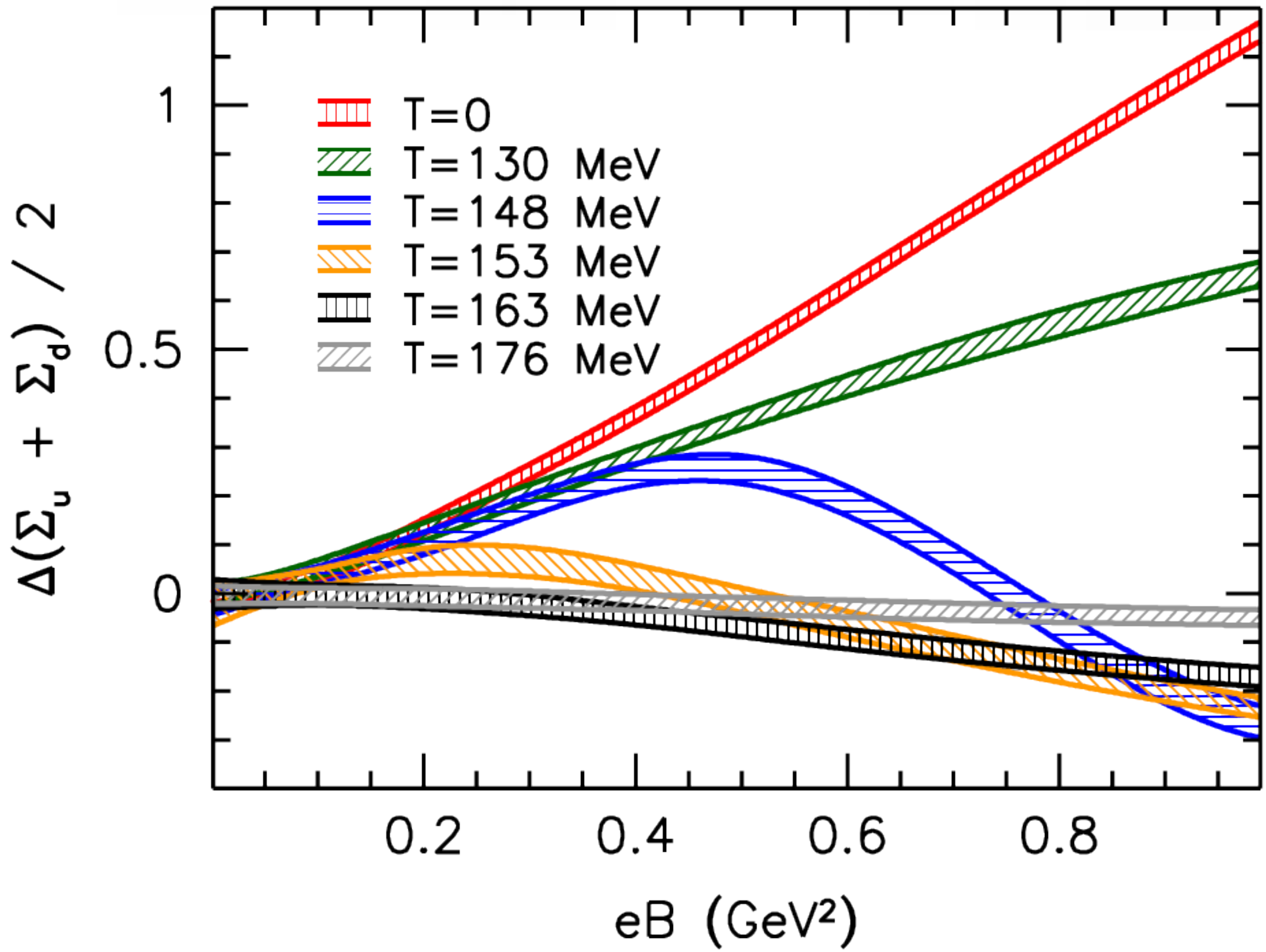


- Very low temperatures, $T \ll \lambda_{QCD}$
  - Ground state in not affected much
  - Color is confined, lowest energy states are glueballs
  - Chiral symmetry is broken ($T \ll \lambda_{QCD} \ll m_{dyn}$)
- Intermediate temperatures, $\lambda_{QCD} \ll T \ll m_{dyn}$
  - Color is deconfined; gluons are thermally populated
  - Chiral symmetry is still broken ($\lambda_{QCD} \ll T \ll m_{dyn}$)
- Moderately high temperatures, $m_{dyn} \ll T \ll \sqrt{|eB|}$
  - Chiral symmetry is restored ($m_{dyn} \ll T$)

PREDICTED PHASE DIAGRAM



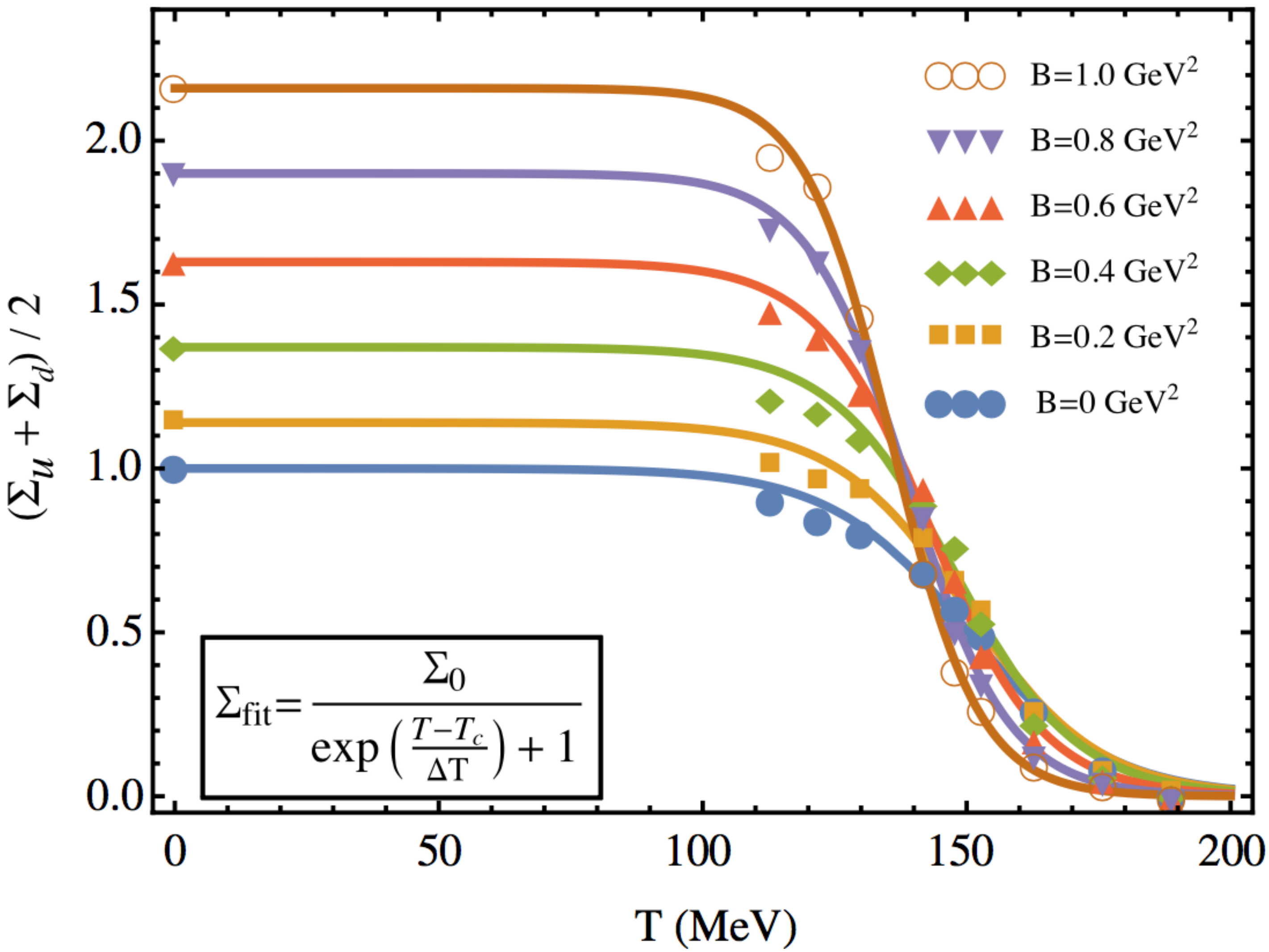
INVERSE CATALYSIS AT T≠0



[Bali et al., Phys. Rev. D**86**, 071502 (2012)]

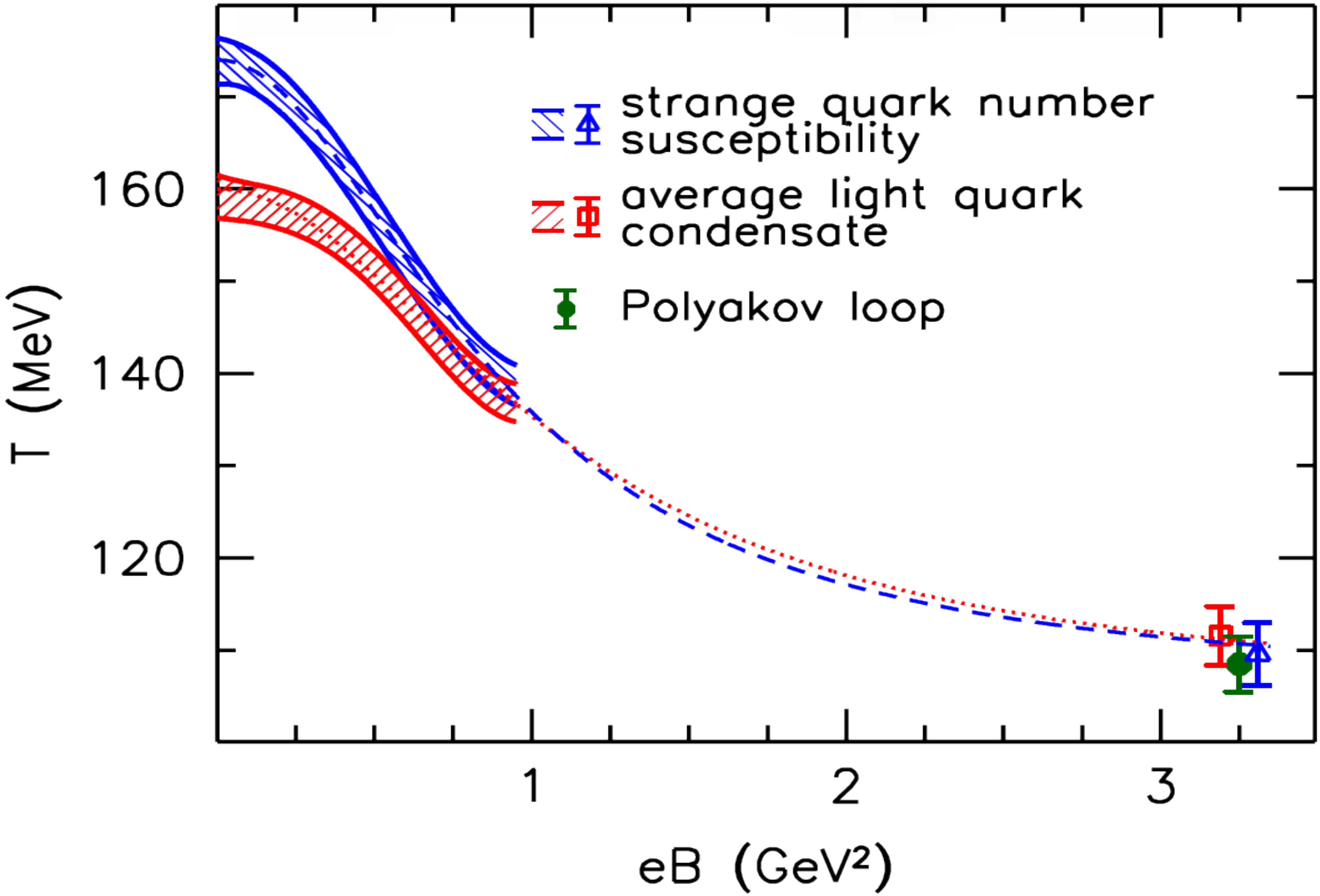
Inverse Catalysis at T≠0

- The temperature dependence at several fixed values of B



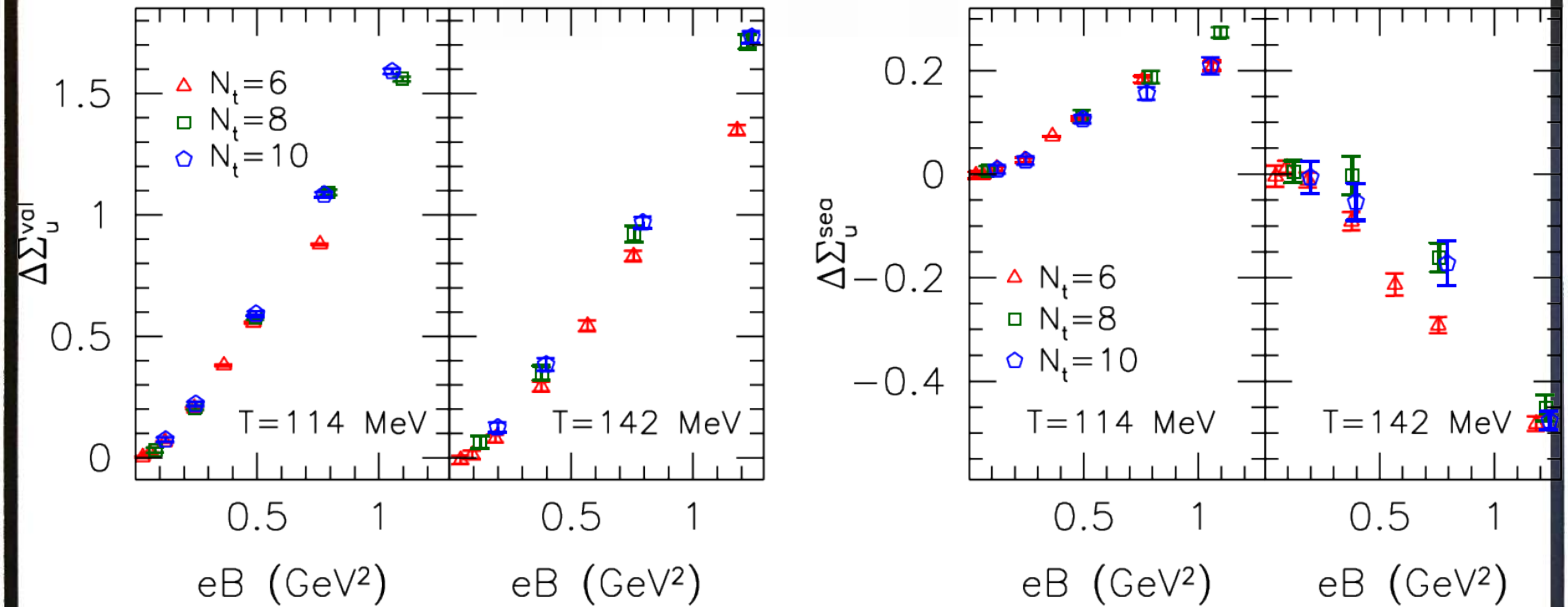
- Confinement strongly affects the low-temperature region

DEPENDENCE OF T_C VS. B



[Bali et al., JHEP 02, 044 (29012)], [Bali et al., PRD**86**, 071502 (2012)], [G. Endrodi, JHEP 1507 (2015) 173]

Valence vs. Sea

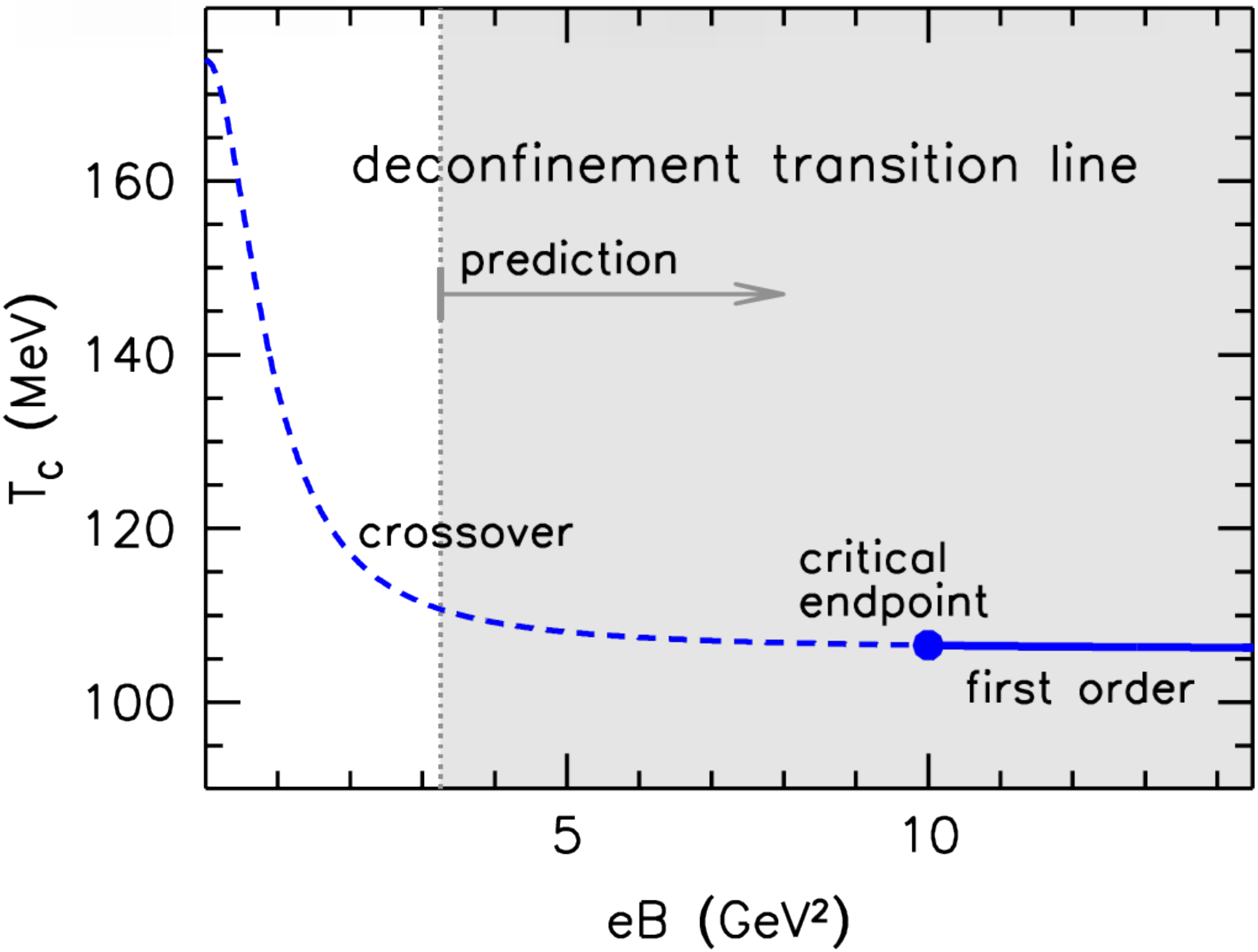


[Bruckmann, G. Endrodi, T. G. Kovacs, JHEP 04 (2013) 112]

- Gluon screening (?)
- Polyakov loops (?)

} or, perhaps, something else (?)

SUPER-STRONG *B*: PREDICTION



[Cohen & Yamamoto, PRD**89**, 054029 (2014)], [G. Endrodi, JHEP 1507 (2015) 173]

PREDICTED PHASE DIAGRAM