Central Clearing and the Sizing of Default Funds*

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Abstract

We develop a model of central clearing and demonstrate that the current standard for collecting default funds, known as the Cover II rule, is intrinsically vulnerable. Although default funds allow members to share risk ex-post, an inherent externality induces members to take excessive risk ex-ante. Notably, regulating the size of the default fund can mitigate such externality. We solve for an optimal default fund that trades off the cost of funding collateral with the extent of risk-shifting. The optimal default fund covers the default costs of a fraction, rather than of a fixed number, of clearing members.

Keywords: Central counterparties (CCPs), default funds, loss mutualization, externality, risk-shifting

JEL: G20, G23, G28, D61

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1 Introduction

Central clearing is a crucial post-crisis reform to reduce counterparty risk in over-the-counter markets. Through a process called novation, the contractual obligations between the trading parties are replaced with two equivalent positions between each of the original trading parties and the clearinghouse. The clearinghouse requires its members to post a default fund, and uses these collective resources as loss absorbing collateral, thereby insulating its members against counterparty risk losses. The volume of cleared contracts has been growing steadily. By the end of 2017, the clearing rate for interest rate swaps had reached 87 percent in the US and 62 percent in the EU (Financial Stability Board, 2017), and the clearing rate for credit default swaps (CDS) had reached 55 percent (Bank for International Settlements, 2017).

There is, at present, considerable debate over the clearinghouse arrangements. A key requirement is that members contribute to a loss-mutualizing default fund. Under the prevailing rule, known as the “Cover II”, the total default fund posted by members should be sufficient to cover the losses caused by the defaults of the two largest clearing members.\(^1\) A member’s default losses that exceed its initial margins and default fund contribution are absorbed first by the CCP’s equity capital, and then by the default fund contributions of surviving members on a pro-rata basis.\(^2\)

In this paper, we study the optimal sizing of default fund contributions. We first show that the loss-mutualization arrangement is intrinsically vulnerable. While the default funds allow members to share risk ex-post, an inherent externality induces members to take excessive risk ex-ante. We then demonstrate how to mitigate excessive risk-taking behavior by regulating the size of the default fund. We solve for a socially optimal default fund level that trades off the funding costs and the extent of risk-shifting. Notably, the optimal default fund should cover the default costs of a fraction rather than of a fixed number of clearing members. Our results thus imply that current regulatory standards such as “Cover II” should be

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\(^1\)The Committee on Payment and Settlement Systems (CPSS) and the Technical Committee of the International Organization of Securities Commissions (IOSCO) issued principles which require CCPs to maintain a default fund large enough to cover the default of two members in extreme yet plausible market scenarios. Clearinghouses in the US must abide by a Cover I system at a minimum, while international and systemic US clearinghouses have to comply with the CPSS-IOSCO regulatory guidelines. For instance, the “Cover II” rule is adopted by major derivative clearinghouses including ICE Clear Credit, CME Clearing, ICE Clear, and LCH.Clearnet (Armakola and Laurent, 2017).

\(^2\)There is not yet a universally agreed upon loss allocation rule. ICE Clear Credit, the leading CDS clearinghouse, adopts a pro-rata basis for futures, options, and cleared CDS contracts (ICE, 2016).
strengthened.

We develop a model that captures the main economic forces behind the determination of default fund requirements. We consider a market consisting of risk-neutral sellers of credit protection, risk-averse protection buyers, and a regulator. A protection seller’s default risk depends on its own risk-taking and is independent of other sellers’ defaults. Central clearing facilitates risk-sharing among protection sellers, and guarantees that their payment obligations to protection buyers are fulfilled. The risk-averse protection buyers pay a premium to the sellers for the elimination of counterparty risk. With this premium, protection sellers participate in central clearing despite the cost of posting default fund contributions. The regulator, who aims at maximizing the aggregate value of all members and the CCP, chooses the size of the default fund contribution fully cognizant of the risk-taking response by the protection sellers.

Using the model, we study the efficiency of the loss-mutualization arrangement and the “Cover II” rule. Loss-mutualization diversifies counterparty risk among members. However, this risk-sharing benefit comes with a dark side. Sharing the pool of default funds creates dependency among members. When members take excessive risk to earn potentially higher returns, a negative externality arises. The degree of the externality links directly to the size of the default fund. A higher default fund contribution, although more costly, can mitigate the externalities by incentivizing members to take safer investments. As such, in collecting default fund contributions, the regulator faces a trade-off between preventing members from excessive risk-taking and keeping the funding cost low. We study this trade-off and identify the size of the default fund that emerges in equilibrium. To the best of our knowledge, our study is the first to formalize this mechanism in the setting of central clearing.

We provide an explicit expression for the optimal default fund requirement that alleviates the inefficiency. For clearinghouses consisting of a sufficiently large number of members, we show that this optimal fund should cover the default costs of a fraction of members, provided that the financing costs are not too high. While such a default fund size is higher than that prescribed by the prevailing “Cover II” rule, it leads members to choose safer investments and avert negative externalities on each other. This finding is in line with Murphy and Nahai-Williamson (2014) who argue that the “Cover II” standard is far from prudent.\(^3\)

\(^3\)They argue that higher levels of financial resources may be needed to ensure the robustness of the clearinghouse: “Perhaps a simple backstop to cover 2 could be considered, such as demanding that the
If instead, the cost of funding collateral is high, requiring a level of default fund that is
incentive-compatible for members would be too costly. In this case, members engage in
risk-taking, leaving the clearinghouse in a vulnerable position. These predictions are also
consistent with legal studies, e.g., Yadav (2013) who argues that risk-taking incentives lead
clearing members to pursue risky payoffs at the expense of the clearinghouse.

The proposed sizing of default funds is robust against entry and exit of members in the
clearing business. In addition, the formulation of the equilibrium problem has a nontrivial
methodological contribution. Because the externality among members introduces a mixing
parameter in the binomial distribution of defaults, the combinatorial techniques employed
in our solution method may apply to a broader class of equilibrium problems characterized
by the presence of domino effects.

Our paper adds to the growing literature on central clearing and its impact on counter-
party risk and financial stability. Several studies have shown that central clearing improves
market conditions relative to bilateral OTC trading along several dimensions. Acharya and
Bisin (2014) show that central clearing can correct an inefficiency arising from the lack of
portfolio transparency in OTC markets; Zawadowski (2013) argues that a clearinghouse ef-
effectively forces banks to contribute to bailing out defaulting counterparties; Biais, Heider,
and Hoerova (2012, 2016) show that CCPs can mutualize counterparty risk by appropriately
setting margins. On the other hand, studies have highlighted various aspects of inefficiency
in central clearing. Duffie and Zhu (2011) and Duffie, Scheicher, and Vuillemey (2015) show
respectively, theoretically and empirically, that central clearing can increase counterparty
risk if the clearing process is fragmented across multiple CCPs; Antinolfi, Carapella, and
Carli (2016) show that loss-mutualization may weaken the incentives to acquire and reveal
information about counterparty risk; Koeppl (2012) and Koeppl and Monnet (2010) high-
light a type of inefficiency related to market liquidity; and Pirrong (2014) argues that central
clearing reforms may redistribute risk rather than reduce risk. Unlike these studies, we ana-
yze the risk-taking incentives of protection sellers in the CDS market, and highlight a novel
form of inefficiency associated with the loss mutualization mechanism.

A closely related paper is that of Stephens and Thompson (2014). They model the CDS

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default fund in addition meets the requirement that it is larger than some fixed percentage of the ‘cover all’
requirement (Murphy and Nahai-Williamson, 2014, pg. 17).”

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4Loon and Zhong (2014) and Bernstein, Hughson, and Weidenmier (2014) both provide empirical evi-
dence that supports the reduction of counterparty risk in central clearing.
market and show that premium-driven insured parties choose to contract with risky insurers, thereby exacerbating counterparty risk in the market. In an extension of the model, they study the role of CCPs and show that co-insurance would make the market riskier as risky insurers crowd out safe insurers. They propose that CCPs condition the size of the insurer’s default fund contribution on its credit quality. In line with their study, we formalize the CCP loss mutualization arrangement and demonstrate how protection sellers would engage in risk-taking. Complementary to their proposed solution, we show that when the insurers’ risk-taking is not contractible, simply regulating the size of default fund contribution can alleviate the inefficiency.

Our paper focuses on the “Cover II” rule; as such, it relates to the studies that stress test the effectiveness of the current standard in absorbing default losses in distressed market scenarios. Using CDS data and an Eisenberg-Noe model, Paddrik and Young (2017) show that the simultaneous failure of two members could cause network contagion and further lead to insufficient funds at the CCP. Menkveld (2016) shows that crowded trades of dealers could amplify losses of CCPs in stressed scenarios. Building on this idea, Campbell and Ivanov (2016) show that the losses could be more substantial if the exposures of large CCP members are positively correlated than if they are independent. Overall, these studies imply that the “Cover II” standard is inadequate because it does not account for compounding sources of risk. Our study is supportive of this conclusion and provides a theoretical model to pin down the optimal size of the default fund.

Finally, we add to the nascent literature on default fund requirements. Menkveld (2017) analyzes systemic liquidation within a crowded trades environment and sets the default fund as the minimum level of funds needed to cover default losses in extreme yet plausible conditions. Ghamami and Glasserman (2017) provide a calibration framework and show that lower default fund requirements reduce the cost of clearing but make CCPs less resilient. Capponi, Cheng, and Sethuraman (2017) analyze the optimal balance of equity and default fund requirements from the clearinghouse point of view in a model economy consisting of safe and risky clearing members. While most of these studies follow exogenously specified risk measures to determine default fund requirements, our study endogenizes members’ choice of risk and focuses on the socially optimal cover rule.

The paper proceeds as follows. Section 2 introduces the baseline model with binary risk. Section 3 analyzes the game between the regulator and the clearing members. Section
generalizes to an environment in which members can choose risk-taking in a continuum domain. Section 5 concludes. Technical proofs are delegated to the Appendix.

2 Baseline Model

In this section, we introduce the baseline model in which clearing members have a binary choice of risk. We show that members have an incentive to take excessive risk due to an inherent externality associated with loss mutualization.

2.1 Environment

There are two dates, \( t = 0, 1 \). The environment has \( N \) risk-neutral dealers, who sell CDS contracts to protection buyers. The protection buyers are identical and risk averse with mean-variance preferences, i.e., \( U(X) = E[X] - \gamma \text{Var}[X] \), where \( \gamma > 0 \) measures the risk aversion.

At \( t = 0 \), buyers and sellers enter into insurance contracts. The price of the contract is normalized to 1 and is paid by the buyer to the seller at \( t = 0 \). The contract specifies that the seller delivers a payment \( \delta \) to the buyer at \( t = 1 \) upon the occurrence of a credit event. To simplify the setup, we assume that the credit event occurs with certainty.\(^5\)

Sellers choose whether to invest in a riskier (\( r \)) or a safer (\( s \)) investment. The riskiness of the investment captures, for example, the insufficient hedging of counterparty exposure as in Zawadowski (2013), the failure to exert effort in risk management as in Biais, Heider, and Hoerova (2016), or the choice of distressed counterparties as in Wang (2015). The choice of riskiness is unobservable and non-contractible.

At time \( t = 1 \), payoffs are realized and payments are allocated. With probability \( q \in (0, 1) \), the investment fails and the seller obtains zero payoff. With probability \( 1 - q \), the investment succeeds and the seller obtains a positive payoff \( R \). The expected return of the investment is thus \( (1 - q)R \), which we denote as \( \mu \).

\(^5\)In practice, the occurrence of a credit event is stochastic. While one could assign a probability to the credit event, doing so would introduce technical complications without deepening the economic insights. Furthermore, to make the insurance contracts meaningful, we implicitly assume that the buyers are not able to self-insure nor do they have access to alternative assets that deliver a future payment upon the credit event.
Assumption 1 Compared to a riskier investment \((r)\), a safer investment \((s)\) has a lower probability of failure but also a lower payoff when successful, i.e.,

\[
0 < q_s < q_r, \quad \delta < R_s < R_r. \tag{1}
\]

In addition, a safer investment has a higher expected payoff, with the expected return differential \(\mu_s - \mu_r\) bounded from above by \(\delta(q_r - q_s)\), i.e.,

\[
0 < \mu_s - \mu_r < \delta(q_r - q_s). \tag{2}
\]

Condition (1) captures the risk-return trade-off between the riskier and the safer investments. Condition (2) informs the socially optimal investment. To the extent that the socially optimal investment is one that maximizes the expected payoff in the economy, the safer investment is the optimal choice; hence, we refer to the risk profile \(a_i^* = s, i = 1, \ldots, N\), as the optimal benchmark. However, Condition (2) also imposes that the expected return differential is not too high, such that when a protection seller with a failing investment can get away with the payment \(\delta\), she prefers the riskier investment despite the lower expected payoff (as seen in (3) below).

### 2.2 Bilateral Trading

We begin by analyzing the utilities of market participants when trading is not guaranteed by a CCP.

**Protection Sellers** Denote the investment choice of a protection seller by \(a \in \{s, r\}\). If the investment is successful, the seller survives; he pays the contractually agreed upon amount \(\delta\) to the buyer, and keeps the residual \(R_a - \delta\), \(a \in \{s, r\}\). If the investment fails, the seller receives zero payoff and defaults on the buyer; this is the source of counterparty risk. The expected profit of a seller in a bilaterally trading (BT) market is given by

\[
V_{BT} = \max_{a \in \{s, r\}} (1 - q_a)(R_a - \delta) = \mu_r - (1 - q_r)\delta. \tag{3}
\]
Condition (2) in Assumption 1 implies that the protection seller chooses the riskier investment when trading bilaterally.

**Protection Buyers** Under bilateral trading, the buyer only receives the promised payment $\delta$ if the seller survives, an event which occurs with probability $1 - q_r$. If instead the seller defaults, the buyer receives a zero payoff. The expected utility for a risk-averse buyer in a bilaterally trading (BT) market is given by

$$U_{BT}(1_{\text{seller survives}}\delta) = (1 - q_r)\delta - \gamma q_r (1 - q_r)^2$$

where $1_A$ denotes the indicator function of the event $A$.

### 2.3 Central Clearing

When protection sellers become members of the CCP, their default risks are insulated from the protection buyers. We next describe the loss allocation mechanism and the utilities of market participants under central clearing.

**Default Fund and Waterfall** Central clearing pools counterparty risks among members. Buyers and sellers establish direct contractual relationships with the CCP. Acting as the buyer to every seller and the seller to every buyer, the CCP mutualizes losses caused by the default of its members and guarantees that the promised payments be delivered to the protection buyer. To do so, the CCP requires that each member posts a prefunded default fund $F \in (0, \delta]$ at time $t = 0$. The default fund is segregated and thus imposes an opportunity cost $\beta F$ to the seller, where $\beta \in (0, 1)$ denotes the cost of funding the collateralized position.

We focus on the default funds and do not model initial margins, which usually serve as the first line of defense against default losses. Hence, the payment shortfall should be interpreted as that which exceeds the initial margin requirements. This setup allows us to focus exclusively on the risk-shifting incentives triggered by default funds. Furthermore, we consider a default waterfall structure in which the clearinghouse’s equity has seniority over

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Footnote 6: While protection buyers could also be members of the CCP, we concentrate on the risk-taking incentives of protection sellers because they have access to risky investment and are subject to default. Consequently, the CCP has risk exposure only to the protection sellers, not to buyers.
the default funds of surviving members in absorbing losses.\textsuperscript{7} Losses exceeding the defaulting member’s fund are allocated proportionally to the default funds of surviving members. As such, the default fund presents the possibility for a surviving member to incur a loss when others default, creating an externality among members.

The total default funds collected by the CCP cover the payment shortfall caused by the defaults of two members, consistent with the CPSS-IOSCO regulatory guidelines.

**Assumption 2** *The total default fund contributions satisfy the "Cover II" rule, i.e.,*

\[ 2\delta \leq NF. \] (5)

The “Cover II” rule sets a lower bound on $F$. If more than two members default, the total default funds may not be sufficient to cover the total losses. In this case, the remaining losses are covered by the CCP’s equity.

**Assumption 3** *The CCP contributes equity capital and incurs an equity loss of $(N_d\delta - NF)^+$, where $N_d$ is the number of defaulting members, i.e., $N_d = \sum_i 1_{i \text{ defaults}}$. Assumption 3 guarantees that the CCP has enough resources to satisfy the obligations to the buyers in all states of the economy.*

**Protection Buyers** When a CCP guarantees the trade, the payment obligation of the CDS contract is honored with certainty, so the buyers are hedged against counterparty risk. Buyers value this protection because they are risk averse. Hence, they are willing to pay an additional premium, $f$, to trade with a seller guaranteed by a CCP. Such a premium leaves a buyer indifferent between trading with the seller bilaterally, or through the CCP, i.e., the premium $f$ satisfies

\[ U_{CCP}(\delta - f) = U_{BT}(1_{\text{seller survives } \delta}) \iff f = q_r \delta + \gamma q_r (1 - q_r) \delta^2. \] (6)

The premium is the certainty equivalent that the buyer is willing to give up for being insulated against counterparty risk. This premium consists of two parts: the expected

\textsuperscript{7}A well-known example is the Korea Exchange (KRX), the Korean CCP. The default of a clearing member in December 2013 generated losses that exceeded the defaulter’s collateral. According to the rules of KRX, these losses were absorbed first using the default fund contributions posted by surviving members.
loss due to the counterparty’s default $q_r\delta$, and the compensation for the variance reduction $\gamma q_r(1 - q_r)\delta^2$. The higher the risk aversion $\gamma$, the higher the additional compensation buyers are willing to offer.\(^8\)

**Protection Sellers** When trading through the CCP, the protection seller posts a default fund $F$. The seller also invests the premium $f$, which scales up the size of his investment. For a given default fund $F$ and investment choices by other sellers $a_{-i}$, seller $i$ chooses the riskiness of his investment to maximize expected profits,

$$V_i(a_{-i}; F) = \max_{a_i \in \{r,s\}} \mathbb{E} \left[ 1_{i \text{ survives}} \left[ (1 + f)R_{a_i} - \delta + \left( F - \frac{N_d}{N-N_d}(\delta - F) \right)^+ \right] \right]. \quad (7)$$

To understand the objective function (7), let us analyze the two possible outcomes at $t = 1$.

1. If seller $i$ survives (with probability $1 - q_{a_i}$), he obtains investment proceeds $(1 + f)R_{a_i}$ and delivers the promised payment $\delta$ to the protection buyer. The default fund posted by $i$ is used to absorb the payment shortfalls of defaulting members, and the residual amount if any, $\left( F - \frac{N_d}{N-N_d}(\delta - F) \right)^+$, is returned to $i$.

2. If seller $i$ defaults (with probability $q_{a_i}$), he obtains zero payoff; his segregated default fund $F$ contributes to his promised payment $\delta$ to the protection buyer.

The last term in Eq. (7) captures the loss mutualization mechanism. If $N_d$ members default, the total shortfall is $N_d\delta$. Each defaulted member first absorbs the loss using his own default fund, $F$, and the remaining shortfall is shared equally by the surviving members. Hence, a surviving member $i$ is charged the amount $\frac{N_d}{N-N_d}(\delta - F)$ capped at the default fund segregated.

For a seller that participates in central clearing, his total expected profit includes the expected profit accrued at $t = 1$ given by Eq. (7), net of the default fund requirement as well as the incurred opportunity cost, i.e.,

$$V_i^{CCP}(a_{-i}; F) = -(1 + \beta)F + V_i(a_{-i}; F). \quad (8)$$

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\(^8\)In solving for the premium given by Eq. (6), we implicitly assume that sellers have full bargaining power; this allows the risk-sharing gains from central clearing to be fully reflected in the sellers’ expected profits.
We assume that protection buyers are sufficiently risk averse and pay a high enough premium, such that sellers are willing to participate in the CCP. Throughout the paper, we impose the following assumption.

**Assumption 4** *Buyers are sufficiently risk averse, i.e., their risk aversion satisfies*

\[
\gamma > \frac{\beta + q_r}{\mu_r q_r (1 - q_r) \delta} - \frac{1}{(1 - q_r) \delta}.
\]  

(9)

The following proposition shows that central clearing is an equilibrium outcome.

**Proposition 1** *A seller expects a higher profit when participating in central clearing than trading bilaterally with a buyer, i.e.,* \( V_{CCP}^i (a_i - i; F) - V_{BT} > 0, \forall F \in (0, \delta], \forall a_i \).

Since the prefunded position is costly, a seller obtains the lowest profit when charged with the highest possible default fund, i.e., \( F = \delta \). In this case, the default fund posted by the seller is sufficient to cover the promised payment to the buyer even in the absence of a loss mutualization mechanism. Under Assumption 4, the premium given by Eq. (6) is high enough to guarantee that the expected profit of the member under the CCP is greater than the expected profit under bilateral trading in Eq. (3). Hence, the seller is better off being a member of the CCP, and the market participants as a whole benefit from central clearing.

Proposition 1 shows that, although clearing members are risk neutral, there is still value created from risk-sharing. When the clearinghouse guarantees trades, the protection buyers are fully hedged against counterparty risk. The value of the risk-sharing benefit is reflected in the premium that buyers offer to the risk-neutral protection sellers.

## 3 Risk-Shifting and the Optimal Default Fund

In this section, we analyze the strategic interactions between a regulator and clearing members using a Stackelberg game. The regulator moves first by choosing the size of the default fund that maximizes social welfare. The clearing members are the followers, who respond to the regulator’s choice of \( F \) by deciding between a riskier and a safer investment to maximize their individual profits.
**Definition 1** A Nash equilibrium between clearing members and the regulator is a set of members’ risk profiles \(a^e := (a_1^e, \ldots, a_N^e) \in \{r, s\}^N\) and a default fund contribution set by the regulator, \(F^e\), such that:

1. Taking the default fund, \(F^e\), and other members’ risk profiles, \(a_{-i}^e\), as given, \(a_i^e\) solves member \(i\)’s optimization problem in (7).

2. Taking the risk profile of members \(a^e\) as given, the regulator chooses a default fund subject to Assumption 2 to maximize the total value of members and the equity of CCP:

\[
F^e = \arg \max_F \left\{ \sum_i V_i(F) - \mathbb{E} \left[ (\mathcal{N}_d \delta - NF)^+ \right] - N(1 + \beta)F \right\},
\]

where \(V_i(F)\) is given by Eq. (7) and \(\mathcal{N}_d\) is the number of defaulted members.

We compute the equilibrium using backward induction, first solving for the best response of members to a given default fund \(F\), then backing out the regulator’s choice of \(F\).

### 3.1 Members’ Investment Choices

We start by comparing members’ equilibrium investment choice with the socially optimal benchmark. Recall that the safer investment has a higher expected payoff. As the buyers are guaranteed with the full payment and thus have a constant utility, the risk profile \(a_i^e = s, \forall i\) maximizes the total payoff of all market participants, i.e., is the socially optimal benchmark. We call the situation in which the equilibrium choice of sellers is the riskier investment—thus deviating from the optimal benchmark—risk-shifting.

**Definition 2** A Nash equilibrium \((a_1, a_2, \ldots, a_N)\) for the members’ risk profiles is Pareto dominating, if, for any other Nash equilibrium \((b_1, b_2, \ldots, b_N)\), it holds that \(V_i(a_{-i}) \geq V_i(b_{-i})\), \(\forall i = 1, 2, \ldots, N\), and the above inequality holds strictly for at least one \(i\).

Suppose member \(i\) survives, and \(g\) of the remaining \(N - 1\) members choose the safer investment. The refund that member \(i\) expects to obtain from the CCP is given by

\[
\psi(g; F) := \mathbb{E} \left[ \left( F - \frac{\mathcal{N}_d}{N - \mathcal{N}_d} (\delta - F) \right)^+ \right| i \text{ survives, } g \text{ members other than } i \text{ choose safer} \right].
\]

(11)
The term $F - \psi(g; F)$ therefore represents the expected contribution of member $i$ to the loss mutualization, i.e., to the absorption of defaulting members’ payment shortfalls. A low refund $\psi(g; F)$ indicates that the negative externalities on $i$ caused by the defaults of other members are high. To elaborate the refund term $\psi(g; F)$ and the combinatorial techniques used in our framework, we show in Lemma A2 in the appendix that $\psi(g; F)$ admits the explicit expression given by

$$
\psi(g; F) = \sum_{k=N-1-[NF]}^{N-1} f_g(k) \left( F - \frac{N - (k + 1)}{k + 1} (\delta - F) \right),
$$

where $[\cdot]$ denotes the floor function (giving the greatest integer less than or equal to the argument). The term $F - \frac{N - (k + 1)}{k + 1} (\delta - F)$ represents the amount of refund to member $i$ in the event that $k$ members among the remaining $N - 1$ members survive. The term $f_g(k)$ represents the probability of such an event when $g$ of the remaining $N - 1$ members choose safer and $N - 1 - g$ of them choose riskier investment, i.e.,

$$
f_g(k) := \sum_{m=0}^{k} \binom{g}{m} (1 - q_s)^m q_s^{g-m} \binom{N - 1 - g}{k-m} (1 - q_r)^{k-m} q_r^{N-1-g-(k-m)}
$$

Note that when the number of survivors is below $N - 1 - [NF]$, the total default fund is not sufficient to cover the total default shortfalls, and thus the refund to member $i$ is zero.

The next proposition summarizes our main results on the investment choice of the clearing members.

**Proposition 2** For a given default fund $F$, the equilibrium risk profiles are given by

$$
a^e = \begin{cases} 
  r, \forall i & F < \hat{F} \\
  r, \forall i, \text{ or } s, \forall i & \hat{F} \leq F \leq \bar{F} \\
  s, \forall i & \bar{F} < F
\end{cases}
$$
Figure 1. Equilibrium investment choices. This figure illustrates how the equilibrium risk profiles, $a^e$, change as we vary the size of the default fund, $F$. In region A, the unique equilibrium features risk-shifting; in region B, we have two equilibria and the optimal benchmark is the Pareto-dominating equilibrium; in region C, the unique equilibrium coincides with the optimal benchmark. The cutoff values, $\hat{F}$ and $\bar{F}$, satisfy

$$
\frac{\mu_s - \mu_r}{q_r - q_s} = \frac{\delta - \psi(N - 1; \hat{F})}{1 + f},
$$

and

$$
\frac{\mu_s - \mu_r}{q_r - q_s} = \frac{\delta - \psi(0; \bar{F})}{1 + f},
$$

and $0 < \hat{F} < \bar{F} < \delta$.

The cutoff values $\hat{F}$ and $\bar{F}$ satisfy the following equations\(^9\)

\[
\frac{\mu_s - \mu_r}{q_r - q_s} = \frac{\delta - \psi(N - 1; \hat{F})}{1 + f}, \quad \frac{\mu_s - \mu_r}{q_r - q_s} = \frac{\delta - \psi(0; \bar{F})}{1 + f}.
\] (14)

Moreover, $\psi(\cdot; \cdot)$ is a strictly increasing function in both arguments. For $F \in [\hat{F}, \bar{F}]$, the “all safer” equilibrium Pareto dominates the “all riskier” equilibrium.

Proposition 2 states that there exist cases in which the loss mutualization mechanism induces members to take excessive risk due to an inherent externality among members. Figure 1 illustrates the relation between the equilibrium investment choices and the size of the default fund. If $F < \hat{F}$ (in Region A), even if all other members choose the safer investment, member $i$ strategically deviates to choose the riskier investment; hence, “all riskier” is the unique inefficient equilibrium. Notably, the adoption of CCPs exacerbates the degree of risk-shifting: Without CCP, each seller invests one unit of riskier investment, whereas with CCP, the riskier investment scales up to $1 + f$. By not implementing the right size of default fund, the introduction of a CCP poses a trade-off between risk-sharing and risk-shifting.

In addition, Proposition 2 states that members’ incentives in risk-shifting decrease as the size of default fund increases. Despite the externalities, it is possible to achieve socially

\(^9\)If $\frac{\mu_s - \mu_r}{q_r - q_s} \geq \frac{\delta}{1 + f}$, it is the degenerate case in which neither equation in (14) has a solution; hence we focus on the case in which $\frac{\mu_s - \mu_r}{q_r - q_s} < \frac{\delta}{1 + f}$ and both cutoff values $\hat{F}$ and $\bar{F}$ exist.
optimal risk-taking in equilibrium. Particularly, if the default fund is set higher than the cutoff value \( \hat{F} \), all members choosing safer investment is an equilibrium and is the unique Pareto-dominating equilibrium. If the default fund is raised further and exceeds the cutoff value \( \bar{F} \), all members choosing safer investment is the unique equilibrium. In sum, once the size of the default fund is set properly, members avert the externalities and achieve a lower default probability.

### 3.2 Default Fund: A Tool to Mitigate Risk-Shifting

Having established the role of the default fund in correcting members’ risk-shifting incentives, we move on to study how the regulator calibrates the size of the default fund to mitigate such inefficiency. Particularly, we solve for the optimal size of the default fund that satisfies criterion (10) given the strategic response of members in (13).

From Assumption 3, the regulator’s objective function in (10) may be rewritten as follows:

\[
\sum_i V_i(F) - \mathbb{E}[(N_d \delta - N\beta F)^+] - N(1+\beta)F = \sum_i (1+f)\mathbb{E}[R_{a_i}] - N\beta F - N\delta := NW(F) \tag{15}
\]

where we use \( W(F) \) to denote the regulator’s objective function averaged at the individual member level.

From Proposition 2, it suffices to consider either “all safer” or “all riskier” profiles. We apply the equilibrium refinement concept of Pareto dominance. When \( \hat{F} \leq F \), members all choose the safer investment, because it is either the unique equilibrium (region C in Figure 1) or the Pareto-dominating equilibrium (region B in Figure 1). Hence the equilibrium risk profile switches from riskier to safer investments at the boundary between regions A and B, i.e., when \( F = \hat{F} \).

Due to this discontinuity in risk choice at \( \hat{F} \), the function \( W(F) \) has a

---

\(^{10}\)To obtain the threshold value of \( \hat{F} \), notice that \( \psi(N-1;F) - \delta + \frac{\mu_r - \mu_s}{q_r - q_s} \) is a strictly increasing function of \( F \) by Lemma A4 in the Appendix. Moreover, \( \psi(N-1;\delta) = \delta \). Thus if condition \( \frac{\psi(N-1;\hat{F}) - \delta}{1+f} + \frac{\mu_r - \mu_s}{q_r - q_s} < 0 \) holds, there exists a unique threshold value \( \hat{F} \). Otherwise, the switch from “all safer” to “all riskier” equilibrium does not occur, in which case the “Cover II” rule is large enough to make a riskier investment not attractive. In the sequel, we only focus on the (interesting) case that the threshold value \( \hat{F} \) exists.
Figure 2. Regulator’s objective function. This figure plots the regulator’s objective $NW(F)$, where $F$ ranges from the lower bound $\frac{2\delta}{N}$ to the upper bound $\delta$. The graph shows that a default fund given by the “Cover II” requirement may not be socially optimal. Rather, a higher value of $F$ that corrects for risk-shifting yields the highest total value. Model parameters: $\gamma = 0.14$, $\mu_r = 2.7$, $\mu_s = 2.755$, $N = 20$, $\delta = 2$, $q_r = 0.1$, $q_s = 0.05$, and $\beta = 0.1$. The optimal default fund is $F^* = \hat{F} = 0.69$. From ICE Clear Credit (https://www.theice.com/clear-credit/participants), the number of clearing members is between 20 and 30.

The optimal default fund is $F^e = \hat{F} = 0.69$. From ICE Clear Credit (https://www.theice.com/clear-credit/participants), the number of clearing members is between 20 and 30.

piecewise linear structure with one discontinuity at $\hat{F}$:

$$W(F) = \begin{cases} 
(1 + f)\mu_r - \beta F - \delta, & F \in \left[\frac{2\delta}{N}, \hat{F}\right) \\
(1 + f)\mu_s - \beta F - \delta, & F \in \left[\hat{F}, \delta\right].
\end{cases} \tag{16}$$

Because $\mu_s > \mu_r$, $W(F)$ exhibits a positive jump at $\hat{F}$ as we increase $F$, with the jump size given by $(1 + f)(\mu_s - \mu_r)$. Moreover, for both cases in Eq. (16), the expressions are linear and strictly decreasing in $F$. Thus the maximum of $W(F)$ over the set of feasible values for $F$ is attained either at the lower bound $\frac{2\delta}{N}$, or at the switch point $\hat{F}$. We summarize this result in the following proposition; see also Figure 2 for an illustration.

Proposition 3  The default fund that maximizes the regulator’s objective function (10), sub-
ject to the “Cover II” requirement (5), is given by

\[ F^e = \begin{cases} 
    \hat{F}, & \text{if } (1 + f)(\mu_s - \mu_r) > \beta \left( \hat{F} - 2\delta/N \right), \\
    \frac{2}{N}\delta, & \text{otherwise.}
\end{cases} \]  

(17)

where \( \hat{F} \) is given by (14).

The regulator aims to maximize the total value of agents in the economy, accounting for members’ incentives to shift risk. The optimal sizing of default funds reflects the following trade-off. A higher value of \( F \) corrects for risk-shifting: As \( F \) increases to \( \hat{F} \), we move from region A to region B in Figure 1, where members’ choices switch from riskier to safer investment and the expected investment payoff of each member increases by \((1 + f)(\mu_s - \mu_r)\). Nonetheless, increasing \( F \) raises the funding cost of each member from \( \beta \frac{2\delta}{N} \) to \( \beta\hat{F} \). If the payoff increase exceeds the increase in funding cost, it is optimal to set the default fund at \( \hat{F} \) beyond the “Cover II” rule; otherwise, the regulator will find it optimal to stick to the “Cover II” lower bound.

3.3 The Optimal Cover Rule

If the “Cover II” rule is not socially optimal, then how many defaults should the total default fund cover? A generalized Cover X rule for a given number of participating members \( N \) is

\[ X(N) := \frac{NF^e(N)}{\delta}, \]  

(18)

where \( F^e(N) \) is the equilibrium size of the default fund given in Proposition 3. When \( X(N) > 2 \), our model provides a rationale to charge a default fund larger than the current regulatory requirement prescribed by the “Cover II” rule; see also Figure 3 for an illustration.

The exact Cover X rule depends on the number of members. For example, the Cover X rule for \( N = 6 \) is less than 2, so “Cover II” is sufficient in preventing risk-shifting. However, clearinghouses usually consist of a large number of participating members. The ICE Clear Credit, for instance, consists of approximately 30 clearing members. For these clearinghouses, our model predicts that the optimal default fund should cover more than two members.

Interestingly, while the optimal cover number clearly depends on the number of mem-
Figure 3. Optimal covering number. This figure shows the optimal cover number $X(N)$ given in equation (18) as a function of the number of clearing members $N$. We use the same parameter settings as in Figure 2. Left panel: the optimal cover number $X(N)$; Right panel: the optimal cover ratio $X(N)/N$. The graph suggests that the optimal default fund should cover payment shortfalls caused by a 34.7% fraction of members’ defaults.

For numbers $N$, the ratio $X(N)/N$ shows little variation with respect to $N$, especially when $N$ is sufficiently large. The next proposition characterizes the asymptotic behavior of the optimal cover ratio as the number of clearing members grows large. Specifically, it shows that the ratio $X(N)/N$ converges to a constant as the number of members tends to infinity. Under certain conditions (e.g., the marginal opportunity cost of default funds $\beta$ is not too high), this limit is a positive number in $(0, 1)$, suggesting that the optimal cover number should be proportional to the number of members.

**Proposition 4** In the limiting case of a large CCP network—as $N \to \infty$—the optimal cover number...
ratio has the following analytical expression:

\[
\frac{X(N)}{N} = \begin{cases} 
1 - \frac{(1 + f)(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)\delta}, & \text{if } \frac{(\mu_s - \mu_r)(1 + f)}{\beta\delta} > 1 - \frac{(1 + f)(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)\delta} \\
0, & \text{otherwise}
\end{cases}
\]  

(19)

where the trading premium associated with CCP guarantee, \( f \), is given by Eq. (6).

The optimal cover ratio \( \frac{X(N)}{N} \) decreases with buyers’ risk aversion \( \gamma \) and the expected return differential \( \mu_s - \mu_r \), and it increases with the failure risk and return of the riskier investment \( q_r \) and \( R_r \). The optimal cover ratio has a nonlinear relation with the default shortfall \( \delta \).

Proposition 4 also presents the comparative statics for the optimal cover ratio, which reduce to the trade-off between encouraging risk-sharing and preventing risk-shifting. First, the optimal cover ratio decreases with the buyers’ risk aversion. As buyers become more risk averse, their valuation of risk-sharing through CCP increases. Consequently, buyers are willing to offer a higher premium to encourage the protection sellers to join the CCP. If the risk-sharing channel dominates, the risk-shifting channel becomes less critical. As a result, a lower default fund is desired.

Second, the optimal cover ratio also decreases with the expected return differential \( \mu_s - \mu_r \). When the riskier investment becomes less attractive, sellers’ incentives for risk-taking drop. Hence, a high default fund is not necessary. Relatedly, the optimal cover ratio increases with \( q_r \) and \( R_r \). When the default probability of the riskier investment is high, preventing sellers’ risk-shifting becomes the prevailing criterion in determining the default fund contribution. Similarly, when the stakes for taking the risk (\( R_r \)) are higher, sellers’ incentives for risk-shifting are stronger, so the required default fund increases accordingly.

Finally, the optimal cover ratio has a profound and nonlinear dependence on the payment shortfall \( \delta \). To understand the economic forces, let us first keep the trading premium for CCP guarantee, \( f \), fixed. We see that the cover ratio increases with \( \delta \). If we only consider the risk-shifting channel, a higher \( \delta \) exacerbates the risk-shifting incentive and thus requires a higher cover ratio to counteract this effect. There is, however, an offsetting force that introduces nonlinearity in the relationship. The premium paid by the buyers responds to \( \delta \) endogenously. The premium \( f \) captures the value from risk-sharing: As \( \delta \) increases, the risk-
averse buyers assign more value to the elimination of counterparty risk and are thus willing to pay a higher premium for CCP guarantee. Since protection sellers can further scale up the size of their investments, they are satisfied with a lower level of risk-taking; hence, a lower default fund requirement is sufficient. Our result shows that when the risk-sharing effect dominates, the cover ratio decreases with $\delta$.

The above results shed light into the design of regulatory frameworks for default fund requirements. As the number of clearing members grows large, the cover rule that the regulator should adopt is simple—rather than covering a fixed number of clearing members as prescribed, for instance, by the “Cover II” rule, the regulator should cover a fixed fraction of the members. Major US derivative clearinghouses consist of more than 20 members; this is the case, for instance, for the major CDS and interest rates swaps clearinghouses, respectively ICE Clear Credit and LCH. Notably, our proposed sizing of default fund applies to different CCPs with various numbers of clearing members; it is also robust concerning entry and exit of members in the clearing business.\footnote{In May 2014, the Royal Bank of Scotland announced the wind-down of its clearing business due to increasing operational costs. This action was followed by State Street, BNY Mellon, and more recently Nomura, each of whom shut down part or all of their clearing business. Our analysis indicates that these events should only slightly alter the default fund requirements, and in practical circumstances, the resulting change is small enough to be ignored.}

4 Extension to Continuous Choice of Risk-Taking

Having illustrated how default funds could alleviate risk-taking for a binary choice of risk, we next extend the analysis to a more general case of a continuous choice of risk. Such a setup allows us to track the marginal impact of default funds on members’ risk-taking.

Unlike the baseline model, members make risk-taking decisions on a continuous domain. Member $i$ chooses the riskiness of his investment, $a_i \in [0, 1]$. The payoff, denoted by $\tilde{R}_i$, is a random variable given by

$$
\tilde{R}_i(a_i) = \begin{cases} 
(R_r - R_s)a_i + R_s, & \text{in good state with probability } 1 - a_i \\
0, & \text{in bad state with probability } a_i,
\end{cases}
$$

where $0 < R_s < R_r$. The payoffs are i.i.d. across members. In a good state, which occurs
with probability $1 - a_i$, the investment yields a positive payoff, $(R_r - R_s)a_i + R_s$, a value increasing with $a_i$. In a bad state, which occurs with probability $a_i$, the investment yields a zero payoff. Parameter $a_i$ measures the level of risk-taking: A higher $a_i$ implies a higher payoff in the good state but also a lower probability for the good state to occur. This setup is similar to those of Holmstrom and Tirole (2001) and Acharya, Shin, and Yorulmazer (2010) in which the technology for risky investment has diminishing returns to scale with respect to risk-taking.

As a benchmark, we solve for the socially optimal risk-taking, $a^*$, which maximizes the aggregate value of all members. Given that there are no deadweight losses, the optimal risk-taking maximizes the expected payoff from the investment.

$$a^* = \arg \max_{a \in [0,1]} \mathbb{E}[\tilde{R}(a)] = \arg \max_{a \in [0,1]} (1 - a) [(R_r - R_s)a + R_s]$$

(21)

A first-order condition gives

$$a^* = \begin{cases} \frac{R_r - 2R_s}{2(R_r - R_s)}, & \text{if } R_r > 2R_s \\ 0, & \text{otherwise} \end{cases}$$

(22)

Like in the baseline model, the seller has a payment obligation, $\delta$, to the buyer. If the seller receives an investment payoff smaller than $\delta$, he defaults and transfers his asset to the buyer. While for small values of $a$ the seller may default even in a good state (for example when $a = 0$ if $R_s < \delta$), the following assumption ensures that a member has enough resources to pay the buyer in the good state when investing at the socially optimal risk level $a^*$.

Assumption 1-C The payoff parameters satisfy

$$\frac{1}{2}R_r > \max\{\delta, R_s\},$$

(23)

so that $a^* = \frac{R_r - 2R_s}{2(R_r - R_s)} \in (0,1)$ and $\tilde{R}(a^*) > \delta$.

12We label the assumptions in this section with “-C” to indicate that they are direct extensions of the assumptions in Section 2 with the same number.
**Bilateral Trading** As before, we start by analyzing the case in which protection sellers and buyers trade bilaterally. A seller chooses the risk-taking parameter to maximize his expected profit. In a good state, the seller receives a positive payoff, pays \(((R_r - R_s)a + R_s - \delta)^+\) to the buyer and keeps the residual amount, if any. In a bad state, the seller receives 0 payoff; he is unable to deliver the promised payment \(\delta\) and defaults. Hence, the seller has a profit \((\tilde{R} - \delta)^+\) and maximizes the expected profit in a bilateral trading (BT) market,

\[
V_{BT} = \max_{a \in [0, 1]} \mathbb{E} \left[ (\tilde{R}(a) - \delta)^+ \right].
\] (24)

Notice that if \(a = 1\) or if \((R_r - R_s)a + R_s < \delta\), the seller obtains zero profit with certainty. A profit-maximizing seller would only consider the set of risk choices under which he obtains a positive expected profit, that is, only when \(a \in \left( \frac{(\delta - R_s)^+}{R_r - R_s}, 1 \right)\). Denote \(a_{BT}\) the optimal choice of a seller in a bilateral trade. By applying a first-order condition to (24), we obtain

\[
a_{BT} = \frac{R_r - 2R_s + \delta}{2(R_r - R_s)}. \] (25)

A direct comparison with \(a^*\) in Eq. (22) shows that \(a^* < a_{BT} < 1\) and \(V_{BT} > 0\).

The buyer is exposed to counterparty risk and gets a zero payoff if the seller defaults, an event which occurs with probability \(a_{BT}\). The expected utility of the buyer takes a similar form as in Eq. (4), i.e.,

\[
U_{BT} = (1 - a_{BT})\delta - \gamma a_{BT} (1 - a_{BT})\delta^2. \] (26)

**Central Clearing** Next, we turn to the case that the sellers join the CCP. As in the binary case, the protection buyers are willing to pay a premium to trade with a seller through the CCP. The premium is given by

\[
f = a_{BT}\delta + \gamma a_{BT} (1 - a_{BT})\delta^2. \] (27)

This premium allows the seller to scale up his investment from 1 to 1 + \(f\). We henceforth adjust Assumption 4 as follows,
Assumption 4-C  Buyers are sufficiently risk averse, i.e., their risk aversion satisfies
\[\gamma > \frac{\beta + a_{BT}}{\mu_{BT}a_{BT}(1 - a_{BT})\delta} - \frac{1}{(1 - a_{BT})\delta},\]  
(28)

where \(\mu_{BT} = (1 - a_{BT})(R_r - R_s)a_{BT} + R_s\).

With sufficient risk aversion, protection buyers value the risk-sharing benefit from central clearing such that the sellers are willing to join the CCP, despite the costly default fund requirement \(F \in (0, \delta)\).

Each clearing member \(i\) strategically chooses his risk profile to maximize expected profit. The seller’s investment is scaled up by \(1 + f\) where the premium \(f\) is given by Eq. (27). Using a similar argument as in the case of bilateral trading, the seller has zero profit with certainty when \(a = 1\) or when \((1 + f)(a(R_r - R_s) + R_s) < \delta\). Thus, we only consider risk choices \(a \in A = \left(\frac{[\delta/(1+f) - R_s]}{R_r - R_s}, 1\right)\) when analyzing the seller’s risk-taking under central clearing. In a good state, the seller is able to pay \(\delta\) to the buyer; in a bad state, the seller exhausts his segregated default fund \(F\) to partially cover the payment \(\delta\) and leaves a residual loss of \(\delta - F\) to be covered jointly by the surviving members.

Risk-Shifting  We consider the risk choices that arise in the symmetric equilibrium of the game, i.e., in which all members choose the same risk-taking strategy. Given the risk choices of other members, \(a_{-i} = \bar{a}\), and the default fund contribution, \(F\), the expected profit of member \(i\) at \(t = 1\) is
\[V_i(a_{-i} = \bar{a}; F) = \max_{a \in [0, 1]} \mathbb{E}\left[1_{(1+f)\hat{R} \geq \delta}\left((1 + f)\bar{R} - \delta + \left(F - \frac{N_d}{N - N_d}(\delta - F)\right)^{+}\right)\right],\]
(29)

where \(\phi_N(\bar{a}; F)\) is obtained by evaluating the function \(\psi(g; F)\) defined in (12) at \(g = N - 1\) and \(q_s = \bar{a}\), i.e.,
\[\phi_N(\bar{a}; F) = \sum_{k=N-1-[\frac{NF}{\bar{a}}}^{N-1} \binom{N-1}{k} (1 - \bar{a})^k \bar{a}^{N-1-k} \left(F - \frac{N - (k + 1)}{k + 1}(\delta - F)\right).\]
(30)

The term \(\phi_N(\bar{a}; F)\) captures the refund that member \(i\) expects to obtain from the CCP,
i.e., the portion of the default fund segregated from member $i$ that is not used to absorb default losses. If seller $i$ participates in central clearing, his total expected profit is defined the same as in the baseline model; the expected profit includes the expected profit accrued at $t = 1$, net of the default fund contribution and the associated opportunity cost.

Because of our symmetric configuration of clearing members, we restrict our attention to symmetric equilibria. A symmetric equilibrium among members is a risk profile $a^e \in A$ such that, member $i$’s best response when all other members choose $a^e$ is also $a^e$; i.e., no unilateral deviation is profitable for members. The following proposition characterizes the risk-taking strategy in a symmetric equilibrium.

**Proposition 5** Given a default fund $F \in \left[ \frac{2\delta}{N}, \delta \right]$, there exists symmetric response of members such that $a_i(F) = a^e(F), \forall i$ and $a^e(F)$ is the interior solution obtained from taking the first-order condition of Eq. (29), i.e.,

$$a^e(F) = \frac{R_r - 2R_s + \frac{\delta - \phi_N(a^e(F);F)}{1+f}}{2(R_r - R_s)}.$$  \hspace{1cm} (31)

For any such response $a^e(F)$, it holds that

$$a^* < a^e(F) < a_{BT},$$  \hspace{1cm} (32)

with $a^*$ given in (22) and $a_{BT}$ in (25). Moreover, assuming that $\frac{R_r-R_s}{\delta}$ is sufficiently large,\(^\text{13}\) then such symmetric equilibrium is unique, this unique $a^e(F)$ is strictly decreasing in $F$, and $\lim_{F \to \delta} a^e(F) = a^*$.

A direct comparison of members’ individual strategic investment choice $a^e$ given in Eq. (31) and the socially optimal outcome $a^*$ given in Eq. (22) reveals the economic mechanism. Since the refund from the CCP is smaller than the default shortfall ($\phi_N(a;F) < \delta$), members strategically take excessive risk. In Figure 4a, we compare $a^e(F)$ and $a^*$; in Figure 4b, we plot $\frac{da^e}{dF}$ and illustrate the decreasing pattern of $a^e(F)$.

\(^{13}\)This is a technical condition that rules out the multiplicity of equilibria. Violation of this condition would result in equilibria that do not have the expected monotonicity properties (e.g., a higher default fund may result in greater risk-taking).
Figure 4. Members’ risk-taking and default fund. This figure demonstrates the relationship between member’s risk-taking and the default fund requirement. In Subfigure 4a, the blue dotted line plots the symmetric strategic response $a^e$ as a function of $F$ with $a^e$ strictly decreasing in $F$. The black dashed line plots the socially optimal risk-taking $a^* = 0.05$. Subfigure 4b plots the derivative of $a^e$ with respect to $F$. Model parameters: $\gamma = \frac{5}{3}, R_r = 19, R_s = 9, \beta = 0.03, N = 20$, and $\delta = 1$.

Optimal Cover Rule We next show that a default fund higher than the “Cover II” rule is needed to regulate members’ risk-taking. As in Section 3.2, the regulator chooses the size of the default fund to maximize the total value of all members and the CCP, anticipating members’ risk-taking strategies characterized in Proposition 5. We refer to the equilibrium choices of the regulator as the constrained optimal default fund.

Proposition 6 The regulator’s objective function is given by

$$\sum_i V_i(a^e(F); F) - E[(N_d\delta - NF)^+] - N(1+\beta)F = N\left((1 + f)E\left[\hat{R}(a^e)\right] - \beta F - \delta\right) := NW(F)$$

(33)
and the equilibrium size of default fund set by the regulator is

\[
F^e = \arg \max_{F \in [\frac{\delta}{N}, \delta]} \text{NW}(F) = \arg \max_{F \in [\frac{\delta}{N}, \delta]} \left(1 + f\right) \mathbb{E} \left[\tilde{R}(a^*)\right] - \beta F - \delta .
\]  

(34)

\(F^e\) is either the “Cover II” level, \(\frac{\delta}{N}\), or an interior solution obtained from taking the first-order condition of (34).

Figure 5. The optimal size of the default fund. This figure illustrates the optimal size of the default fund. Subfigure 5a plots the regulator’s objective function. Subfigure 5b plots the marginal benefit of increasing the size of the default fund \((1 + f) \frac{d\mathbb{E}[\tilde{R}(a^*)]}{dF}\) (the blue-dotted curve) and the marginal cost \(\beta\) (the horizontal black dashed line). Model parameters: \(\gamma = \frac{5}{3}, R_r = 19, R_s = 9, \beta = 0.03, N = 20,\) and \(\delta = 1\). The optimal default fund is \(F^e \approx 0.38\).

Proposition 6 delineates the trade-off for the regulator to size the equilibrium (constrained optimal) default fund. The regulator balances the marginal benefit of increasing the default fund against the marginal cost of funding. The optimal default fund may be higher than the “Cover II” rule. Figure 5 presents a scenario in which the “Cover II” rule is not socially optimal. As the default fund increases, the objective function of the regulator increases and peaks at \(F^e \approx 0.38\). This value corresponds to a Cover 7.6 rule for a 20-member clearing arrangement.

Analogous to the binary case, we can show that the ratio between the optimal default fund
\( F^e(N) \) and default shortfall \( \delta \) (or equivalently, the optimal cover ratio, \( X(N)/N \)) converges to a constant as the number of members \( N \) grows large. This limit falls in \((0,1)\) when the marginal opportunity cost of default funds \( \beta \) is not too high.

**Proposition 7** In the limiting case of a large CCP network—as \( N \to \infty \)—the unique symmetric risk-taking strategy of members as a function of the default fund requirement admits the following representation

\[
a^e(F) \to \begin{cases} 
  a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}, & \text{if } \frac{F}{\delta} \leq a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}, \\
  \frac{1 + a^*}{2} - \sqrt{\left(\frac{1 - a^*}{2}\right)^2 - \frac{\delta - F}{2(1 + f)(R_r - R_s)}}, & \text{otherwise.}
\end{cases}
\]

(35)

In the first case, \( a^e(F) \) converges to a value independent of \( F \), and thus increasing default fund is not effective in regulating members’ risk-taking. Because maintaining default fund collateralized position is costly, a regulator would set a zero default fund. The intuition is as follows. When the ratio \( \frac{F}{\delta} \) is lower than the threshold, the total amount of default funds would be exhausted in absorbing default losses, so members expect to get zero refund from the CCP. This corresponds to setting \( \phi_N(a^e, F) = 0 \) in Eq. (31). Consequently, each member chooses \( a^* + \frac{\delta}{2(1 + f)(R_r - R_s)} \), regardless of the default fund.

In the second case, the total default funds will not be exhausted. Each surviving member expects to receive a positive refund from the CCP \( (\phi_N(a^e, F) > 0) \), which induces the members to reduce risk-taking. In this case, \( a^e(F) \) is strictly decreasing with \( F \), thus setting a positive \( F \) is potentially optimal; denote by \( F(a) = \delta - 2(1 + f)(R_r - R_s)(a - a^*)(1 - a) \) the inverse function of \( a^e(F) \). We next derive the constrained optimal risk-taking and optimal size of the default fund.

**Proposition 8** The constrained optimal risk-taking is \( a^*_0 = \frac{(1 + \beta)a^* + \beta}{1 + 2\beta} \). The corresponding default fund requirement is \( F(a^*_0) \). If a positive default fund induces the members to take lower risk \( (a^*_0 < a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}) \) and if the benefit of stepping up the default fund exceeds the cost \( (W(F(a^*_0)) > W(0)) \), then the equilibrium risk-taking is \( a^*_0 \) and the default fund is
positive at $F(a^*_0)$; otherwise, the default fund requirement is zero. Formally,

$$
\begin{align*}
\begin{cases}
    a^e = a^*_0 = \frac{(1 + \beta)a^* + \beta}{1 + 2\beta}, & F^e = \delta - \frac{\beta(1 + \beta)(1 + f)R^2_r}{2(1 + 2\beta)^2(1 + 2\beta)(1 + f)(R_r - R_s)}, \\
    a^e = a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}, & F^e = 0
\end{cases}
\end{align*}
$$

if $\frac{(1 + f)R_r\beta}{(1 + 2\beta)\delta} < 1$ and $W(F^e) > W(0)$

(36)

Proposition 8 presents a simple Cover-X rule. As long as the benefit of stepping up the default fund dominates the funding cost, the optimal Cover-X rule, $X(N)/N = F^e/\delta$ converges to a constant, i.e.,

$$
X(N)/N \to 1 - \frac{\beta(1 + \beta)(1 + f)R^2_r}{2(1 + 2\beta)^2(1 + 2\beta)(R_r - R_s)\delta}.
$$

If the marginal opportunity cost $\beta$ is sufficiently low, Proposition 8 indicates that the constrained socially optimal risk choice $a^*_0$ is a weighted average of $a^*$ and 1, which converges to $a^*$ as $\beta$ reduces to 0. Because of the vanishing opportunity cost of default fund collateral, the socially optimal level $F^e$ converges to $\delta$.

Similar to the baseline model, as the buyers become more risk averse (hence are willing to pay a higher premium $f$), inducing the sellers to select the optimal risk choice $a^*_0$ requires a lower default fund. A leverage effect arises: Because the premium $f$ is used by the sellers to scale up the size of their investments, thereby improving the reward per unit of risk-taking, the incentive for excessive risk-taking declines.

An essential enhancement of this extension from the baseline model is that changes in the opportunity cost of default funds affect the optimal default fund. An increase in the cost of financing the default fund position raises the desired level of each member’s risk-taking, which in turn implies a lower default fund requirement. This extension thus suggests that the optimal sizing of the default fund is sensitive to variations in funding cost.

5 Conclusion and Policy Implications

The determination of default fund contribution for central clearing has been the subject of extensive regulatory debate. Current requirements prescribe that default fund contributions
should be sufficient to absorb the losses from the default by the two largest clearing members. There is, however, no analysis of the optimality of this rule, or of alternative rules that improve welfare. Our paper fills this critical gap and introduces a parsimonious model to study the main economic incentives behind the determination of default fund requirements. We propose a model of central clearing to study the equilibrium and socially optimal choice of risk-taking by protection sellers of CDS contracts. We show that the CCP can mitigate the resulting inefficiency through a suitable sizing of the default fund, which balances the regulation of ex-ante risk-taking of members with the opportunity cost of default funds.

Based on our analysis, if the clearinghouse consists of a sufficiently large number of protection sellers, the total default fund contributions should cover the shortfalls of a fraction, rather than a fixed number, of members. Our result indicates that the currently imposed “Cover II” standards are optimal only if the marginal opportunity costs of default funds are sufficiently high. If these costs are low, however, then the Cover II rule would not be sufficiently stringent, in that clearing members would engage in excessive risk-taking leading to a socially suboptimal outcome.

Our findings produce novel implications for regulating clearinghouses. Our cover rule, covering the payment shortfalls generated by the default of a proportion of clearing members, is robust to the number of clearing members. The optimal proportion depends, in an explicit way, on the relation between the premium earned by the member who undertakes high-risk projects and the payment shortfall incurred at default. Because of its simplicity, the proposed rule may also serve as a benchmark against more complex rules for default fund determination based on simulated scenarios for stress testing.

Notably, our analysis suggests that the size of default funds is negatively related to the marginal opportunity cost of default funds. Large amounts of capital may be tied up with the clearinghouse in the current low interest-rate environment. Thus, it may be socially desirable to limit the charge of default funds when marginal funding costs are low. In contrast, our analysis also suggests that as the cost of funding capital rises, the clearinghouse has a higher incentive to reduce default funds until achieving the “Cover II” standard. This may have systemic risk implications if a period of elevated funding costs is followed by high market stress, given that the clearinghouse may be vulnerable, i.e., not sufficiently capitalized under these market scenarios.
References


Proof of Propositions and Technical Lemmas

Proof of Proposition 1

It follows from Assumption 4 and the expression of the premium \( f \) given in Eq. (6) that
\[
\mu_r f > (q_r + \beta)\delta. \tag{A1}
\]

Consider the situation in which all members except for \( i \) choose the riskier investment. Using (7), (8), and (11), we obtain that the expected profit of member \( i \) when he trades with the CCP satisfies
\[
V_{i}^{CCP}(a_{-i}; F) \geq -(1 + \beta)F + (1 - q_r)(\psi(0; F) - \delta) + (1 + f)\mu_r. \tag{A2}
\]

The above inequality follows from the fact that the right-hand side is the value of member \( i \) when he chooses the riskier investment, whereas the left-hand side is the optimal value obtained from solving the optimization problem in (7). Using Lemma A1 below, we obtain that the right-hand side of Condition (A2) takes the lowest value when \( F \) is maximum, i.e., \( F = \delta \). In addition, it follows directly from the definition of \( \psi(g; F) \) given in Eq. (11) that \( \psi(g; \delta) = \delta \). Evaluating the right-hand side of Condition (A2) at \( F = \delta \), we obtain that
\[
V_{i}^{CCP}(a_{-i}; F) \geq -(1 + \beta)\delta + (1 + f)\mu_r,
\]
for any \( F \). Hence, the expected profit of member \( i \) when trading through CCP satisfies
\[
V_{i}^{CCP}(a_{-i}; F) \geq \mu_r f - (1 + \beta)\delta + \mu_r > (\beta + q_r)\delta - (1 + \beta)\delta + \mu_r = V_{BT},
\]
where the last inequality above follows directly from Condition (A1), and the last equality follows from Eq. (3).

Lemma A1 The function \( h(F) := -(1 + \beta)F + (1 - q_r)\psi(0; F) \) is strictly decreasing in \( F \) over \([0, \delta]\).

Proof. From the definition of \( \psi(g; F) \) in Eq. (11),
\[
\psi(0; F) = \mathbb{E} \left[ \left( F - \frac{N_d}{N-N_d}(\delta - F) \right)^+ \right| (N-N_d) \geq 1 \] = \mathbb{E} \left[ \left( \frac{N}{N-N_d}(F - \delta) + \delta \right)^+ \right| (N-N_d) \geq 1 \right].
\]

Since \( \left( \frac{N}{N-N_d}(F - \delta) + \delta \right)^+ \) is almost surely convex in \( F \), we can conclude that \( \psi(0; F) \) is convex as well. As a result, it follows that \( h(F) \) is convex in the interval \([0, \delta]\). To complete the proof, it suffices to show that \( h'(\delta-) < 0 \).
To this end, we focus on the sub-interval $F \in \left[\frac{N-1}{N}\delta, \delta]\right]$. Then it follows that

$$\psi(0; F) = \delta + (F - \delta) \mathbb{E}\left[\frac{N}{N - N_d} \middle| N - N_d \geq 1\right].$$

Because $N - N_d$ follows a binomial distribution with parameter $(N, 1 - q_r)$, we know that $N - N_d - 1| (N - N_d \geq 1)$ follows a binomial distribution with parameter $(N - 1, 1 - q_r)$. Hence,

$$\psi(0; F) = \delta + (F - \delta) \mathbb{E}\left[\frac{N}{N - N_d} \middle| N - N_d \geq 1\right]$$

$$= \delta + (F - \delta) \sum_{k=0}^{N-1} \frac{N}{1+k} \binom{N-1}{k} (1 - q_r)^k q_r^{N-1-k}$$

$$= \delta + (F - \delta) \frac{1}{1 - q_r} \sum_{k=1}^{N-1} \binom{N}{k} (1 - q_r)^{k+1} q_r^{N-1-k}$$

$$= \delta + (F - \delta) \frac{1}{1 - q_r} \sum_{m=1}^{N} \binom{N}{m} (1 - q_r)^m q_r^{N-m} = \delta + (F - \delta) \frac{1}{1 - q_r}.$$

Taking derivatives with respect to $F$, we obtain that

$$h'(\delta -) = -(1 + \beta) + (1 - q_r) \frac{d}{dF} \psi(0; F) = -(1 + \beta) + (1 - q_r) \frac{1 - q_r^N}{1 - q_r} = -(\beta + q_r^N) < 0.$$

Therefore $h(F)$ is strictly decreasing in $F$ over $[0, \delta]$. ■

**Proof of Proposition 2**

We first present some lemmas to fix notations and technical results.

**Lemma A2** Suppose member $i$ survives, and $g$ of the remaining $N - 1$ members choose the safer investment. Then, for any $F \in \left[\frac{2\delta}{N}, \delta\right]$, we have that

$$\psi(g; F) := \sum_{k=N-1-[\frac{N\delta}{F}]}^{N-1} f_g(k) \left( F - \frac{N - (k + 1)}{k + 1} \left( \delta - F \right) \right).$$

(A3)

Above, $[\cdot]$ denotes the floor function (giving the greatest integer less than or equal to the argument),
and for each \( k \),

\[
f_g(k) := \sum_{m=0}^{k} \binom{g}{m} (1-q_s)^m q_s^{g-m} \times \binom{N-1-g}{k-m} (1-q_r)^{k-m} q_r^{N-1-g-(k-m)}
\]

is a positive constant.

**Proof.** Suppose that the default fund \( F \) is such that

\[
\frac{l\delta}{N} \leq F < \frac{(l+1)\delta}{N}, \text{ or equivalently } 1 - \frac{l+1}{N} < 1 - \frac{F}{\delta} \leq 1 - \frac{l}{N},
\]

(A4)

for some integer \( l = 2, 3, \ldots, N-1 \). Then, when member \( i \) survives, the contribution of member \( i \) to absorb losses arising when other members default is given by

\[
F - \psi(g; F) = \min \left( F, \frac{N_{d}}{N-N_{d}} (\delta - F) \right) = F \min \left( 1, \left( \frac{N}{N-N_{d}} - 1 \right) \left( \frac{\delta}{F} - 1 \right) \right)
\]

\[
= (\delta - F) \left( \frac{N}{N-N_{d}} - 1 \right) 1_{N-N_{d} \geq (1-\frac{\delta}{\epsilon})N} + F 1_{N-N_{d} < (1-\frac{\delta}{\epsilon})N} \tag{A5}
\]

If \( F \) satisfies Condition (A4), we have

\[
N - (l+1) < \left( 1 - \frac{F}{\delta} \right) N \leq N - l.
\]

Thus \( N-N_{d} \geq (1-\frac{\delta}{\epsilon})N \) if and only if \( N-N_{d} \geq N - l \). Hence, if the remaining \( N-1 \) members have fewer than or equal to \( l \) defaults, member \( i \) will pay less than \( F \). (See the two cases determined by the indicator in Eq. (A5).) But, if the number of defaults is \( l \) or larger and \( F = \frac{\delta}{N} \), member \( i \)'s default fund will be exhausted completely.

Suppose all members except for \( i \) choose the safer investment. If member \( i \) survives, his expected contribution to losses caused by other members is

\[
(\delta - F) \mathbb{E} \left[ \left( \frac{N}{N-N_{d}} - 1 \right) 1_{N_{d} \leq l} | \text{member } i \text{ survives} \right] + F \text{ Pr}(N_d > l | \text{member } i \text{ survives})
\]

\[
= (\delta - F) \sum_{k=N-(l+1)}^{N-1} \binom{N-1}{k} \frac{N-1-k}{k+1} (1-q_s)^k q_s^{N-1-k} + F \sum_{k=0}^{N-(l+2)} \binom{N-1}{k} (1-q_s)^k q_s^{N-1-k}
\]

\[
= (\delta - F) \sum_{k=N-(l+1)}^{N-2} \binom{N-1}{k+1} (1-q_s)^k q_s^{N-1-k} + F \sum_{k=0}^{N-(l+2)} \binom{N-1}{k} (1-q_s)^k q_s^{N-1-k}. \tag{A6}
\]
The index \( k \) in the above summation keeps track of the number of surviving members other than member \( i \). If at most \( l \) members default and we know that \( i \) survives, then the number of remaining surviving members (other than \( i \)) can be any number between \( N - 1 - l \) and \( N - 1 \). If instead more than \( l \) members default and we know that \( i \) survives, then the number of remaining surviving members (other than \( i \)) can be any number between \( 0 \) and \( N - 1 \).

Likewise, if all members except member \( i \) choose the riskier investment, then if member \( i \) survives, his expected contribution is

\[
(\delta - F)\mathbb{E}\left[\left(\frac{N}{N - N_d} - 1\right)1_{N_d \leq l}\mid \text{member } i \text{ survives}\right] + F \Pr(N_d > l\mid \text{member } i \text{ survives})
\]

\[
= (\delta - F) \sum_{k=N-(l+1)}^{N-2} \binom{N-1}{k+1} (1 - q_r)^k q_r^{N-1-k} + F \sum_{k=0}^{N-(l+2)} \binom{N-1}{k} (1 - q_r)^k q_r^{N-1-k}. \quad (A7)
\]

In general, if there are \( g \) members (for \( g = 0, 1, \ldots, N - 1 \)) among the remaining \( N - 1 \) choosing the safer investment, then the number of surviving ones among these \( N - 1 \) members, \( N - 1 - N_d \), is the sum of the number of surviving members choosing the safer investment and that of the surviving members choosing the riskier investment. Specifically, the probability that there are \( k \) surviving members is given by

\[
f_g(k) := \sum_{m=0}^{k} \binom{g}{m} (1 - q_s)^m q_s^{g-m} \times \binom{N - 1 - g}{k - m} (1 - q_r)^{k-m} q_r^{N-1-g-(k-m)}. \]

It follows that, if member \( i \) survives, his expected contribution to the losses of other members is

\[
(\delta - F)\mathbb{E}\left[\left(\frac{N}{N - N_d} - 1\right)1_{N_d \leq l}\mid \text{member } i \text{ survives}\right] + F \cdot \Pr(N_d > l\mid \text{member } i \text{ survives})
\]

\[
= \sum_{k=N-(l+1)}^{N-1} f_g(k) \frac{N - (k + 1)}{1 + k} (\delta - F) + F \sum_{k=0}^{N-(l+2)} f_g(k)
\]

\[
= \sum_{k=N-(l+1)}^{N-1} f_g(k) \left(\frac{N - (k + 1)}{k + 1}(\delta - F) - F\right) + F, \quad (A8)
\]

where the last equality comes from the law of total probabilities \( \sum_{k=0}^{N-1} f_g(k) = 1. \)

**Lemma A3** For any \( F \in \left[\frac{\delta}{N}, \delta\right] \), the function \( \psi(g; F) \) is strictly increasing in \( g \) (the number of members that choose the safer investment), i.e.,

\[
0 < \psi(0; F) < \psi(1; F) < \ldots < \psi(N - 1; F) < F < \delta.
\]

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Proof. Using Eq. (11), we have that the default fund contribution of a surviving member $i$ is

$$F - \psi(g; F) = \mathbb{E}\left[\min\left(F, \frac{N_d - N_d(\delta - F)}{N}\right)\right].$$

(A9)

We will show that the right-hand side of Eq. (A9) is strictly decreasing in $g$ for $g = 0, 1, 2, \ldots, N - 1$, where $g$ is the number of members other than $i$, who choose the safer investment. We know from Lemma A2 that the expression $\psi(g; F)$ only depends on the default probabilities $q_r$ and $q_s$. We next consider a probability model for the default events that is convenient for our analysis (the results would remain the same under any alternative specification which yields default probabilities $q_r$ and $q_s$ upon the choice of a riskier and safer investment, respectively). Concretely, for each member $j$, we assume the existence of a random variable $\epsilon_j$, uniformly distributed over the interval $(0, 1)$, such that conditional on member $j$ choosing the safer investment, he defaults if and only if $\epsilon_j < q_s$, and conditional on $j$ choosing the riskier investment, he defaults if and only if $\epsilon_j < q_r$. We assume that $\epsilon_1, \epsilon_2, \ldots, \epsilon_N$ are independent. Because the random variables $\epsilon_i$’s are uniform, this means that the default probability of a member $j$ conditional on a safer investment is $\mathbb{P}(\epsilon_j < q_s) = q_s$, and the default probability of a member $j$ conditional on a riskier investment is $\mathbb{P}(\epsilon_j < q_r) = q_r$. Moreover, these default events are independent because the random variable $\epsilon_i$’s are independent. Hence, this probabilistic model is consistent with specified conditional default probabilities. Because the event $\{\epsilon_j < q_s\}$ is contained in the event $\{\epsilon_j < q_r\}$, the function $g \mapsto N_d$ is non-increasing in $g$, i.e., in the number of members other than $i$ who choose the safer investment. Moreover, if a member $j$ were to switch from the safer investment to the riskier investment, the probability of observing an additional default is $q_r - q_s$, i.e., $\mathbb{P}(N_d(g - 1) - N_d(g) = 1) = q_r - q_s$. This means that $g \mapsto N_d$ is non-increasing in $g$, and strictly decreasing in $g$ with a positive probability.

Notice that as $N_d$ ranges from $N - 1$ to 0, the quantity $\frac{N_d}{N - N_d}(\delta - F)$ ranges from $(N - 1)(\delta - F)$ to 0. Hence the random variable $\min\left(F, \frac{N_d}{N - N_d}(\delta - F)\right)$ is decreasing in $g$ and strictly decreasing with a positive probability. As a result, the expected value of this random variable is strictly decreasing in $g$.

Lastly, we conclude from Eq. (11) that $\psi(N - 1; F) < F$. This is because, with a positive probability, at least one member defaults and the expected contribution of a surviving member $i$ towards other members’ default losses is positive. Moreover, $\psi(0; F) > 0$, i.e., the expected refund of member $i$ is positive; to understand this, note that with a positive probability, no members default, and member $i$ receives a full refund from the CCP.

Lemma A4 For any fixed $g = 0, 1, \ldots, N - 1$, the function $\psi(g; F)$ is piecewise linear and strictly increasing in $F$ in the interval $[\frac{6s}{N}, \delta]$. In particular, $\psi(g; \delta) = \delta$.

Proof. First, from Lemma A2, Eq. (A3), $\psi(g; F)$ is linear and strictly increasing in $F$, within any subinterval $F \in (\frac{l\delta}{N}, \frac{(l+1)\delta}{N})$ with $l = 2, 3, \ldots, N - 1$. Hence, we conclude that $\psi(g; F)$ is piecewise
linear. Second, the nonnegative random variable \( \min(F, \frac{N_d}{N} (\delta - F)) \) is almost surely continuous in \( F \) and bounded by \( \delta \). By the dominated convergence theorem, the expected value of this random variable, i.e., \( \psi(g; F) \), is continuous. Therefore, the function \( \psi(g; F) \) is strictly increasing and piecewise linear in the interval \( F \in \left[ \frac{2}{N}\delta, \delta \right] \). It follows directly from Eq. (11) that \( \psi(g; \delta) = \delta \).

**Proof of Proposition 2**  For a given default fund \( F \), we characterize the conditions under which member \( i \) chooses the riskier investment, given all possible risk profiles by the other members. We will show that “all riskier investment” and “all safer investment” strategies are the only possible equilibria.

To understand the risk choice of member \( i \), let us consider different scenarios of other members’ risk profiles. Suppose \( g \) of the remaining members choose the safer investment and the remaining \( N - 1 - g \) members choose the riskier investment, where \( g \in \{0, \ldots, N-1\} \). If member \( i \) chooses the safer investment, his expected profit at \( t = 1 \) is

\[
\mathbb{E}\left[ \mathbb{1}_{i \text{ survives}} ((1 + f)R_s - \delta + \psi(g; F)) \bigg| i \text{ survives, } g \text{ members other than } i \text{ choose safer} \right] \\
= (1 + f)\mu_s + (1 - q_s)(\psi(g; F) - \delta),
\]

where we have used the definition of \( \psi(g; F) \) given in Eq. (11). If member \( i \) chooses instead the riskier investment, his expected profit at \( t = 1 \) is

\[
\mathbb{E}\left[ \mathbb{1}_{i \text{ survives}} ((1 + f)R_r - \delta + \psi(g; F)) \bigg| i \text{ survives, } g \text{ members other than } i \text{ choose safer} \right] \\
= (1 + f)\mu_r + (1 - q_r)(\psi(g; F) - \delta).
\]

Hence, if among the remaining members \( g \) of them choose the safer and \( N - 1 - g \) choose the riskier investment, member \( i \) chooses the riskier (safer, resp.) investment if and only if

\[
(1 + f)\frac{\mu_s - \mu_r}{q_r - q_s} < (>, \text{resp.}) \delta - \psi(g; F).
\]

(A10)

In cases when the left- and right-hand sides of the inequality (A10) coincide, member \( i \) is indifferent between choosing the riskier or safer investment. Using inequality (A10) and recalling that \( \psi(g; F) \) is increasing in \( g \) from Lemma A3, we can distinguish the following cases:

1. \( (1 + f)\frac{\mu_s - \mu_r}{q_r - q_s} < \delta - \psi(N - 1; F) \). Then every member will choose the riskier investment regardless of the other members’ choices. Hence, the “all riskier investment” strategy is the unique equilibrium among members.

2. \( (1 + f)\frac{\mu_s - \mu_r}{q_r - q_s} > \delta - \psi(0; F) \). Then every member will choose the safer investment regardless of
other members’ choices. Hence, the “all safer investment” strategy is the unique equilibrium among members.

3. \( \delta - \psi(0; F) \geq (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} \geq \delta - \psi(N - 1; F) \). Then there are two equilibria: the “all riskier investment” strategy and the “all safer investment” strategy. By Lemma A4, \( \psi(g; F) \) is increasing in \( F \). This implies the existence of thresholds \( \hat{F} \) and \( \bar{F} \), defined respectively as the maximum value of \( F \) at which all members choose the riskier investment, i.e., \( (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} = \delta - \psi(N - 1; \hat{F}) \), and the minimum value of \( F \) at which all members choose the safer investment, i.e., \( (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} = \delta - \psi(0; \bar{F}) \). We need to show that there cannot exist any other equilibria. We argue by contradiction. Suppose there exists an equilibrium risk profile consisting of \( g \) members who choose the safer investment and \((N - g) \) members who choose the riskier investment, for some \( g = 1, 2, \ldots, N - 1 \). Then, any member choosing the riskier investment faces other \( g \) members choosing the safer investment and \((N - g - 1) \) members choosing the riskier investment. It follows from the inequality (A10) that in order for the risky member to have no incentive to unilaterally deviate to the safer investment, it must hold that

\[
(1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} \leq \delta - \psi(g; F). \tag{A11}
\]

Using analogous reasoning, any member choosing the safer investment faces other \( g - 1 \) members who choose the safer investment and \( N - g \) members who choose the riskier investment. Using the inequality (A10), for this member to not deviate to a riskier investment it must hold that

\[
(1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} \geq \delta - \psi(g - 1; F). \tag{A12}
\]

But (A11) and (A12) cannot hold simultaneously because \( \psi(g - 1; F) < \psi(g; F) \) (see Lemma A3). Hence, we obtain a contradiction and thus complete the proof for the first statement of the proposition.

The second statement of the proposition (that \( \psi(\cdot; \cdot) \) is a strictly increasing function in both of its arguments) follows directly from Lemma A3 and Lemma A4.

Finally, we prove the third statement of the proposition, i.e., that the “all safer” investment profile Pareto dominates the “all riskier” investment profile in the default fund region \( \delta - \psi(0; F) \geq (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} \geq \delta - \psi(N - 1; F) \), where both the “all safer” and the “all riskier” strategy profiles are equilibria. Recall that member \( i \)'s expected profits at \( t = 1 \) under the “all safer” investment profile is

\[
V_i(s; F) = (1 + f) \mu_s - (1 - q_s) (\delta - \psi(N - 1; F)).
\]

Likewise, his expected profits at \( t = 1 \) under the “all riskier” investment profile is

\[
V_i(r; F) = (1 + f) \mu_r - (1 - q_r) (\delta - \psi(0; F)).
\]
Taking the difference between these two terms, we obtain

\[ V_i(s; F) - V_i(r; F) = (1 + f)(\mu_s - \mu_r) + (1 - q_s)\psi(N - 1; F) - (1 - q_r)\psi(0; F) - (q_r - q_s)\delta. \]

Because \((1 + f)(\mu_s - \mu_r) \geq (q_r - q_s)(\delta - \psi(N - 1; F))\), the above expression is bounded from below by \((1 - q_r)(\psi(N - 1; F) - \psi(0; F)) > 0\), where the positivity follows from Lemma A3. Hence, among the possible equilibria, the “all safer” investment profile Pareto dominates the “all riskier” investment profile.

**Proof of Proposition 4**

We first derive the limit of \(\psi(N - 1; F)\) as \(N \to \infty\). From Eq. (14) which pins down \(\hat{F}\), the limit of \(\psi(N - 1; F)\) directly informs the limit of \(\hat{F}\). From Eq. (A3), we have

\[
\psi(N - 1; F) = \sum_{k=N-1}^{N-1} N(N-1\ldots k)(1-q_s)^k q_s^{N-1-k} \left( \delta - \frac{N(\delta - F)}{k+1} \right),
\]

where \(l = \lfloor \frac{NF}{\delta} \rfloor\), and

\[
v_l^{(N)}(a) = \sum_{k=N-1-l}^{N-1} N(N-1\ldots k)(1-a)^k a^{N-1-k}, \quad u_l^{(N)}(a) = \sum_{k=N-1}^{N} N(N-1\ldots k)(1-a)^k a^{N-1-k}.
\]

We next derive the expressions for \(v_l^{(N)}\) and \(u_l^{(N)}\). Suppose \(X\) follows a Binomial distribution with parameter \((N - 1, a)\) and \(Y\) follows a Binomial distribution with parameter \((N, a)\), then

\[
v_l^{(N)}(a) = P(X \leq l) = P\left( \sqrt{N - 1} \left( \frac{X}{N-1} - a \right) \leq \sqrt{N - 1} \left( \frac{l}{N-1} - a \right) \right),
\]

\[
u_l^{(N)}(a) = \frac{P(Y \leq l)}{1-a} = P\left( \sqrt{N} \left( \frac{Y}{N} - a \right) \leq \sqrt{N} \left( \frac{l}{N} - a \right) \right).
\]

By the central limit theorem, both \(\sqrt{N - 1} \left( \frac{X}{N-1} - a \right)\) and \(\sqrt{N} \left( \frac{Y}{N} - a \right)\) converge in distribution to a Gaussian distribution with mean 0 and variance \(a(1-a)\). On the other hand, using the inequality...
\[ \frac{NF - \delta}{\delta} < l = \left\lfloor \frac{NF}{\delta} \right\rfloor \leq \frac{NF}{\delta} \] \text{ we can conclude that}

\[
\lim_{N \to \infty} \sqrt{N - 1} \left( \frac{l}{N - 1} - a \right) = \lim_{N \to \infty} \sqrt{N} \left( \frac{l}{N} - a \right) = \begin{cases} 
\infty, & \text{if } \frac{F}{\delta} > a, \\
0, & \text{if } \frac{F}{\delta} = a, \\
-\infty, & \text{if } \frac{F}{\delta} < a.
\end{cases}
\]

Using the fact that a zero-mean Gaussian distribution is smaller than \( \infty \) with probability one, and smaller than 0 with probability 1/2, we obtain that

\[
\lim_{N \to \infty} v_l^{(N)}(a) = 1_{\{F > a\}} + \frac{1}{2} 1_{\{F = a\}}, \quad \lim_{N \to \infty} u_l^{(N)}(a) = \frac{1_{\{F > a\}} + \frac{1}{2} 1_{\{F = a\}}}{1 - a}.
\]

Based on the above limiting results, we obtain that

\[
\lim_{N \to \infty} \psi(N - 1; F) = \delta \cdot 1_{\{F > q_s\}} \left( 1 - \frac{1 - \frac{F}{\delta}}{1 - q_s} \right), \quad (A17)
\]

where we note that \( \delta \cdot 1_{\{F > q_s\}} \frac{1}{2} (1 - \frac{1 - \frac{F}{\delta}}{1 - q_s}) = 0 \). Using (14), the candidate optimal threshold \( \hat{F} \) at which members move away from the “all riskier” equilibrium is defined as the unique root to the equation \( \psi(N - 1; \hat{F}) = \delta - (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} \); from the above result in (A17), the limit of \( \hat{F} \) converges to the solution of the equation:

\[
\delta \cdot 1_{\{F > q_s\}} \left( 1 - \frac{1 - \frac{F}{\delta}}{1 - q_s} \right) = \delta - (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s}.
\]

Hence, under the condition that \( (1 + f) \frac{\mu_s - \mu_r}{q_r - q_s} \leq \delta \), we obtain that

\[
\lim_{N \to \infty} \hat{F} = \delta - \frac{(1 + f)(\mu_s - \mu_r)(1 - q_s)}{q_r - q_s}.
\]

Hence, by (17) we obtain that as \( N \to \infty \)

\[
\frac{x^e(N)}{N} = \frac{F^e(N)}{\delta} \rightarrow \begin{cases} 
\hat{F}, & \text{if } \frac{(\mu_s - \mu_r)(1 + f)}{\beta \delta} > 1 - \frac{(1 + f)(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)\delta} \\
0, & \text{otherwise}.
\end{cases} \quad (A18)
\]

To obtain comparative statics on the limit of the optimal cover ratio \( \frac{\hat{F}}{\delta} \), we take derivatives:
• with respect to $\gamma$:
\[
\frac{\partial (\hat{F}/\delta)}{\partial \gamma} = -q_r(1 - q_r)(\mu_s - \mu_r)(1 - q_s) \delta < 0;
\]

• with respect to $\mu_s - \mu_r$ (while fixing $q_r, q_s$):
\[
\frac{\partial (\hat{F}/\delta)}{\partial (\mu_s - \mu_r)} = -\frac{(1 + f)(1 - q_s)}{(q_r - q_s)\delta} < 0;
\]

• with respect to $q_r$ (while fixing $\mu_s - \mu_r$):
\[
\frac{\partial (\hat{F}/\delta)}{\partial q_r} = \frac{(1 + f)(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)^2\delta} - \frac{(\delta + \gamma(1 - 2q_r)\delta^2)(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)\delta}
\]
\[
= \frac{(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)^2\delta} \left(1 + q_r\delta + \gamma q_r(1 - q_r)\delta^2 - (q_r - q_s)(\delta + \gamma(1 - 2q_r)\delta^2)\right)
\]
\[
> \frac{(\mu_s - \mu_r)(1 - q_s)}{(q_r - q_s)^2\delta} \left(1 + q_s\delta + \gamma(q_r^2 + q_s - 2q_rq_s)\delta^2\right) > 0;
\]

• with respect to $\delta$:
\[
\frac{\partial (\hat{F}/\delta)}{\partial \delta} = \frac{(\mu_s - \mu_r)(1 - q_s)}{q_r - q_s} \left(\frac{1}{\delta^2} - \gamma q_r(1 - q_r)\right).
\]

Notice that the above expression can be either positive or negative.

This proves Proposition 4.

**Proof of Proposition 5**

We first state and prove additional technical properties of the members’ equilibrium response $a^e$ as a function of $F$:

• $a^e(F)$ is strictly decreasing and $a^e(\delta^−) := \lim_{F \uparrow \delta} a^e(F) = a^*$, for $F \neq \frac{l\delta}{N}, l = 2, 3, \ldots, N - 1$;

• $a^e$ is an infinitely differentiable function of $F$ and $\frac{da^e}{dF} < 0$;

• for $F = \frac{l\delta}{N}$, $l = 3, 4, \ldots, N - 1$, $\frac{da^e}{dF}(F+) - \frac{da^e}{dF}(F−) < 0$, where $\frac{da^e}{dF}(F+)$ and $\frac{da^e}{dF}(F−)$ are respectively the right and left derivatives of $a^e(F)$ at $F$. 

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Next we prove the proposition together with the above technical properties. Recall that a symmetric equilibrium is given by \( a^e_i = a^e \) for all \( i = 1, 2, \ldots, N \). From the member’s expected profit given by (29), we obtain that the equilibrium risk choice \( a^e \) satisfies a first-order condition given by

\[
H(a; F) = \frac{\delta - \phi_N(a; F)}{2(1 + f)(R_r - R_s)} + a^* - a = 0,
\]

where we recall that the expression of \( a^* \) is given in Eq. (22). We first show the root of the above equation exists in the interval \((a^*, a_{BT})\), and there is no root in the interval \([0, a^*] \) or \([a_{BT}, 1]\). To this end, we recall that \( 0 < \phi_N(a; F) < F < \delta \) (see Lemma A3), so for any \( a < a^* \),

\[
H(a; F) > \frac{\delta - \phi_N(a; F)}{2(1 + f)(R_r - R_s)} > 0.
\]  

(A19)

On the other hand, for any \( a \geq a_{BT} \),

\[
H(a; F) < \frac{\delta}{2(1 + f)(R_r - R_s)} + a^* - a < a_{BT} - a \leq 0,
\]  

(A20)

using Assumption 1-C and recalling the expression of \( a_{BT} \) given in (25). Hence, from (A19) and (A20) we conclude that, for any fixed \( F \in (\delta, \delta) \), there exists at least one \( a^e \) in \((a^*, a_{BT})\) such that \( H(a^e; F) = 0 \), and no root exists outside of the interval \((a^*, a_{BT})\).

To shed light into the uniqueness of \( a^e \), we show that when \( \frac{R_r - R_s}{\delta} \) is sufficiently large, \( H(a; F) \) is strictly decreasing in \( a \) over the interval \((a^*, a_{BT})\) for every \( F \). Because Assumption 1-C implies that \( a_{BT} = 1 - \frac{R_r - \delta}{2(R_r - R_s)} < 1 - \frac{R_r - \delta}{2R_r} = \frac{1}{2} + \frac{\delta}{2R_r} < \frac{3}{4} \), it suffices to show that \( H(a; F) \) is strictly decreasing in \( a \) over \([0, \frac{3}{4}]\) for any given \( F \). To this end, we first observe that Lemma A3 implies \( \phi_N(q_r; F) = \psi(0; F) < \psi(N - 1; F) = \phi_N(q_r; F) \) for \( 0 < q_r < q_r < 1 \), so that \( \phi_N(a; F) \) is strictly decreasing in \( a \). Therefore, the monotonicity of \( H(a; F) \) in \( a \) is fully determined by the strictly increasing function \(-\frac{\phi_N(a; F)}{2(1 + f)(R_r - R_s)}\) and the strictly decreasing function \(-a\). Using Lemma A5, we deduce that the derivative of \(-\frac{\phi_N(a; F)}{2(1 + f)(R_r - R_s)}\) with respect to \( a \) is bounded from above by \(-\frac{\phi_N(a; F)}{2(1 + f)(R_r - R_s)}\) for some positive constant \( M \) that does not depend on \( \delta, R_r \) or \( R_s \). Thus, if \( \frac{R_r - R_s}{\delta} \) is sufficiently large, then \( \frac{R_r - R_s}{\delta} > \frac{M}{2(1 + f)} \) and the partial derivative of \( H(a, F) \) with respect to \( a \) is strictly smaller than 0 for all \( a \in [0, \frac{3}{4}] \), i.e. \( H(a; F) \) is strictly decreasing. This guarantees the uniqueness of \( a^e \). Recall from Eq. (30), we have \( \phi_N(a^e; \delta) = \delta \). Because \( H(a^e; F) = 0 \) when \( \phi_N(a^e; F) = \delta \), it follows immediately that \( a^e(\delta) = a^* \).

Next, we prove the monotonicity of \( a^e \) in \( F \). First, recall the definition of \( \phi_N(a; F) \) given in Eq. (30). By Lemma A4, for each \( a \in (0, 1) \) we have that \( \phi_N(a; F) \) is strictly increasing in \( F \) over the interval \((\frac{3}{4}, \delta)\). Hence, \( H(a; F) \) is strictly decreasing in \( F \) over the same domain. Letting
\[ \frac{2\delta}{N} \leq F_1 < F_2 < \delta, \text{ we obtain} \]
\[ 0 = H(a^e(F_2);F_2) = H(a^e(F_1);F_1) > H(a^e(F_1);F_2). \]

Because \( H(a;F_2) \) is strictly decreasing in \( a \) when \( R_r - R_s \) is sufficiently large, it follows that \( a^e(F_2) < a^e(F_1) \). This completes the proof that \( a^e(F) \) is monotonic in \( F \).

The infinite differentiability of \( a^e(F) \) in \( F \) follows from implicit differentiation and Lemma A4, from which it follows directly that \( H(a;F) \) is infinitely differentiable in \( F \) if \( F \neq \frac{l\delta}{N} \) for \( l = 2, 3, \ldots, N-1 \). This, in conjunction with the monotonicity of \( a^e(F) \) in \( F \), implies that

\[ \frac{da^e}{dF}(F) < 0, \text{for any } F \in \left( \frac{2\delta}{N}, \delta \right) \quad \text{(A21)} \]

except for the discrete set of points \( \frac{l\delta}{N}, l = 3, 4, \ldots, N-1 \).

Lastly, we use implicit differentiation to prove that \( a^e(F) \) is not differentiable at \( F = \frac{l\delta}{N} \), and that the left derivative is larger than the right derivative at these points. For any \( F \in \left( \frac{l\delta}{N}, \frac{(l+1)\delta}{N} \right) \) with \( l = 3, \ldots, N-2 \), we have

\[ \frac{da^e}{dF}(F) = -\left. \frac{\partial H}{\partial a} \right|_{(a,F)=(a^e(F),F)} = -\frac{u^{(N)}(a^e(F))}{2(1+f)(R_r - R_s) + \delta \cdot v^{(N),l}(a^e(F)) - (\delta - F)u^{(N),l}(a^e(F))}, \quad \text{(A22)} \]

where \( v^{(N)}_l \) and \( u^{(N)}_l \) are the functions defined in Eq. (A14). Thus, the right derivative at \( F = \frac{l\delta}{N} \) is given by

\[ \frac{da^e}{dF} \left( \frac{l\delta}{N}^+ \right) = -\frac{u^{(N)}(a^e(\frac{l\delta}{N}))}{2(1+f)(R_r - R_s) + \delta \cdot v^{(N),l}(a^e(\frac{l\delta}{N})) - (\delta - \frac{l\delta}{N})u^{(N),l}(a^e(\frac{l\delta}{N}))}. \quad \text{(A23)} \]

Similarly, for the left derivative at \( F = \frac{l\delta}{N} \) we obtain that

\[ \frac{da^e}{dF} \left( \frac{l\delta}{N}^- \right) = -\frac{u^{(N)}(a^e(\frac{l\delta}{N}))}{2(1+f)(R_r - R_s) + \delta \cdot v^{(N),l-1}(a^e(\frac{l\delta}{N})) - (\delta - \frac{l\delta}{N})u^{(N),l-1}(a^e(\frac{l\delta}{N}))}. \quad \text{(A24)} \]

Next, we compare (A23) with (A24). First, we show that \( \frac{da^e}{dF}(\frac{l\delta}{N}^+) \) and \( \frac{da^e}{dF}(\frac{l\delta}{N}^-) \) have the same
denominators. A direct computation shows that

$$
\left[ \delta \cdot v_i^{(N)y}(a^e\left(\frac{l\delta}{N}\right)) - \delta \frac{N-l}{N} u_i^{(N)y}(a^e\left(\frac{l\delta}{N}\right)) \right] - \left[ \delta \cdot v_i^{(N)y}(a^e\left(\frac{l\delta}{N}\right)) - \delta \frac{N-l}{N} u_i^{(N)y}(a^e\left(\frac{l\delta}{N}\right)) \right]
$$

$$
= \delta \left( \left( \frac{N-l}{N-1} \right) \left[ (1-a)^{N-l-1}l^y \right] - \frac{N-l}{N} \left[ (1-a)^{N-1}l^y \right] \right)_{a=a^e\left(\frac{l\delta}{N}\right)}
$$

$$
= \delta \left( 1 - a \right)^{N-l-1}l^y|_{a=a^e\left(\frac{l\delta}{N}\right)} \cdot \left( \left( \frac{N-1}{l} \right) - \frac{N-l}{N} \left( \frac{N}{l} \right) \right) = 0.
$$

On the other hand, recall that when \( R_e - R_s \) is large enough, it follows from Eq. (A21) that both \( \frac{da^e}{dF} \left( \frac{l\delta}{N}^+ \right) \) and \( \frac{da^e}{dF} \left( \frac{l\delta}{N}^- \right) \) are negative. Because \( u_i^{(N)}(a^e\left(\frac{l\delta}{N}\right)), u_i^{(N)}(a^e\left(\frac{l\delta}{N}\right)) > 0 \), then the (same) denominator of \( \frac{da^e}{dF} \left( \frac{l\delta}{N}^+ \right) \) and \( \frac{da^e}{dF} \left( \frac{l\delta}{N}^- \right) \) must be positive. Thus,

$$
\frac{da^e}{dF} \left( \frac{l\delta}{N}^+ \right) - \frac{da^e}{dF} \left( \frac{l\delta}{N}^- \right) = \frac{-\left[ u_i^{(N)}(a^e\left(\frac{l\delta}{N}\right)) - u_i^{(N)}(a^e\left(\frac{l\delta}{N}\right)) \right]}{2(1+f)(R_e - R_s) + \delta \cdot v_i^{(N)}, (a^e\left(\frac{l\delta}{N}\right)) - \delta \frac{N-l}{N} u_i^{(N)}, (a^e\left(\frac{l\delta}{N}\right))} < 0.
$$

**Lemma A5** For any \( F \in [\frac{2\delta}{N}, \delta] \), there exists a constant \( M > 0 \) such that

$$
\sup_{a \in [0, \frac{3}{4}]} \left| \frac{\partial}{\partial a} \phi_N(a; F) \right| < \delta M,
$$

where \( M \) is a positive constant that does not depend on \( R_e, R_s \) or \( \delta \).

**Proof.** Without loss of generality, suppose that \( F \in [\frac{2\delta}{N}, \frac{l(l+1)\delta}{N}] \) for some \( l = 2, 3, \ldots, N - 1 \), so that \( \left[ \frac{NF}{\delta} \right] = l \). Observe that the expression of \( \phi_N(a; F) \) given in Eq. (30) can be obtained from Eq. (A13) by setting \( q_s = a \), and observe that the functions \( v_i^{(N)}(a) \) and \( u_i^{(N)}(a) \) defined in Eq. (A14) do not depend on \( F \) and \( \delta \). It is then seen that both of these functions admit continuous first-order derivatives in \( a \) over the closed interval \([0, \frac{3}{4}]\). This completes the proof. \( \blacksquare \)

**Proof of Proposition 6**

The regulator’s objective function \( W(F) \) is differentiable everywhere in \( F \) except for the set of kinks \( \left\{ \frac{l\delta}{N}; l = 2, 3, \ldots, N - 1 \right\} \). Hence, if the equilibrium default fund \( F^e \) is not at one of the kinks, it must be the point at which the marginal benefit from increasing \( F \) equals the marginal
positive and jumps upward as \( F \) ranges over the set of kinks \( \frac{l\delta}{N} \), for \( l = 3, 4, \ldots, N - 1 \). This can be seen from the fact that, for any \( F \neq \frac{l\delta}{N} \),

\[
B'(F) = -2(1 + f)(R_r - R_s)(a^e(F) - a^*) \frac{da^e}{dF}(F), \tag{A26}
\]

where we know from Proposition 5 that \( \frac{da^e}{dF}(F) \) has a negative jump as \( F \) crosses over \( \frac{l\delta}{N} \), for \( l = 3, 4, \ldots, N - 1 \). Because \( W(F) \) is locally convex at each of these points, none of them can be a local maximum of \( W(F) \). The maximum of \( W(F) \) can only be attained either at the extremes of the interval \( \frac{2\delta}{N} \) and \( \delta \), or at an interior point in \( (\frac{2\delta}{N}, \delta) \) off the set \( \{ \frac{l\delta}{N}, l = 3, 4, \ldots, N - 1 \} \). If it occurs at an interior point, then the optimal \( F \) can be determined by the first-order condition, owing to the smoothness of \( W(F) \) off those kinks.

Next, we show that if the marginal benefit is higher than the opportunity cost when \( F \) equals its lower bound \( \frac{2\delta}{N} \) (\( B'(\frac{2\delta}{N}) \geq \beta \)), then we must have \( \frac{2\delta}{N} < F^c < \delta \). We will show that

\[
B'\left(\frac{2\delta}{N} + \right) > \beta \iff \frac{(a^e(\frac{2\delta}{N}) - a^*) \cdot u_2^{(N)}(a^e(\frac{2\delta}{N}))}{1 + \frac{\delta}{2(1 + f)(R_r - R_s)}[v_2^{(N),'}(a^e(\frac{2\delta}{N})) - \frac{N-2}{N}u_2^{(N),'}(a^e(\frac{2\delta}{N}))]} > \beta, \tag{A27}
\]

where \( u_2^{(N)}(a) = \sum_{k=N-3}^{N-1} \binom{N-1}{k} (1 - a)^k a^{N-1-k} \) and \( u_2^{(N)}(a) = \sum_{k=N-2}^{N} \binom{N}{k} (1 - a)^{k-1} a^{N-k} \).

Using (A22) for \( F \in (\frac{2\delta}{N}, \frac{3\delta}{N}) \) (so \( l = 2 \))

\[
- \frac{da^e}{dF}(F) = \frac{u_2^{(N)}(a^e(F))}{2(1 + f)(R_r - R_s) + \delta [v_2^{(N),'}(a^e(F)) - \frac{N-2}{N}u_2^{(N),'}(a^e(F))]}.
\tag{A28}
\]

Using (A26), (A28), the continuity of \( a^e(F) \) in \( F \), and the continuity of \( u_2^{(N)}(a), u_2^{(N),'}(a), v_2^{(N),'}(a) \) in \( a \), we know that, as \( F \downarrow \frac{2\delta}{N} \),

\[
B'\left(\frac{2\delta}{N} + \right) = \frac{(a^e(\frac{2\delta}{N}) - a^*) \cdot u_2^{(N)}(a^e(\frac{2\delta}{N}))}{1 + \frac{\delta}{2(1 + f)(R_r - R_s)}[v_2^{(N),'}(a^e(\frac{2\delta}{N})) - \frac{N-2}{N}u_2^{(N),'}(a^e(\frac{2\delta}{N}))]}.
\tag{A29}
\]
Therefore, the inequality on the right-hand side of (A27) is equivalent to $B'(\frac{2\hat{\delta}}{N}+) - \beta > 0$. Hence, the objective function $W(F)$ is locally increasing in a small neighborhood right of $\frac{2\hat{\delta}}{N}$. Using a similar argument and the fact that $a^e(\delta-) = a^*$ (see Proposition 5), we obtain that $B'(\delta-) - \beta = -\beta < 0$. That is, the objective function $W(F)$ is locally decreasing in a small neighborhood left of $\delta$. It follows that the value $F^e$ at which the maximum is achieved is an interior point, i.e., $F^e < \delta$.

**Proof of Proposition 7**

For a given $F \in (0, \delta]$, we choose $N$ such that $F \geq \frac{2\hat{\delta}}{N}$. Using Eq. (31), any optimal risk choice $a^e_N(F)$ satisfies the first-order condition:

$$a = \frac{R_r - 2R_s + \frac{\delta - \phi_N(a; F)}{1+f}}{2(R_r - R_s)} = a^* + \frac{\delta - \phi_N(a; F)}{2(1+f)(R_r - R_s)}.$$  

Above, we have added the subscript $N$ in $a^e_N(F)$ to stress its dependence on $N$. Because $0 \leq \phi_N(a; F) \leq \delta$ (this follows directly from the definition of $\phi_N(a; F)$ given in Eq. (30)), we obtain that

$$a^* \leq a^e_N(F) \leq a^* + \frac{\delta}{2(1+f)(R_r - R_s)}. \quad (A30)$$

Even for a fixed $N$ there might be multiple values of $a^e_N(F)$, we will show that all these values converge to the same limit as $N$ goes to $\infty$, and this limit only depends on $F$. To begin with, recall from the proof of Proposition 5 that $a^e_N(F)$ is a root of the equation $H_N(a; F) = 0$, where

$$H_N(a; F) = a^* - a + \frac{\delta - \phi_N(a; F)}{2(1+f)(R_r - R_s)}.$$

As $N \to \infty$, using the definition of $\phi_N(a; F)$ given in Eq. (30) and recall that $\phi_N(a; F)$ equals $\psi(N - 1; F)$ by evaluating $q_s$ at $a$, we obtain from (A17) that

$$\lim_{N \to \infty} \phi_N(a; F) = \delta \cdot 1_{\{\frac{F}{\delta} > a\}} \left(1 - \frac{1 - \frac{F}{\delta}}{1 - a}\right). \quad (A31)$$

Using the above limiting results, we obtain that

$$H(a; F) = \lim_{N \to \infty} H_N(a; F) = \begin{cases} a^* - a + \frac{\delta}{2(1+f)(R_r - R_s)}, & \text{if } \frac{F}{\delta} \leq a, \\ a^* - a + \frac{\delta - F}{2(1+f)(R_r - R_s)(1-a)}, & \text{if } \frac{F}{\delta} > a. \end{cases} \quad (A32)$$

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Straightforward analysis based on computing derivatives indicates that \( H(a, F) \) is continuous in \( a \) for all \( a \in (0, 1) \). Since \( H_N(a, F) \) solves the equation \( H_N(a, F) = 0 \) in the interval \([a^*, a^* + \frac{\delta}{2(1+f)(R_r - R_s)}]\) (see (A30)), we expect that its limit, if exists, also solves the equation \( H(a; F) = 0 \) over the same interval. Guided by this insight, we solve for roots of the equation \( H(a; F) = 0 \) inside the interval \([a^*, a^* + \frac{\delta}{2(1+f)(R_r - R_s)}]\). We distinguish two cases:

1. \( a \geq \frac{F}{\delta} \). Using the expression of \( H(a; F) \), we find that the only candidate root to \( H(a; F) = 0 \) is \( a = a^* + \frac{\delta}{2(1+f)(R_r - R_s)} \). This becomes a feasible root to \( H(a; F) = 0 \) if and only if

\[
\frac{F}{\delta} \leq a^* + \frac{\delta}{2(1+f)(R_r - R_s)}. \tag{A33}
\]

2. \( a < \frac{F}{\delta} \). Using the expression of \( H(a; F) \), we find two candidate roots to \( H(a; F) = 0 \), namely

\[
a_{\pm}(F) = \frac{1 + a^*}{2} \pm \sqrt{\left(\frac{1 - a^*}{2}\right)^2 - \frac{\delta - F}{2(1+f)(R_r - R_s)}}. \tag{A34}
\]

We notice that one of the two roots satisfies \( a_{+}(F) > \frac{1 + a^*}{2} \). However, it follows from

\[
a^* + \frac{\delta}{(1+f)(R_r - R_s)} < \frac{R_r - 2R_s + \delta}{2(R_r - R_s)} = \frac{1}{2} + \frac{\delta - R_s}{2(R_r - R_s)} < 1
\]

(the first inequality follows from the definition of \( a^* \) given in Eq. (21) and the last inequality is a consequence of Assumption 1-C) that

\[
2a^* + \frac{\delta}{(1+f)(R_r - R_s)} < a^* + 1 \iff a^* + \frac{\delta}{2(1+f)(R_r - R_s)} < \frac{1 + a^*}{2},
\]

This implies that \( a_{+}(F) \) is outside the interval of admissible values. Hence, \( a_{-}(F) \) is the only viable candidate solution. It can be observed from expression (A34) that \( a_{-}(F) \) is strictly decreasing in \( F \). Moreover, it is straightforward to verify that \( a_{-}(F) < \frac{F}{\delta} \) holds if and only if

\[
\frac{F}{\delta} > a^* + \frac{\delta}{2(1+f)(R_r - R_s)}. \tag{A35}
\]
Altogether, we conclude that the unique symmetric risk-taking strategy of members in the limiting case of a large CCP network is given by

\[
a^e(F) \rightarrow \begin{cases} 
\frac{\delta}{2(1 + f)(R_r - R_s)}, & \text{if } \frac{F}{\delta} \leq a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}, \\
\frac{1 + a^*}{2} - \sqrt{\left(1 - a^*\right)^2 - \frac{\delta - F}{2(1 + f)(R_r - R_s)}}, & \text{otherwise.}
\end{cases}
\]

(A36)

It can be seen from the above equation that \(a^e(F)\) is a continuous function of \(F\), strictly decreasing in the interval \((a^* \delta + \frac{\delta^2}{2(1 + f)(R_r - R_s)}, \delta)\), and admitting the limit \(a^e(\delta^-) = a^*\).

We complete the proof by showing that \(\lim_{N \to \infty} a^e_N(F) = a^e(F)\). By the Bolzano-Weierstrass theorem, the bounded, infinite sequence \(\{a^e_N(F)\}_N\) always has at least one limit point as \(N\) goes to \(\infty\). By choosing a specific subsequence, we obtain that \(\{a^e_N(F)\}_N\) has a unique limit point in \([a^*, a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}]\). The proof is completed once we demonstrate that the limit point of the subsequence is equal to \(a^e(F)\) (regardless of the way in which we select the subsequence). To that end, let us denote by

\[
a^e_\infty(F) = \lim_{N \to \infty} a^e_N(F)
\]

the limit of the symmetric response function. Then, for any small \(\epsilon > 0\) there exists a positive integer \(N_\epsilon\) such that

\[
0 < a^e_\infty(F) - \epsilon < a^e_N(F) < a^e_\infty(F) + \epsilon < 1
\]

holds for any \(N \geq N_\epsilon\). Because \(\phi_N(a; F)\) is strictly decreasing in \(a\) (see the proof of Proposition 5 and Lemma A3), it follows from (31) that for all \(N \geq N_\epsilon\)

\[
\frac{\delta - \phi_N(a^e_\infty(F) - \epsilon; F)}{2(1 + f)(R_r - R_s)} < a^e_N(F) - a^* < \frac{\delta - \phi_N(a^e_\infty(F) + \epsilon; F)}{2(1 + f)(R_r - R_s)}.
\]

(A38)

Taking the limits of all terms in (A38) as \(N\) goes to \(\infty\) and using (A37), we obtain that

\[
\frac{\delta - \phi(a^e_\infty(F) - \epsilon; F)}{2(1 + f)(R_r - R_s)} \leq a^e_\infty(F) - a^* \leq \frac{\delta - \phi(a^e_\infty(F) + \epsilon; F)}{2(1 + f)(R_r - R_s)},
\]

(A39)

where \(\phi(a; F) = \lim_{N \to \infty} \phi_N(a; F)\). Letting \(\epsilon\) go to 0, and using the fact that \(\phi(a; F)\) is continuous in \(a\) (see (A31)), we deduce that the limiting value \(a^e_\infty(F)\) solves the equation \(H(a; F) = 0\) over the interval \([a^*, a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}]\). Because we already know that \(a^e(F)\) is the only such solution, we can conclude that \(a^e_\infty(F) = a^e(F)\). This completes the proof of Proposition 7.
Proof of Proposition 8

It follows from Proposition 7 and Eq. (34) that, as $N \to \infty$, we have

$$\frac{x_s(N)}{N} = \frac{F_s(N)}{\delta} = \frac{1}{\delta} \arg \max_{F \in [\delta, \frac{\delta^2}{2}]} W(F) = \frac{1}{\delta} \arg \max_{F \in [\delta, \frac{\delta^2}{2}]} (B(F) - \beta F - \delta)
\rightarrow \frac{1}{\delta} \arg \max_{F \in (0, \delta]} ((1 + f)(1 - a^e(F))(a^e(F)(R_r - R_s) + R_s) - \beta F - \delta). \quad (A40)$$

To find the maximizer on the right-hand side of (A40), we first recall from Proposition 7 that, for $F \leq a^*\delta + \frac{\delta^2}{2(1 + f)(R_r - R_s)}$, $a^e(F)$ does not depend on $F$; in this case, choosing a positive $F$ would only present an opportunity cost and reduce the objective function by $\beta F$. Hence,

$$\arg \max_{0 < F \leq a^*\delta + \frac{\delta^2}{2(1 + f)(R_r - R_s)}} ((1 + f)(1 - a^e(F))(a^e(F)(R_r - R_s) + R_s) - \beta F - \delta) = 0,$$

with maximum value equal to $W(0)$.

On the other hand, the range of the objective function in (A40) for $F \in (a^*\delta + \frac{\delta^2}{2(1 + f)(R_r - R_s)}, \delta)$ is the same as the range of the function

$$G(a) := (1 + f)(1 - a)(a(R_r - R_s) + R_s) - \beta F(a) - \delta, \quad a \in \left(a^*, a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}\right), \quad (A41)$$

where $F(a)$ is the inverse of $a^e(F)$ in (A36) for $F \in (a^*\delta + \frac{\delta^2}{2(1 + f)(R_r - R_s)}, \delta)$, and is given by

$$F(a) = \delta - 2(1 + f)(R_r - R_s)(a - a^*)(1 - a). \quad (A42)$$

Plug Eq. (5) into Eq. (A41) and take derivative with respect to $a$, we obtain

$$G'(a) = 2(1 + f)(R_r - R_s)[(1 + \beta)(a^* - a) + \beta(1 - a)]. \quad (A43)$$

We can then conclude from the above expression that $G'(a)$ is strictly decreasing in $a$. Hence, $G(a)$ is concave in $a$. Moreover, since $G'(a^*) > 0$, the maximum of $G(a)$ over $(a^*, a^* + \frac{\delta}{2(1 + f)(R_r - R_s)})$ is either $a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}$ or the global maximizer of $G(a)$ if the maximizer is an interior point of the interval. To proceed, we first determine the global maximizer by taking the first-order condition...
\[ G'(a) = 0, \text{ and obtain } a_0^* = \frac{\beta + (1 + \beta)a^*}{1 + 2\beta}. \] Thus,

\[
\sup_{a \in [a^*, a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}]} G(a) = \begin{cases} 
G(a_0^*), & \text{if } a_0^* < a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}, \\
G \left( a^* + \frac{\delta}{2(1 + f)(R_r - R_s)} \right), & \text{otherwise}.
\end{cases}
\]

We then obtain that

\[
F^e = \arg \max_{F \in \left[ \frac{\delta}{2r}, \delta \right]} W(F) = \begin{cases} 
F(a_0^*), & \text{if } G(a_0^*) > W(0) \text{ and } a_0^* < a^* + \frac{\delta}{2(1 + f)(R_r - R_s)}, \\
0, & \text{otherwise}.
\end{cases}
\] (A44)

Writing the condition in the first line of Eq. (A44) more explicitly, we obtain that this is equivalent to

\[
\frac{(1 + f)R_r \beta}{(1 + 2\beta)\delta} < 1 \text{ and } W(F(a_0^*)) > W(0) \text{ with default fund } F^e = F(a_0^*) = \delta - \frac{\beta(1 + \beta)(1 + f)R_r^2}{2(1 + 2\beta)^2(R_r - R_s)},
\]

where we plug in the relation that \( G(a_0^*) = W(F(a_0^*)) \). This completes the proof of Proposition 8.