Solution to Homework 1

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1 Problem 1

Let $Ax = \lambda x$, where x denotes the eigen-vector, and λ denotes the corresponding eigen-value. It follows that

$$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} x = 0. \tag{1}$$

Let $|A - \lambda I| = 0$, where I is the identity matrix. It follows that

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0. \tag{2}$$

By solving Eqs. (1) and (2), we obtain

$$\begin{array}{ll} \lambda_1 = 1, & x_1 = (0,0,1)^T, \\ \lambda_2 = 1, & x_2 = (-0.707, 0.707, 1)^T, \\ \lambda_3 = 3, & x_3 = (0.707, 0.707, 0)^T. \end{array} \tag{3}$$

Note that all eigen-vectors are normalized to length 1.

2 Problem 2

Following from $\cos\theta = \frac{x^T y}{\|x\|_2 \cdot \|y\|_2} = \frac{2}{3}$, we obtain $\theta = \arccos\frac{2}{3}$.

3 Problem 3

Let $x = (x_1, x_2, \cdots, x_n)^T$, the maximum entry of x be x_m, and $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$. It follows that

$$\|x\|_p \geq (|x_m|^p)^{\frac{1}{p}} = \|x\|_\infty, \quad \lim_{p\to\infty} \|x\|_p \geq \|x\|_\infty. \tag{4}$$

On the other hand,

$$\|x\|_p \leq (n|x_m|^p)^{\frac{1}{p}}, \quad \lim_{p\to\infty} \|x\|_p \leq \|x\|_\infty. \tag{5}$$

It shows

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty. \tag{6}$$

4 Problem 4

It can be verified the permutation of rows (columns) will not change the matrix norm. Without loss of generality, we can assume

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \tag{7}$$

where B is a submatrix of A. It follows that

$$\|A\|_p = \sup_x \frac{\|Ax\|_p}{\|x\|_p} \geq \sup_y \frac{\|Ay\|_p}{\|y\|_p}, \tag{8}$$

where the entries of vector y corresponding to C and E are set to zeros, and the entries corresponding to B and D are denoted as z, namely $y = (z, 0, \cdots, 0)$. Moreover,

$$\sup_y \frac{\|Ay\|_p}{\|y\|_p} = \sup_z \frac{\left\| \begin{pmatrix} B \\ D \end{pmatrix} z \right\|_p}{\|z\|_p} \geq \sup_z \frac{\|Bz\|_p}{\|z\|_p} = \|B\|_p, \tag{9}$$

which completes the proof.

5 Problem 5

Let rank$(A) = k$, and $(v_1, v_2, \cdots, v_k)$ be the basis of $A = (a_1, \cdots, a_r) \in \mathbb{R}^{m\times r}$. It can be verified that each a_i can be expressed as a linear combination of $\{v_j\}_{j=1}^k$.

Let $B = (b_1, \cdots, b_n) \in \mathbb{R}^{r\times n}$, and $b_i = (b_{1i}, b_{2i}, \cdots, b_{ri})^T$. It follows that $AB = (Ab_1, Ab_2, \cdots, Ab_n)$, and $Ab_i = a_1 b_{1i} + a_2 b_{2i} + \cdots + a_r b_{ri}$.

Since $\{v_j\}_{j=1}^k$ is the basis of A, each Ab_i can also be expressed as a linear combination of $\{v_j\}_{j=1}^k$. It shows the rank of AB is no larger than k, i.e., rank$(AB) \leq$ rank(A).

Similarly, we can prove rank$(AB) \leq$ rank(B). Thus we obtain rank$(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.