Assignment 2.2 - Differentiating the Inverse

Complete the problems below and turn them it at the beginning of class on Tuesday, October 13.

1. Show that the function $f(x) = x + \ln(x)$ (restricted to $x > 0$) has an inverse.
2. If the inverse of $f(x)$ from problem 1 is called $g(x)$, find the equation of the tangent line to the curve $g(x)$ at the point (1, 1).
3. Show that the function $f(x) = x^3 - 1$ has an inverse.
4. If the inverse of $f(x)$ from problem 3 is called $g(x)$, find the equation of the tangent line to the curve $g(x)$ at the point (7, 2).
5. For the function $f(x) = \frac{x-1}{x+1}$, evaluate $\left(f^{-1}\right)'(3)$.
6. Show that the function $f(x) = 1 + x + x^2 + x^3$ has an inverse.
7. For the function in problem 6, find $\left(f^{-1}\right)'(40)$

Solutions:

1. $f(x) = x + \ln(x)$ is continuous when restricted to $x > 0$. The derivative $f'(x) = 1 + \frac{1}{x}$ is always positive when restricted to $x > 0$. This means that the original function $f(x) = x + \ln(x)$ is always increasing when restricted to $x > 0$. This means that the original function $f(x) = x + \ln(x)$ is one-to-one and thus has an inverse.
2. To find the equation of the tangent line, we first have to have the slope of the tangent line. We use the Inverse Function Theorem to do this. $g'(1) = \frac{1}{f'(1)} = \frac{1}{1 + \frac{1}{1}} = \frac{1}{2}$. Now we just need to plug this and the point into the point-slope form of a line. This will give us $y - 1 = \frac{1}{2}(x - 1)$. When simplified we get $y = \frac{1}{2}x - \frac{1}{2} + 1 = \frac{1}{2}x + \frac{1}{2}$.
3. $f(x) = x^3 - 1$ is continuous since it is a polynomial. The derivative $f'(x) = 3x^2$ is always positive. This means that the original function $f(x) = x^3 - 1$ is always increasing. This means that the original function $f(x) = x^3 - 1$ is one-to-one and thus has an inverse.
4. To find the equation of the tangent line, we first have to have the slope of the tangent line. We use the Inverse Function Theorem to do this.

$g'(7)=\frac{1}{f'(2)}=\frac{1}{3(2)^2}=\frac{1}{12}$. Now we just need to plug this and the point into the point-slope form of a line. This will give us $y-2=\frac{1}{12}(x-7)$.

When simplified we get $y=\frac{1}{12}x-\frac{7}{12}+2=\frac{1}{12}x+\frac{17}{12}$.

5. Once again we will need to use the Inverse Function Theorem that says $\left(f^{-1}(y)\right)'=\frac{1}{f'(x)}$ where (x, y) is a point on the original function $f(x)$. We know y is 3 from the problem and we thus will need to find the x-coordinate that goes with the y-coordinate 3. Thus we need to solve $3=\frac{x-1}{x+1}$. Solving this will get us
$$\begin{aligned}3(x+1)&=x-1\\3x+3&=x-1\\2x&=-4\\x&=-2\end{aligned}$$
. Now we need to find the derivative of the original function $f(x)=\frac{x-1}{x+1}$ and then we will plug the x-value and derivative function into the Inverse Function Theorem.
$$f'(x)=\frac{1(x+1)-(x-1)1}{(x+1)^2}=\frac{x+1-x+1}{(x+1)^2}=\frac{2}{(x+1)^2}$$
Plugging into the theorem, we get
$$\left(f^{-1}\right)'(3)=\frac{1}{\left(\frac{2}{(-2+1)^2}\right)}=\frac{1}{\left(\frac{2}{(-1)^2}\right)}=\frac{1}{\left(\frac{2}{1}\right)}=\frac{1}{2}.$$

6. $f(x)=1+x+x^2+x^3$ is continuous since it is a polynomial. The derivative $f'(x)=1+2x+3x^2$ is always positive since it is a quadratic function that opens upward and has a vertex y-coordinate (lowest point) that is positive. This means that the original function $f(x)=1+x+x^2+x^3$ is always increasing. This means that the original function $f(x)=1+x+x^2+x^3$ is one-to-one and thus has an inverse.

7. Once again we will need to use the Inverse Function Theorem that says $\left(f^{-1}(y)\right)'=\frac{1}{f'(x)}$ where (x, y) is a point on the original function $f(x)$. We know y is 3 from the problem and we thus will need to find the x-coordinate that goes with the y-coordinate 40. Thus we need to solve $40=1+x+x^2+x^3$. Use the abilities of our graphing calculator, we can find that x is 3. We already know that the derivative of the function is

$f'(x) = 1 + 2x + 3x^2$. Plugging the derivative and $x = 3$ into the theorem, we get $\left(f^{-1}\right)'(40) = \frac{1}{1 + 2(3) + 3(3)^2} = \frac{1}{34}$.