ANNUITIES

Objectives:
   After completing this section, you should be able to do the following:
- Calculate the future value of an ordinary annuity.
- Calculate the amount of interest earned in an ordinary annuity.
- Calculate the total contributions to an ordinary annuity.
- Calculate monthly payments that will produce a given future value.

Vocabulary:
   As you read, you should be looking for the following vocabulary words and their definitions:
- ordinary annuity
- simple annuity
- Christmas club
- tax-deferred annuity (TDA)
- present value of an annuity

Formulas:
   You should be looking for the following formulas as you read:
- future value of an ordinary annuity
- total contribution to an annuity
- interest earned on an annuity
- present value of an annuity

An *annuity* is defined by merriam-webster.com as "a sum of money payable yearly or at other regular intervals". Wikipedia defines an *annuity* as "any recurring periodic series of payment".
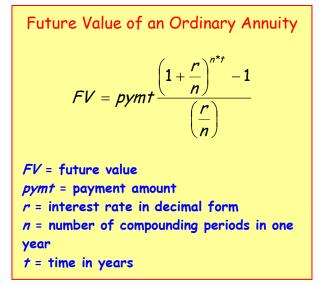
Some examples of annuities are regular payments into a savings account, monthly mortgage payments, regular insurance payments, etc. Annuities can be classified by when the payments are made. Annuities whose payments are made at the end of the period are called *ordinary annuities*. Annuities whose payments are made at the beginning of the period are called *annuity-due*. In this class we will only work with ordinary annuities.

ordinary annuity

annuity due

We will need to be able to calculate the future value of our annuities. In order to do this we will need a formula to calculate future value if we know the amount of the payment, the interest rate and compounding period, and the number of payments.

Future Value of an Ordinary Annuity

$$FV = pymt\dfrac{\left(1+\dfrac{r}{n}\right)^{n*t} - 1}{\left(\dfrac{r}{n}\right)}$$

FV = **future value**
pymt = **payment amount**
r = **interest rate in decimal form**
n = **number of compounding periods in one year**
t = **time in years**

NOTE: The payment period and the compounding period will always match in our problems. Annuities which have the same payment and compounding period are called simple annuities.

simple annuity

Example 1:
Find the future value of an ordinary annuity with $150 monthly payments at $6\frac{1}{4}\%$ annual interest for 12 years.

Solution:
For this problem we are given payment amount ($150), the interest rate (.0625 in decimal form), the compounding period (monthly or 12 periods per year), and finally the time (12 years). We plug each of these into the appropriate spot in the formula

$$FV = pymt\dfrac{\left(1+\dfrac{r}{n}\right)^{n*t} - 1}{\left(\dfrac{r}{n}\right)}.$$ This will give us

$$FV = 150\frac{\left(1 + \frac{.0625}{12}\right)^{12*12} - 1}{\left(\frac{.0625}{12}\right)}$$

$$FV = 32051.04651$$

In this class we will round using standard rounding. This will make the future value $32051.05.

A Christmas club account is a short-term special savings account usually set up at a bank or credit union in which a person can deposit regular payments for the purposes of saving money for Christmas purchases.

Christmas club

Example 2:

On March 9, Mike joined a Christmas club. His bank will automatically deduct $210 from his checking account at the end of each month, and deposit it into his Christmas club account, where it will earn $5\frac{1}{4}\%$ annual interest. The account comes to term on December 1. Find the following:

  a. Find the future value of Mike's Christmas club account.
  b. Find Mike's total contribution to the account.
  c. Find the total interest earned on the account.

Future Value of an Ordinary Annuity

$$FV = pymt\frac{\left(1 + \frac{r}{n}\right)^{n*t} - 1}{\left(\frac{r}{n}\right)}$$

FV = future value
pymt = payment amount
r = interest rate in decimal form
n = number of compounding periods in one year
t = time in years

Solution:

  a. For this part we will use the future value formula for an ordinary annuity. The payment amount is 210. The interest rate in decimal form is .0525. The number of compounding period in one year is 12 (monthly payments). The amount of time in years (t) is $\frac{9}{12}$. We will be making payments for 9 months (end of March, end of April, end of May, end of June, end of July, end of August, end of September, end of October, and finally end of November). This will give us

$$FV = 210\frac{\left(1+\dfrac{.0525}{12}\right)^{\left(12*\frac{9}{12}\right)}-1}{\left(\dfrac{.0525}{12}\right)}$$

$$FV = 1923.414866$$

In this class we will round using standard rounding. This will make the future value $1923.41.

b. To find the total amount of Mike's contribution, we only need to take the amount of each monthly payment and multiply by the number of payments per year and finally multiply by the number of years. This formula can be seen in the box on the left. This will give us
$210*12*\dfrac{9}{12} = 1890$. Thus the total contribution made by Mike is $1890.

> **Annuity Total Contribution**
> total contribution $= pymt * n * t$
>
> *pymt* = payment amount
> *n* = number of compounding periods in one year
> *t* = time in years

c. To find the interest earned by Mike, we need to use the interest formula to the left. This basically takes the total contribution and subtracts it from the future value. The difference between these two numbers will be the interest earned on the annuity. Plugging into the formula we get
$1923.41 - 210*12*\dfrac{9}{12} = 1923.41 - 1890 = 33.40$.
Thus Mike earns $33.40 in interest on this account.

> **Annuity Interest Formula**
> $I = FV - pymt * n * t$
>
> *I* = interest
> *FV* = future value
> *pymt* = payment amount
> *n* = number of compounding periods in one year
> *t* = time in years

A tax-deferred annuity (TDA) is an annuity in which you do not pay taxes on the money deposited or on the interest earned until you start to withdraw the money from the annuity account.

tax-deferred annuity

Example 3:

John Jones recently set up a tax-deferred annuity to save for his retirement. He arranged to have $50 taken out of each of his biweekly checks; it will earn $8\frac{3}{8}$% annual interest. He just had his thirty-fifth

birthday, and his ordinary annuity comes to term when he is sixty-five. Find the following:

a. Find the future value of John's annuity.
b. Find John's total contribution to the annuity.
c. Find the total interest earned on the annuity.

Solution:

a. For this part we will use the future value formula for an ordinary annuity. The payment amount is 50. The interest rate in decimal form is .08375. The number of compounding period in one year is 26 (bi-weekly payments – every two weeks) . The amount of time in years (t) is calculated by taking the age at which John's annuity comes to term and subtracting John's current age (65-35 = 30). This will give us

$$FV = 50\frac{\left(1+\dfrac{.08375}{26}\right)^{(26*30)} - 1}{\left(\dfrac{.08375}{26}\right)}$$

$$FV = 175186.9942$$

In this class we will round using standard rounding. This will make the future value $175,186.99.

b. To find the total amount of John's contribution, we only need to take the amount of each monthly payment and multiply by the number of payments per year and finally multiply by the number of years. This formula can be seen in the box on the left. This will give us $50 * 26 * 30 = 39000$. Thus the total contribution made by Mike is $39,000.

c. To find the interest earned by John, we need to use the annuity interest formula. This basically takes the total contribution and subtracts it from the future value. The difference between these two numbers will be the interest earned on the annuity. Plugging into the formula we get

$$175185.99 - 50 * 26 * 30 = 175185.99 - 39000 = 136185.99 \, .$$

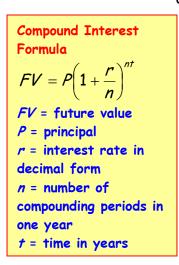Thus John earns $136,185.99 in interest on this account.

Another way to think about your savings prospects is to determine what you monthly expenses are now and put away enough each month so that you will have an annuity that will produce monthly interest equal to your monthly expenses. This next example will look at how to make such a calculation.

Example 4:

Bill and Nancy want to set up a TDA that will generate sufficient interest at maturity to meet their living expenses, which they project to be $1,500 per month.

a. Find the amount needed at maturity to generate $1500 per month interest if they can get $6\frac{3}{4}\%$ annual interest compounded monthly.

b. Find the monthly payment they would have to put into an ordinary annuity to obtain the future value found in part a if their money earns $9\frac{1}{2}\%$ annual interest and the term is 30 years.

Solution:

a. This first question is not an annuity problem at all. It is a basic compound interest problem (see formula to the left), where we do not know the principal or the future value, but we do know the relationship between the two. We know that the amount of interest that needs to be earned in one month is 1500. This means that the future value will need to be the principal plus 1500 ($FV = P + 1500$). We are given r to be .06375. n is 12 since it is monthly compounding. t will be $\frac{1}{12}$ since this is only one month.

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Compound Interest Formula

$$FV = P\left(1 + \frac{r}{n}\right)^{nt}$$

FV = future value
P = principal
r = interest rate in decimal form
n = number of compounding periods in one year
t = time in years

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$$P + 1500 = P\left(1 + \frac{.06375}{12}\right)^{\left(12*\frac{1}{12}\right)}$$

$$P + 1500 = P\left(1 + \frac{.06375}{12}\right)^{1}$$

$P + 1500 = P(1 + .0053125)$ *calculate the fraction*

$P + 1500 = P(1.0053125)$ *calculate the parentheses*

$1500 = P(1.0053125) - P$ *subtract P from both sides*

$1500 = P[(1.0053125) - 1]$ *factor P out of right side*

$1500 = P[.0053125]$ *combine numbers in brackets*

$\dfrac{1500}{.0053125} = P$ *divide both sides by number in brackets*

$282352.9412 = P$

In this class we will round using standard rounding. This will make the needed principal $282,352.94 in order to earn $1500 in interest per month.

b. For this part of the problem, we are being asked to find the monthly payments needed to end up with the future value of $282,352.94 (see part a) in our annuity. We will use the future value of an ordinary annuity formula. We are given the future value of 282352.94. We have an interest rate for this annuity of .095 (different from part a). n is 12 (monthly payments and monthly compounding since a simple annuity). t is given as 30 years. We need to solve for the payments. When we plug all this into the formula we get

$$282352.94 = pymt \frac{\left(1 + \frac{.095}{12}\right)^{(12*30)} - 1}{\left(\frac{.095}{12}\right)}$$

$282352.94 = pymt(2033.035174)$ *calculate the fraction*

$\dfrac{282352.94}{2033.035174} = pymt$ *divide both sides by number in parentheses*

$138.8824667 = pymt$

Normally in this class we use standard rounding, but in this case, we will want to round up to ensure that we end up with the desired future value in 30 years. Thus the monthly payments will need to be $138.89