Continuity

Objectives:
Students will be able to
• Determine where a function is discontinuous (if anywhere)
• Explain why a function is discontinuous at a point (or points)
• Determine the value of a variable to make a function continuous

Up until now, most students have defined a continuous function as one that can be drawn without picking up your pencil. This is not a very rigorous definition, but it is a good intuitive one.

We can use our limits to define the continuity of a function more precisely.

Continuity
A function \( f(x) \) is continuous at a point \( x = a \) if the following are all true:
• The function \( f(x) \) is defined at \( x = a \).
• \( \lim_{{x \to a}} f(x) \) exists.
• \( \lim_{{x \to a}} f(x) = f(a) \)

Example 1:

Using interval notation, indicate where the function \( f(x) \) shown above is continuous.
• What requirement(s) for continuity is the function \( f(x) \) missing?
• Can this function be made continuous by changing the value of the function at $x = 3.25$?

Example 2:
Find all values $x = a$ where the function $f(x) = \frac{x - 8}{x^2 - 25}$ is continuous. At all values of $x = a$ where the function is discontinuous, what is the limit of the function as $x$ approaches $a$?

Example 3:
Find all values $x = a$ where the function $f(x) = 4x^2 + 6x - 4$ is continuous. At all values of $x = a$ where the function is discontinuous, what is the limit of the function as $x$ approaches $a$?

Example 4:
Find all values $x = a$ where the function $f(x) = \begin{cases} 3 + x, & x < 2 \\ 7 - x, & x \geq 2 \end{cases}$ is continuous. At all values of $x = a$ where the function is discontinuous, what is the limit of the function as $x$ approaches $a$?

Example 5:
Find all values $x = a$ where the function $f(x) = \begin{cases} 4x, & x \leq 2 \\ x^2, & x > 2 \end{cases}$ is continuous. At all values of $x = a$ where the function is discontinuous, what is the limit of the function as $x$ approaches $a$?

Example 6:
Find the values of $c$ and $d$ which will make the function $g(x)$ continuous on $(-\infty, \infty)$

$$g(x) = \begin{cases} 2x, & x < 1 \\ cx^2 + d, & 1 \leq x \leq 2 \\ 7x, & x > 2 \end{cases}$$

Results on Continuous Functions:
If \( f(x) \) and \( g(x) \) are continuous at \( a \),

1. \( f(x) + g(x) \) and \( f(x) - g(x) \) are continuous at \( a \)
2. \( f(x) \cdot g(x) \) and \( \frac{f(x)}{g(x)} \) (if \( g(a) \neq 0 \)) are continuous at \( a \)
3. \( (f(x))^q \) is continuous at \( a \) if \( (f(a))^q \) is defined

If \( g(x) \) is continuous at \( a \) and \( f(x) \) is continuous at \( g(a) \), then \( f(g(x)) \) is continuous at \( x = a \).