

ISOSPIN-SYMMETRY BREAKING IN THE πN SYSTEM

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Abstract

There are only two independent isospin amplitudes for pion-nucleon scattering if isospin is assumed to be a good symmetry. In those cases in which isospin symmetry is broken (but time-reversal symmetry is valid) six more are needed to express an on-shell amplitude and two more to express an off-shell amplitude or potential. An independent set of these operators is constructed and analyzed.

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I. Introduction

The charge states of the pion-nucleon system are characterized by \mathbf{t} , the pion isospin operator, and $\boldsymbol{\tau}/2$, the nucleon isospin operator. If the interaction of the pion with a nucleon is assumed to be charge independent, then the amplitudes or potentials describing the interactions must depend only on the scalar combinations of these operators and on the identity operator. The only isospin-conserving combinations which can be formed from these operators are $a\mathbf{1}$ and $b\mathbf{t} \cdot \boldsymbol{\tau}$ where a and b are space- and spin-dependent coefficients. These two operators describe eight different reactions (or ten reactions in the absence of time-reversal symmetry, see Table 1), and so the predictive power of the symmetry is very great. It is known, however, that charge independence is broken. Examples are the $\pi^+\pi^0$ mass difference, the np mass difference, the Coulomb interaction between charged pions and the proton, magnetic interactions, and more exotic mechanisms such as those arising from $\rho\omega$ and $\pi\eta$ mixing. To describe these mechanisms we must introduce additional operators. If isospin-symmetry is totally broken an additional six amplitudes are needed to characterize all eight of the reactions, where we assume that charge conservation and time-reversal symmetry hold. It is the purpose of this article to define and analyze these isospin-symmetry-breaking amplitudes.

The article is arranged as follows. In Sec.II we count the number of independent amplitudes for several systems, with particular attention given to the pion-nucleon system. As a warm-up exercise the operators describing three simpler systems ($N\alpha$, $\pi\alpha$, and KN) are constructed in Sec.III. In Sec.IV the operators for the πN system are defined (Table 2) and their matrix elements evaluated in both charge (Table 3) and isospin (Table 4) bases. The contribution of some of the simpler isospin-breaking mechanisms to these amplitudes is examined in Sec.V. In Sec.VI we give an introductory account of what current experiments tell us about the isospin-breaking amplitudes. The article closes with appendices on on-shell *vs.* off-shell amplitudes, unitarity, and further mathematical aspects of the isospin operators.

II. The Number of Independent Amplitudes

A multiplet of isospin T is a set of $2T + 1$ particles with integer-spaced charges but with otherwise identical quantum numbers. The particles need not be degenerate in mass, and interactions of different members of the multiplet need not be identical. In this section we count the number of independent isospin amplitudes for scattering and charge-exchange of two particles belonging to isotopic multiplets T_1 and T_2 . The system has $(2T_1 + 1) \times (2T_2 + 1)$ possible channels. For example, the KN system has $2 \times 2 = 4$ and the πN has $3 \times 2 = 6$ channels [1]. The channels may be grouped into $2(T_1 + T_2) + 1$ sets according to their total charge Q . C_Q is defined to be the number of channels in the set with charge Q . (If we choose $T_1 \leq T_2$, then $C_Q \leq 2T_1 + 1$.) The number of isospin amplitudes necessary to describe elastic and charge-exchange reactions among channels of a given Q equals the total number of elastic and charge-exchange processes connecting them. This number is $N_Q = \frac{1}{2}C_Q(C_Q + 1)$ if we assume that the isospin amplitudes corresponding to $i \rightarrow j$ and $j \rightarrow i$ are equal (time-reversal symmetry of the isospin operators). If the isospin amplitudes are not symmetric under time-reversal, then $N_Q = C_Q^2$. It is, of course, possible to have time-reversal symmetry satisfied for the overall amplitude provided that the *product* of the isospin amplitude and the space-spin coefficient is invariant. Such space-spin coefficients may be constructed for the potentials (or off-shell T -matrices) describing the NN [2,3], πN , and KN systems. For parity-conserving on-shell amplitudes, however, no such coefficients exist for the πN and KN systems (see Appendix 1).

We denote by $N = \sum_Q N_Q$ the total number of elastic and charge-exchange reactions among all of the channels. For example, in KN scattering there are four channels: K^+p , K^+n , K^0p , and K^0n . Since $C_1 = 2$ and $C_2 = C_0 = 1$, we find $N = 3 + 1 + 1 = 5$; there are five independent isospin amplitudes which are invariant under time-reversal. This is increased to $N = 4 + 1 + 1 = 6$ when isospin amplitudes which are odd are used in conjunction with an odd space-spin coefficient. Finally, for pion-nucleon scattering there are 3×2 or six channels: π^+p , π^+n , π^-p , π^-n , π^0p , and π^0n . Thus we have $C_2 = C_{-1} = 1$, and $C_1 = C_0 = 2$, and so the number of independent isospin amplitudes is eight ($N = 1 + 1 + 3 + 3$). If time-reversal

odd isospin amplitudes are also used this becomes ten ($N = 1 + 1 + 4 + 4$).

We have mentioned conditions due to time-reversal invariance, parity, and the conservation of charge, but not due to unitarity. This is because unitarity restricts the number of real parameters necessary to describe the *coefficients* of the isospin amplitudes, but places no constraint on the number of independent isospin amplitudes. See Appendix 2.

We will next turn to the explicit construction of isospin operators from which to construct these on- and off-shell amplitudes.

III. Introductory Examples of Isospin Amplitudes

a) $T_1 = \frac{1}{2}, T_2 = 0$

As a warm-up exercise we begin with the scattering of an isospin $\frac{1}{2}$ particle from an isospin 0 particle (such as $N\alpha$ or $N\eta$ scattering, where α is treated as an elementary particle). The only available operators in isospin space are the identity, $\mathbf{1}$, and the isospin operator for the isospin $\frac{1}{2}$ particle, $\boldsymbol{\tau}/2$. Because the representation for an isospin $\frac{1}{2}$ particle is two-dimensional no products of the components of $\boldsymbol{\tau}$ are needed; *e.g.* terms such as $\tau_\alpha\tau_\beta$ may be reduced to either the identity or to a single component of $\boldsymbol{\tau}$.

The operator must have a z-component of zero in order that charge be conserved. The most general charge-conserving operator will be a linear combination of $\mathbf{1}$ and τ_0 . Of these, only $\mathbf{1}$ conserves isospin symmetry. τ_0 violates both isospin symmetry and charge-reflection symmetry. As an example, the operator structure for the Coulomb interaction for $N\alpha$ scattering is $\frac{1}{2}(1 + \tau_0)$. In the charge basis (*i.e.* the basis with τ_0 diagonal) the operators are diagonal with diagonal elements $\mathbf{1} = (1, 1)$ and $\tau_0 = (1, -1)$. There are two independent operators simply because there are two independent diagonal 2×2 matrices, or, more physically, because there are just two independent reactions: $p\alpha \rightarrow p\alpha, n\alpha \rightarrow n\alpha$.

b) $T_1 = 1, T_2 = 0$

The operators available in isospin space for a pair of particles with isospins 1 and 0 (such as the $\pi\alpha$ system) are the identity and the isospin operator for the isospin one particle,

\mathbf{t} . The representation space of a particle with isospin one is three-dimensional, and so nine independent operators are available for the construction of the scattering amplitude in isospin space. These could, for example, be chosen as $\mathbf{1}$, \mathbf{t} , and $(\mathbf{t} \quad \mathbf{t})_M^2$ where $M = 2, 1, 0, -1, -2$. The tensor product is defined as

$$(V^{T_1} \quad U^{T_2})_M^T = \sum_{\alpha, \beta} C_{\alpha\beta M}^{T_1 T_2 T} V_{\alpha}^{T_1} U_{\beta}^{T_2}$$

where V^{T_1} and U^{T_2} are tensor operators of rank T_1 and rank T_2 . $C_{\alpha\beta M}^{T_1 T_2 T}$ is a Clebsch-Gordan coefficient. We have used the conventional spherical basis: $U_{+1} = -(\frac{U_x + iU_y}{\sqrt{2}})$, $U_{-1} = (\frac{U_x - iU_y}{\sqrt{2}})$, and $U_0 = U_z$. Charge conservation limits us to operators with total $T_z = M = 0$, assuming that there is no strangeness change. The operators with $M \neq 0$ will be useful when the second particle has nonzero isospin, as in the case of pion-nucleon scattering. We conclude that the most general charge-conserving amplitude or potential is a linear combination of the three operators [4] $\mathbf{1}$, t_0 , and $(\mathbf{t} \quad \mathbf{t})_0^2$. (The operator $(\mathbf{t} \quad \mathbf{t})_0^1$ is proportional to t_0 , and $(\mathbf{t} \quad \mathbf{t})_0^0$ is proportional to the identity, so these operators are redundant.) As in the previous example, there is only one isospin-conserving operator, $\mathbf{1}$. The tensor product $(\mathbf{t} \quad \mathbf{t})_0^2$ violates isospin symmetry, but is invariant under charge reflection. The remaining operator, t_0 , violates both isospin symmetry and charge-reflection symmetry.

There are three independent operators because there are three independent diagonal matrices, or, more physically, because there are only three independent reactions among the three charge channels: $\pi^+\alpha \rightarrow \pi^+\alpha$, $\pi^0\alpha \rightarrow \pi^0\alpha$, and $\pi^-\alpha \rightarrow \pi^-\alpha$.

$$c) T_1 = \frac{1}{2}, T_2 = \frac{1}{2}$$

As a final warm-up exercise, we study the scattering of two isospin $\frac{1}{2}$ particles, such as a KN system. As discussed in Sec.II, six amplitudes are required to describe all elastic and charge exchange scattering in this system. This is consistent with the fact that there are four elastic reactions and one charge exchange reaction (with two possible directions): K^+p , K^+n , K^0p , K^0n , $K^+n \leftrightarrow K^0p$. An independent set of six operators is $\mathbf{1}$, τ_{10} , τ_{20} , and $(\boldsymbol{\tau}_1 \quad \boldsymbol{\tau}_2)_0^I$, where $I = 0, 1, 2$. The first label on the symbol $\boldsymbol{\tau}$ refers to the particle and the second, the z-component. The tensor product with $I = 1$ is odd under time-reversal symmetry,

and reverses sign under interchange of the initial and final charge labels. As discussed in Appendix 1, the on-shell amplitude is independent of $(\boldsymbol{\tau}_1 \quad \boldsymbol{\tau}_2)_0^1$ while the potential could depend upon it. Of these operators, only the identity and the term $(\boldsymbol{\tau}_1 \quad \boldsymbol{\tau}_2)_0^0$ will conserve isospin symmetry (Class I in the nomenclature of Henley and Miller [2]). The terms τ_{10} and τ_{20} violate isospin symmetry and charge-reflection symmetry (Classes III and IV of Henley and Miller). The term with $I = 2$ violates isospin symmetry, but is invariant under charge reflection. By the Wigner-Eckart theorem the operator of rank two has nonzero matrix elements only between isospin one states of the KN system; it does not mix isospins (Class II of Henley and Miller [2]).

IV. The πN System

We now turn to our major application, the pion-nucleon system. Table 1 itemizes the independent reactions for each total charge Q . Reactions 7 and 8 are time-reversals of each other as are reactions 9 and 10.

Table 1. Charge states for πN Scattering.

R=Reaction no.	Q	Process
1	2	$\pi^+ p \rightarrow \pi^+ p$
2	1	$\pi^+ n \rightarrow \pi^+ n$
3	0	$\pi^- p \rightarrow \pi^- p$
4	-1	$\pi^- n \rightarrow \pi^- n$
5	1	$\pi^0 p \rightarrow \pi^0 p$
6	0	$\pi^0 n \rightarrow \pi^0 n$
7,8	1	$\pi^+ n \leftrightarrow \pi^0 p$
9,10	0	$\pi^- p \leftrightarrow \pi^0 n$

The most general operator in isospin space can be expressed as a function of t_α , τ_β , and the identity operator $\mathbf{1}$. In line with the previous examples it will then contain at most linear factors of the components of $\boldsymbol{\tau}$ and at most quadratic factors of the components of \mathbf{t} .

A complete set of operators [6] and their matrix elements in a charge basis are listed in

Table 2.

Table 2. Isospin Operators and Matrix Elements in the Charge Basis.

i	$(\theta_i)_0^I$	I	$\langle \pi^c N^d \theta_i \pi^a N^b \rangle$
1	$\sqrt{\frac{1}{6}} \mathbf{1}$	0	$\sqrt{\frac{1}{6}} \delta_{ca} \delta_{db}$
2	$\sqrt{\frac{1}{6}} \tau_0$	1	$\sqrt{\frac{2}{3}} b \delta_{ca} \delta_{db}$
3	$\frac{1}{2} t_0$	1	$\frac{1}{2} a \delta_{ca} \delta_{db}$
4	$\sqrt{\frac{1}{2}} (\mathbf{t} \quad \mathbf{t})_0^2$	2	$\sqrt{\frac{5}{6}} C_{a0c}^{121} \delta_{db}$
5, (9), 6	$\frac{1}{2} (\boldsymbol{\tau} \quad \mathbf{t})_0^I$	0, (1), 2	$\sqrt{\frac{3}{2}} C_{\alpha\beta 0}^{11I} C_{a\beta c}^{111} C_{bad}^{\frac{1}{2}\frac{1}{2}}$
7, (10), 8	$\sqrt{\frac{1}{2}} (\boldsymbol{\tau} \quad (\mathbf{t} \quad \mathbf{t})^2)_0^I$	1, (2), 3	$\sqrt{\frac{5}{2}} C_{\alpha\beta 0}^{12I} C_{a\beta c}^{121} C_{bad}^{\frac{1}{2}\frac{1}{2}}$

In the expression for the matrix elements we have defined $\alpha = d - b$ and $\beta = c - a$ where a and c are $T_{\pi z}$ and b and d are T_{Nz} . The operators are classified by their total isospin I . The two rank-0 operators θ_1 and θ_5 are the only operators which conserve isospin. The traditional ‘‘isoscalar’’ operator $\mathbf{1}$ is proportional to θ_1 . The ‘‘isovector’’ operator $\boldsymbol{\tau} \cdot \mathbf{t}$, which transforms as a scalar under isospin rotations, is proportional to θ_5 . Operators of the form of terms 7-10, but with a \mathbf{t} and a $\boldsymbol{\tau}$ reversed, are related by a recoupling to other operators in the set. Operators of the form of term 4 but with I equal to 0 or 1 are linear combinations of operators 1 and 3 as was discussed in Sec.IIIb.

In the charge basis the matrix elements of operators 1-8 are symmetric under the interchange of charge indices: $(a, b) \leftrightarrow (c, d)$, while the matrix elements of operators 9 and 10 reverse sign under the interchange. For time-reversal symmetry to hold the coefficients of operators 1-8 must be even under time reversal while those of operators 9 and 10 must be odd. These last two operators are needed only for off-shell T-matrices or for potentials. See Appendix 1.

The final column of Table 2 indicates the matrix elements of the corresponding operators in the charge basis. Operators 1-4 do not contribute to charge exchange because they do not involve both \mathbf{t} and $\boldsymbol{\tau}$. The most general potential or on-shell amplitude for an individual partial wave is a linear combination of these operators: $V = \sum_{i=1}^{10} v_i \theta_i$ and $T = \sum_{i=1}^8 a_i \theta_i$.

In addition to having simple properties under time reversal, the operators defined in Table 2 are simple under charge reflection:

$$e^{i\pi T_y} \theta_i e^{-i\pi T_y} = (-1)^I \theta_i.$$

Thus in Table 2 operators $i = 1$ and 5 conserve isospin, operators $i = 4, 6,$ and 10 do not conserve isospin but are invariant under charge reflection, while operators $i = 2, 3, 7,$ and 9 neither conserve isospin nor are invariant under charge reflection. Operator 8 does not conserve isospin and is not invariant under charge reflection, but because it has isospin 3 it connects only $I = \frac{3}{2}$ states.

An amplitude (or potential) $\Sigma_i a_i \langle \pi^c N^d | \theta_i | \pi^a N^b \rangle$ may be constructed from the coefficients given in the final column of Table 2. Each set $(a, b; c, d)$ may be identified with a reaction number R defined in Table 1. The T -matrix corresponding to a particular reaction may be written as

$$T_R = \langle \pi^c N^d | T | \pi^a N^b \rangle = \Sigma_i \Theta_{Ri} a_i$$

where we have defined $\Theta_{Ri} = \langle \pi^c N^d | \theta_i | \pi^a N^b \rangle$. The explicit values of Θ_{Ri} for all pion-nucleon elastic and charge-exchange reactions are given in Table 3. A square root must be taken of the magnitude of each entry. For example, the entry $-\frac{1}{20}$ means $-\sqrt{\frac{1}{20}}$.

Table 3. Values of $\Theta_{Ri} = \langle \pi^c N^d | \theta_i | \pi^a N^b \rangle$

Reaction	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
$\pi^+ p \rightarrow \pi^+ p$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{20}$	0	0
$\pi^+ n \rightarrow \pi^+ n$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{30}$	$-\frac{1}{20}$	0	0
$\pi^- p \rightarrow \pi^- p$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{20}$	0	0
$\pi^- n \rightarrow \pi^- n$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{30}$	$-\frac{1}{20}$	0	0
$\pi^0 p \rightarrow \pi^0 p$	$\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{3}$	0	0	$\frac{2}{15}$	$-\frac{1}{5}$	0	0
$\pi^0 n \rightarrow \pi^0 n$	$\frac{1}{6}$	$-\frac{1}{6}$	0	$-\frac{1}{3}$	0	0	$-\frac{2}{15}$	$\frac{1}{5}$	0	0
$\pi^+ n \rightarrow \pi^0 p$	0	0	0	0	$-\frac{1}{6}$	$-\frac{1}{12}$	$-\frac{3}{20}$	$-\frac{1}{10}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$\pi^0 p \rightarrow \pi^+ n$	0	0	0	0	$-\frac{1}{6}$	$-\frac{1}{12}$	$-\frac{3}{20}$	$-\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{4}$
$\pi^- p \rightarrow \pi^0 n$	0	0	0	0	$-\frac{1}{6}$	$-\frac{1}{12}$	$\frac{3}{20}$	$\frac{1}{10}$	$\frac{1}{4}$	$-\frac{1}{4}$
$\pi^0 n \rightarrow \pi^- p$	0	0	0	0	$-\frac{1}{6}$	$-\frac{1}{12}$	$\frac{3}{20}$	$\frac{1}{10}$	$-\frac{1}{4}$	$\frac{1}{4}$

If the amplitudes T_R are known and we wish to determine the coefficient a_i corresponding to a given isospin operator θ_i , we need the inverse of Θ_{Ri} . The operators θ_i have been normalized to make the matrix Θ_{Ri} orthogonal [5]. Consequently, we have

$$a_i = \sum_R T_R \Theta_{Ri}$$

i.e. the inverse of Θ_{Ri} is its transpose. Thus the columns of Table 3 give the linear combinations of the reaction amplitudes needed to extract the coefficients a_i of the individual isospin-space operators.

For the computation of on-shell T -matrices the final two columns of Table 3 may be omitted. In this case rows seven and eight are equal ($\pi^+ n \leftrightarrow \pi^0 p$), as are rows nine and ten ($\pi^- p \leftrightarrow \pi^0 n$).

From the first column of Table 3 we see that the sum of the six elastic reactions gives $\sqrt{6}a_1$. The difference of the three elastic reaction with proton targets and the three with neutron targets isolates $\sqrt{6}a_2$. $2a_3$ is isolated by taking differences in the π^+ and the π^- elastic processes. $2\sqrt{3}a_4$ is determined by the difference of the sum of the charge-pion elastic amplitudes and twice the sum of the π^0 elastic amplitudes. a_5 and a_6 measure the

difference in the sum of the charge-exchange reactions and differences in the elastic scattering amplitudes. The next two isospin amplitudes are similar but require the differences of the charge-exchange amplitudes. The final pair of isospin amplitudes contribute only to charge exchange potentials; they are antisymmetric under time reversal.

Finally, we evaluate the matrix elements of the 10 independent isospin operators in a basis of definite total isospin. The matrix elements are related to those of the charge basis by

$$\langle 1\frac{1}{2}T'T_3|\theta|1\frac{1}{2}TT_3\rangle = \sum_{abcd} C_{abT_3}^{1\frac{1}{2}T} C_{cdT_3}^{1\frac{1}{2}T'} \langle \pi^c N^d|\theta|\pi^a N^b\rangle$$

where the symbols $1\frac{1}{2}$ refer to the pion and nucleon isospin. The matrix elements are given in Table 4. Note that this matrix, like that in Table 3, is orthogonal. As before, the square root of the magnitude of each entry should be taken.

Table 4. Matrix Elements of θ_i in the Isospin Basis

T	T'	T_3	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{54}$	$-\frac{1}{9}$	0	$\frac{1}{3}$	0	$-\frac{10}{27}$	0	0	0
			$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{54}$	$\frac{1}{9}$	0	$\frac{1}{3}$	0	$\frac{10}{27}$	0	0
$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	0	$-\frac{4}{27}$	$\frac{1}{18}$	$-\frac{1}{6}$	0	$\frac{1}{12}$	$-\frac{5}{108}$	0	$\frac{1}{4}$	$-\frac{1}{4}$
			$\frac{1}{2}$	0	$-\frac{4}{27}$	$\frac{1}{18}$	$\frac{1}{6}$	0	$-\frac{1}{12}$	$-\frac{5}{108}$	0	$\frac{1}{4}$
$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{4}{27}$	$\frac{1}{18}$	$-\frac{1}{6}$	0	$\frac{1}{12}$	$-\frac{5}{108}$	0	$-\frac{1}{4}$	$\frac{1}{4}$
			$\frac{1}{2}$	0	$-\frac{4}{27}$	$\frac{1}{18}$	$\frac{1}{6}$	0	$-\frac{1}{12}$	$-\frac{5}{108}$	0	$-\frac{1}{4}$
$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{30}$	$-\frac{1}{20}$	0	0
			$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{54}$	$-\frac{1}{36}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{270}$	$\frac{9}{20}$	0
			$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{54}$	$\frac{1}{36}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{270}$	$-\frac{9}{20}$	0
			$\frac{3}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{20}$	0

Because the operators $\theta_1 = \frac{1}{\sqrt{6}}\mathbf{1}$ and $\theta_5 = \frac{1}{2}(\boldsymbol{\tau} \cdot \mathbf{t})_0^0$ conserve isospin their matrix elements vanish when the initial and final total isospins are unequal. Operator $\theta_8 = \frac{1}{\sqrt{2}}(\boldsymbol{\tau} \cdot (\mathbf{t} \cdot \mathbf{t}))_0^3$ contributes only if both initial and final isospins are $\frac{3}{2}$, as was noted earlier. The operators

$\theta_4 = \frac{1}{\sqrt{2}}(\mathbf{t} \cdot \mathbf{t})_0^2$ and $\theta_6 = \frac{1}{2}(\boldsymbol{\tau} \cdot \mathbf{t})_0^2$ have isospin 2, and their matrix elements vanish if the initial and final isospins are $\frac{1}{2}$. The matrix elements with T and T' interchanged are the same except for the final two amplitudes, for which there is a sign reversal. Under charge reflection $T_3 \rightarrow -T_3$ the matrix elements acquire the phase factor $(-1)^{T+T'-1+I}$, as can be seen from the defining equation earlier in this paragraph.

We end this section by applying the formalism to reproduce some of the standard results in the case that isospin is conserved (for which only a_1 and a_5 are nonzero). In the usual formalism the isospin-conserving T -matrix is expressed as $T = A_s \mathbf{1} + A_v \boldsymbol{\tau} \cdot \mathbf{t}$ where A_s is the ‘‘isoscalar’’ amplitude and A_v is the ‘‘isovector’’ amplitude. This is to be compared with $T = a_1 \theta_1 + a_5 \theta_5$. Since

$$\theta_1 = \sqrt{\frac{1}{6}} \mathbf{1}$$

and

$$\theta_5 = \frac{1}{2}(\boldsymbol{\tau} \cdot \mathbf{t})_0^2 = -\sqrt{\frac{1}{12}} \boldsymbol{\tau} \cdot \mathbf{t}$$

we identify $A_s = \sqrt{\frac{1}{6}} a_1$ and $A_v = -\sqrt{\frac{1}{12}} a_5$. For example, from rows 3, 9, and 1 of Table 3 we then recover the standard forms:

$$T_{\pi^- p \rightarrow \pi^- p} = \sqrt{\frac{1}{6}} a_1 + \sqrt{\frac{1}{12}} a_5 = A_s - A_v$$

$$T_{\pi^- p \rightarrow \pi^0 n} = -\sqrt{\frac{1}{6}} a_5 = \sqrt{2} A_v$$

and

$$T_{\pi^+ p \rightarrow \pi^+ p} = A_s + A_v.$$

The isoscalar and isovector amplitudes are also easily expressed in terms of amplitudes of pure isospin. From Table 4 we obtain, again assuming isospin conservation,

$$T_{\frac{1}{2}} = \sqrt{\frac{1}{6}} a_1 - \sqrt{\frac{1}{3}} a_5 = A_s - 2A_v$$

and

$$T_{\frac{3}{2}} = \sqrt{\frac{1}{6}}a_1 + \sqrt{\frac{1}{12}}a_5 = A_s + A_v.$$

We could, of course, have defined θ_1 to be the identity and θ_5 to be $\boldsymbol{\tau} \cdot \mathbf{t}$ so that a_1 and a_5 were the standard isoscalar and isovector amplitudes. The present definitions have the advantage that the matrices in Tables 3 and 4 are orthogonal so that their inverses are trivial [7].

Further mathematical aspects of the isospin operators θ_i such as matrix representations, trace theorems, and the multiplication table are developed in Appendix 3.

V. Elementary Examples of Isospin-Violating Processes

The classic example of an isospin-violating interaction is the Coulomb potential. For the πN system it is proportional to the product of the nucleon and pion charges, and so has the structure $\frac{1}{2}(1 + \tau_0)t_0$. Expressed in terms of the tensor operators of Table 2 this interaction is $\theta_3 - \sqrt{\frac{1}{3}}\theta_5 + \sqrt{\frac{2}{3}}\theta_6$. It is easily seen from Table 3 that this combination gives the product of the nucleon and pion charge (in units of e^2).

Based on current knowledge of charge-symmetry breaking in nucleon-nucleon scattering [8] we expect that meson-mixing models will also lead to charge-symmetry violations in pion-nucleon scattering. Our first example is $\rho\omega$ mixing. The lowest-order graph is the vector-meson exchange graph which consists of factors corresponding to the ωNN vertex, the ω propagator, the $\omega\rho^0$ coupling, the ρ propagator, and the $\rho\pi\pi$ vertex. The only nontrivial isospin dependence is a factor t_0 which accompanies the $\rho^0\pi\pi$ vertex. Thus the operator is of class 3 in Table 2. Because the exchanged vector mesons are neutral, this process does not contribute in first order to charge exchange. The graph with the ω and ρ^0 interchanged does not occur because G-parity conservation eliminates the strong $\omega\pi\pi$ vertex. (In this model the only isospin breaking is in the $\omega\rho^0$ transition; all other vertices conserve isospin.)

Another similar example is $\eta\pi$ mixing. The pion-nucleon scattering process proceeds, for example, through an eta-production process (π, η) followed by an isospin-symmetry breaking

transition $\eta \rightarrow \pi^0$. The process only occurs if at least one of the pions is neutral; on free nucleons this process is experimentally possible only in charge exchange, although $\pi^0 N \rightarrow \pi^0 N$ could occur within a nuclear target. The amplitude is most easily evaluated in the charge basis. Using the notation $\pi^a N^b \rightarrow \pi^c N^d$ the amplitude is $C_{bad}^{\frac{1}{2}1\frac{1}{2}} \delta_{c0}$ plus a term in which (a,b) is interchanged with (c,d) to enforce time-reversal invariance. The Kronecker delta is necessary because the pion which mixes with the eta is necessarily neutral. In π^0 elastic scattering the η could attach to either the incident or the outgoing leg, so both terms will contribute. In charge-exchange processes only one or the other of the two terms will contribute. Expressed in terms of the operators in Table 2, the $\pi\eta$ mixing graph is of the form $\sqrt{\frac{8}{9}}\theta_2 + \sqrt{\frac{40}{9}}\theta_7$, as is shown in Appendix 4. Because it is a mixture of two of the operators with isospin 1, it reverses its sign under charge reflection.

As we have seen, the $\omega\rho^0$ mixing graph transforms as θ_3 . This process may, however, be preceded or followed (or both) by isospin-conserving strong interactions. These final- and initial-state interactions (FISI) may renormalize the isospin-violating graph, inducing other isospin-breaking operators. Working to lowest order in isospin-breaking interactions, the FISI will conserve isospin, and so will induce only those operators which have the same tensor properties as the original isospin breaking interaction. Since θ_3 has isospin one, we expect that the FISI will introduce at most an additional dependence proportional to θ_2 and θ_7 . Similarly the $\eta\pi^0$ mixing graph, which we have seen is a linear combination of θ_2 and θ_7 , may have an induced θ_3 dependence. Thus both meson-mixing graphs depend, in principle, on the same three isospin-violating operators. It is conceivable that the FISI will not induce all of the possible operators of the same isospin structure as the original graph, however. For example, if the only isospin-conserving behavior in the FISI comes from the term $\theta_1 = \mathbf{1}$ then it is obvious that although the magnitude of the graph may be altered, there will be no new isospin structure induced. We have shown in Appendix 4 that if the other isospin-conserving operator, $\theta_5 = \frac{1}{2}(\boldsymbol{\tau} \cdot \mathbf{t})_0^0$, is non-negligible all three of the rank one ($I = 1$) amplitudes (2,3, and 7) can be expected to be present with the exception of

the time-reversal odd isospin amplitude 9, which cannot be induced by time-reversal even amplitudes.

VI. Tests of Isospin Symmetry Breaking

One advantage of the operators introduced here is that a partial hierarchy of strengths is established according to the tensor ranks. The operators θ_1 and θ_5 have rank 0 and are of order 0 in isospin-symmetry breaking. All of the other operators are at least of first order in the smallness parameter, but can be mixed by the strong interactions with θ_1 and θ_5 . Since the ρ - ω and π - η mixing are of rank 1, they can be spread among all operators of the same rank by isospin-conserving initial- and final-state interactions, and hence they contribute (in first order) to a_2 , a_3 , and a_7 , as is described in the appendix.

The Coulomb interaction, being a combination of rank 0, 1, and 2 operators, is mixed by the isospin-conserving conserving potentials (θ_1 and θ_5) among all operators of these ranks.

Except for mixing due to the Coulomb interaction, which contains a rank-0 component, the coefficients of θ_1 and θ_5 , are free of first-order contributions from isospin-symmetry-breaking terms. This follows because the lowest-order corrections to a rank-0 operator arise when it is combined with the product of at least two isospin-breaking operators (of the same rank) coupled to isospin zero. Hence, such corrections are at least quadratic in the isospin-breaking parameter.

By means of a series of experiments one can, in principle, measure each of the isospin-breaking terms and hope to infer the source of the breaking. A complete analysis of the system and survey of the data is beyond the scope of the present work, but we present the following as an introduction to this subject.

Extensive pion-deuteron scattering experiments have been performed to test isospin conservation in pion-nucleon interactions. In the impulse approximation the scattering is calculated by adding reactions 1 and 2 (or 3 and 4) together. The only contributing amplitudes are a_1 , a_3 and a_4 , because an isospin-zero target cannot have any dependence on τ . By comparing π^+ to π^- scattering, one isolates the coefficient a_3 , but not a_4 , since it has the

same sign relative to a_1 in all cases. From Table 2 one can observe that only the scattering of a π^0 will allow the separation of a_4 from a_1 .

Using the optical theorem, we see that the difference in the total cross sections for π^- and π^+ on the deuteron is given by

$$\sigma^- - \sigma^+ = \frac{16\pi}{k} \text{Im}(a_3(0^\circ)).$$

From the experiment of Pedroni et al. [9], which measured this difference of total cross sections in the 3-3 resonance region, we find that the maximum value of $\text{Im}(a_3(0^\circ))$ is of the order of .03 fm.

By comparing the differential cross section in the same energy region it is possible to get another estimate of a_3 but at finite angles. From the equation

$$y = \frac{\sigma^-(\theta) - \sigma^+(\theta)}{\sigma^-(\theta) + \sigma^+(\theta)} = -2\text{Re}\left(\frac{a_3}{a_1}\right),$$

a condition can be obtained on a_3 , but the absolute value depends on the relative phase of a_3 and a_1 . If they have the same phase there will be a maximal effect observed. While this condition is useful, we must use specific models to interpret the result of Smith et al. [10] that $|y| \lesssim .01$.

By using ${}^3\text{He}$ and ${}^3\text{H}$ targets the charge-exchange reaction cross section at 0° can be measured both on protons and neutrons at the s-p interference dip [11,12], producing a very precise determination of the difference in energy of the minimum for the two cases. As can be seen from Table 3, the difference in the two amplitudes is given by

$$a_{\pi^-p \rightarrow \pi^0n} - a_{\pi^+n \rightarrow \pi^0p} = \sqrt{\frac{3}{5}} a_7 + \sqrt{\frac{2}{5}} a_8.$$

If we assume that a_8 is small, then we can determine the value of a_7 from this measurement. (From mixing of the amplitudes a_8 is of second order, and it is difficult to find a particle exchange to give it directly.)

Tests of the triangle relationship have been made by Wightman *et al.* [13]. These tests compare $2\sigma_{\text{cex}}$ with $(\sqrt{\sigma_{\pi^+p}} + \sqrt{\sigma_{\pi^-p}})^2$ and $(\sqrt{\sigma_{\pi^+p}} - \sqrt{\sigma_{\pi^-p}})^2$ to look for deviations from the relation

$$\sqrt{2}A_{cex} = A_{\pi^+p} - A_{\pi^-p},$$

which results from isospin conservation. Subtracting the right side of this expression from the left, we see that the isospin breaking implied by a failure of this equation is given by

$$-a_3 - \sqrt{\frac{3}{2}} a_6 + \sqrt{\frac{3}{10}} a_7 + \sqrt{\frac{1}{5}} a_8.$$

Using pion-deuteron scattering to determine a_3 and charge exchange to find a_7 , we see that the breaking of the triangle relation will give limits on a_6 .

The scattering of neutral pions can be studied in the nuclear medium [14], which is the only way to determine a_4 , but it is not clear at this point if the experimental techniques and/or theory are adequate to extract a useful difference.

It is the same combination of amplitudes needed for deuteron scattering that is also necessary for the extraction of the sigma term, so that a knowledge of the value of a_4 is essential for the reliable estimation of this quantity. Hence, in the absence of neutral pion scattering there is a correction to be made which can only be determined from model calculations.

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Appendices

Appendix 1. Off-shell *vs.* On-shell Amplitudes

It is important to distinguish between potentials (or off-shell T matrices) and on-shell T -matrices. For an on-shell amplitude only the total energy E , the incident and outgoing beam directions, and the spin projections are independent variables. We now restrict our attention to the scattering of a spin 0 particle and a spin 1/2 particle. We will refer to these particles as π and N , but the results of this paragraph are equally applicable to KN systems, for example. The vectors available for the construction of the on-shell amplitudes are the unit vector in the incident direction $\hat{\mathbf{k}}$, the unit vector in the scattered direction $\hat{\mathbf{k}}'$, and the nucleon spin operator $\boldsymbol{\sigma}$. For an on-shell amplitude the magnitudes of both \mathbf{k} and \mathbf{k}' are fixed by the total energy E although in charge-exchange reactions the magnitudes k and k' need not be equal. It is easy to see that there are no rotational invariants that can be constructed from this set of vectors which are both even under parity and are odd under time reversal. (Note that under time reversal $\mathbf{k} \leftrightarrow -\mathbf{k}'$ and $\boldsymbol{\sigma} \rightarrow -\boldsymbol{\sigma}$, while under parity both \mathbf{k} and \mathbf{k}' reverse signs and $\boldsymbol{\sigma}$ is unchanged.) Thus only those isospin amplitudes which are even under time reversal are needed for the description of the on-shell pion-nucleon amplitude. The off-shell T -matrix depends on the initial and final momenta \mathbf{k}, \mathbf{k}' as well as on $\boldsymbol{\sigma}$. Since the magnitudes of the momenta are now independent we can introduce the factor $k^2 - k'^2$, for example, to obtain a coefficient which is odd under time-reversal but is even under parity. As another related example, a coefficient of the form $V(r)\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r}V(r)$ is even under parity but is odd under time-reversal. We conclude that the most general form for the pion-nucleon potential involves all independent isospin operators, whatever their properties are under time reversal. In contrast, the most general form of the on-shell amplitude for pion-nucleon scattering involves only those isospin operators which are even under time reversal. Since in the NN system there are two independent spin operators, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, the on-shell amplitude could contain a term proportional to

$$(\hat{\mathbf{k}} \times \hat{\mathbf{k}}') \cdot (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2),$$

which is even under parity and odd under time-reversal [3].

Appendix 2. Unitarity

The coefficients of each isospin amplitude may be composed of several partial wave or spin amplitudes. As an elementary example, consider the case in which only the elastic and charge-exchange scattering occurs, and further assume that the partial waves are uncoupled (as in pion-nucleon scattering, but unlike the coupled triplet states in nucleon-nucleon scattering). A unitary S-matrix may be defined by a hermitian (for unitarity), symmetric (for time-reversal invariance) K matrix. Together these conditions require the K matrix to be real and symmetric. Let us consider an example in which orbital angular momentum is conserved. If there are C_Q channels, then the K matrix will be a real and symmetric $C_Q \times C_Q$ matrix. Such a matrix possesses $C_Q(C_Q + 1)/2$ real parameters, a number equal to N_Q , the number of independent isospin amplitudes found in Sec.II.

For example, consider a single partial wave in pion-nucleon scattering below the pion-production threshold. As discussed in Sec.II $C_{-1} = C_2 = 1$ and the $Q = -1$ and $Q = 2$ channels are each described by a single phase shift. There are two channels with $Q = 1$ and two with $Q = 0$, so $C_1 = C_0 = 3$. These parameters are often taken as two phase shifts and a mixing angle. Counting up, we need $(1 + 1 + 3 + 3)$ real parameters per partial wave to describe the S-matrix.

Appendix 3. Further Properties of θ_i .

The anticommutator product of the isospin operators defined in Table 2 forms a closed algebra. The multiplication table is given in Table 5ab. Table 5a gives the closed subalgebra of operators 1-8, and Table 5b gives the additional anticommutators needed when operators 9 and 10 are required. These tables may be used, for example, in constructing iterates of a potential, as in the Born approximation (see Appendix 4). As usual, a square root of the magnitude of each term is necessary.

Table 5a. Coefficients C_{ijk} : $\frac{1}{2}\{\theta_i, \theta_j\} = \sum_k C_{ijk}\theta_k$

k	1	2	3	4	5	6	7	8
i j								
2 2	$\frac{1}{6}$							
2 3					$-\frac{1}{18}$	$\frac{1}{9}$		
2 4							$-\frac{1}{15}$	$\frac{1}{10}$
2 5			$-\frac{1}{18}$					
2 6			$\frac{1}{9}$					
2 7				$-\frac{1}{15}$				
2 8				$\frac{1}{10}$				
3 3	$\frac{1}{6}$			$\frac{1}{12}$				
3 4			$\frac{1}{12}$					
3 5	$-\frac{1}{18}$						$\frac{5}{72}$	
3 6	$\frac{1}{9}$						$-\frac{1}{720}$	$\frac{3}{40}$
3 7					$\frac{5}{72}$	$-\frac{1}{720}$		
3 8						$\frac{3}{40}$		
4 4	$\frac{1}{6}$			$-\frac{1}{12}$				
4 5							$-\frac{1}{24}$	
4 6						$-\frac{1}{24}$	$\frac{1}{48}$	
4 7	$-\frac{1}{15}$						$-\frac{49}{1200}$	$\frac{1}{200}$
4 8	$\frac{1}{10}$						$\frac{1}{200}$	$-\frac{4}{75}$
5 5	$\frac{1}{6}$				$\frac{1}{12}$			
5 6				$-\frac{1}{24}$		$-\frac{1}{48}$		
5 7			$\frac{5}{72}$				$\frac{3}{16}$	
5 8								$-\frac{1}{12}$
6 6	$\frac{1}{6}$			$\frac{1}{48}$	$-\frac{1}{48}$	$-\frac{1}{24}$		
6 7			$-\frac{1}{720}$				$\frac{3}{200}$	$-\frac{9}{400}$
6 8			$\frac{3}{40}$				$-\frac{9}{400}$	$-\frac{8}{75}$
7 7	$\frac{1}{6}$			$-\frac{49}{1200}$	$\frac{3}{16}$	$\frac{3}{200}$		19

Table 5b. Coefficients C_{ijk} : $\frac{1}{2}\{\theta_i, \theta_j\} = \sum_k C_{ijk}\theta_k$

k	1	2	3	4	5	6	7	8	9	10
i	j									
2	9									
2	10									
3										$\frac{1}{16}$
3									$\frac{1}{16}$	
4								$-\frac{1}{48}$		
4								$-\frac{1}{48}$		
5								$\frac{1}{48}$		
5								$\frac{1}{48}$		
6								$-\frac{1}{24}$		
6								$-\frac{1}{24}$		
7									$\frac{3}{40}$	
7								$\frac{3}{40}$		
8									$-\frac{9}{80}$	
8								$-\frac{9}{80}$		
9	9	$-\frac{1}{6}$		$\frac{1}{48}$	$-\frac{1}{48}$	$\frac{1}{24}$				
9	10		$-\frac{1}{16}$				$-\frac{3}{40}$	$\frac{9}{80}$		
10	10	$-\frac{1}{6}$		$\frac{1}{48}$	$-\frac{1}{48}$	$\frac{1}{24}$				

The coefficients for which $i = 1$ are $C_{1jk} = \frac{1}{\sqrt{6}}\delta_{jk}$. The remaining coefficients may be obtained from the symmetry relation $C_{ijk} = C_{jik}$.

An efficient way of representing the matrix elements of $\langle \pi^c N^d | \theta_i | \pi^a N^b \rangle$ in the charge basis is given in Table 3. An alternative method is to express the matrix elements directly as a set of 6×6 matrices corresponding to the six channels. For this purpose it is most convenient to order the channels as π^+p , π^+n , π^0p , π^-n , π^-p , and π^0n . With this ordering the second three channels are obtained from the first three by charge reflection. The matrix

representation of the operators will be in the form of blocks along the diagonal: 1×1 (with $Q=2$), 2×2 (with $Q=1$), 1×1 (with $Q=-1$), and 2×2 (with $Q=0$). The general operator which conserves charge will then be

$$\theta = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B + \mathbf{C} \cdot \boldsymbol{\sigma} & 0 & 0 \\ 0 & 0 & A' & 0 \\ 0 & 0 & 0 & B' + \mathbf{C}' \cdot \boldsymbol{\sigma} \end{pmatrix}$$

where A, B, A', B' are scalar parameters, \mathbf{C}, \mathbf{C}' are vector parameters, (*i.e.* a total of 10 parameters), and the components of $\boldsymbol{\sigma}$ are the Pauli matrices. The primed quantities are equal to the unprimed ones if the operator is even under charge reflection, and they are the negative of the unprimed ones if the operator is odd under charge reflection. If time-reversal invariance is imposed on these operators, they must be symmetric. In this case C_y and C'_y will both be zero, and only the usual eight parameters remain. The explicit form of the matrices, constructed from the Table 3, follows.

$$\theta_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \theta_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\theta_3 = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1 \\ & & & & -1 \\ & & & & & 0 \end{pmatrix} \quad \theta_4 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 1 \\ & & & & 1 \\ & & & & & -2 \end{pmatrix}$$

$$\theta_5 = \frac{1}{\sqrt{12}} \begin{pmatrix} -1 & & & & & \\ & 1 & -\sqrt{2} & & & \\ & -\sqrt{2} & 0 & & & \\ & & & -1 & & \\ & & & & 1 & -\sqrt{2} \\ & & & & -\sqrt{2} & 0 \end{pmatrix} \quad \theta_6 = \frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{2} & & & & & \\ & -\sqrt{2} & -1 & & & \\ & -1 & 0 & & & \\ & & & \sqrt{2} & & \\ & & & & -\sqrt{2} & -1 \\ & & & & -1 & 0 \end{pmatrix}$$

$$\theta_7 = \frac{1}{\sqrt{30}} \begin{pmatrix} -1 & & & & & & & \\ & 1 & -\frac{3\sqrt{2}}{2} & & & & & \\ & -\frac{3\sqrt{2}}{2} & 2 & & & & & \\ & & & 1 & & & & \\ & & & & -1 & \frac{3\sqrt{2}}{2} & & \\ & & & & \frac{3\sqrt{2}}{2} & -2 & & \\ & & & & & & & \end{pmatrix} \quad \theta_8 = \frac{1}{\sqrt{20}} \begin{pmatrix} 1 & & & & & & & \\ & -1 & -\sqrt{2} & & & & & \\ & -\sqrt{2} & -2 & & & & & \\ & & & -1 & & & & \\ & & & & 1 & \sqrt{2} & & \\ & & & & \sqrt{2} & 2 & & \\ & & & & & & & \end{pmatrix}$$

$$\theta_9 = \frac{1}{2} \begin{pmatrix} 0 & & & & & & & \\ & 0 & -1 & & & & & \\ & 1 & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & -1 & 0 & & \end{pmatrix} \quad \theta_{10} = \frac{1}{2} \begin{pmatrix} 0 & & & & & & & \\ & 0 & -1 & & & & & \\ & 1 & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & -1 & & \\ & & & & 1 & 0 & & \end{pmatrix}$$

Notice that the first eight matrices are symmetric while the last two are skew-symmetric. (The matrices could be made hermitian if the last two were multiplied by i). The matrices satisfy the relation $tr(\theta_i^T \theta_j) = \delta_{ij}$ where the superscript T stands for transpose. The anti-commutation relations in Table 5 may be verified directly from this representation from the relation

$$C_{ijk} = \frac{1}{2} Tr(\theta_k^T (\theta_i \theta_j + \theta_j \theta_i)).$$

The matrix product of $\theta_i \theta_j$ also forms a closed algebra although operators 1-8 do not form a subalgebra because the product of two symmetric matrices is not necessarily symmetric. Since any operator in isospin space for the πN system can be written as a linear combination of the operators θ_i we have

$$\theta_i \theta_j = \sum_k D_{ijk} \theta_k.$$

Using the orthogonality relation given above we have

$$D_{ijk} = Tr(\theta_k^T \theta_i \theta_j).$$

The results are presented in Tables 6a and 6b where the format is the same as for Tables 5a and 5b.

Notes for Table 6a. We have given only those elements for which $i \leq j$. The portion of the table for which $k \leq 8$ is identical to Table 5a and is valid even if i and j are interchanged. The new entries in columns 9 and 10 reverse sign under interchange of i and j .

Table 6a. Coefficients D_{ijk} : $\theta_i\theta_j = \sum_k D_{ijk}\theta_k$

k	1	2	3	4	5	6	7	8	9	10
i	j									
2	2	$\frac{1}{6}$								
2	3				$-\frac{1}{18}$	$\frac{1}{9}$				
2	4						$-\frac{1}{15}$	$\frac{1}{10}$		
2	5		$-\frac{1}{18}$						$-\frac{1}{9}$	
2	6		$\frac{1}{9}$						$-\frac{1}{18}$	
2	7			$-\frac{1}{15}$						$-\frac{1}{10}$
2	8			$\frac{1}{10}$						$-\frac{1}{15}$
3	3	$\frac{1}{6}$			$\frac{1}{12}$					
3	4		$\frac{1}{12}$							
3	5	$-\frac{1}{18}$					$\frac{5}{72}$		$\frac{1}{24}$	
3	6	$\frac{1}{9}$					$-\frac{1}{720}$	$\frac{3}{40}$	$\frac{1}{48}$	
3	7				$\frac{5}{72}$	$-\frac{1}{720}$				$\frac{3}{80}$
3	8					$\frac{3}{40}$				$\frac{1}{40}$
4	4	$\frac{1}{6}$		$-\frac{1}{12}$						
4	5					$-\frac{1}{24}$				$\frac{1}{8}$
4	6					$-\frac{1}{24}$	$\frac{1}{48}$			$\frac{1}{16}$
4	7	$-\frac{1}{15}$					$-\frac{49}{1200}$	$\frac{1}{200}$	$\frac{9}{80}$	
4	8	$\frac{1}{10}$					$\frac{1}{200}$	$-\frac{4}{75}$	$\frac{3}{40}$	
5	5	$\frac{1}{6}$			$\frac{1}{12}$					
5	6			$-\frac{1}{24}$		$-\frac{1}{48}$				$\frac{1}{16}$
5	7		$\frac{5}{72}$				$\frac{3}{16}$		$\frac{5}{144}$	
5	8							$-\frac{1}{12}$		
6	6	$\frac{1}{6}$		$\frac{1}{48}$	$-\frac{1}{48}$	$-\frac{1}{24}$				
6	7		$-\frac{1}{720}$				$\frac{3}{200}$	$-\frac{9}{400}$	$-\frac{1}{90}$	
6	8		$\frac{3}{40}$				$-\frac{9}{400}$	$-\frac{8}{75}$	$-\frac{3}{80}$	
7	7	$\frac{1}{6}$		$-\frac{49}{1200}$	$\frac{3}{16}$	$\frac{3}{200}$				24

Notes for Table 6b. Columns 9 and 10 and the rows for which $i > 8$ are identical to those of Table 5b (this includes all of the nonzero elements of Table 5b) and are not altered through the interchange of i and j . The new terms (with $k = 1 - 8$ and $i \leq 8$) change sign under the interchange of i and j .

Table 6b. Coefficients $D_{ijk} : \theta_i \theta_j = \sum_k D_{ijk} \theta_k$

k	1	2	3	4	5	6	7	8	9	10
i	j									
2	9				$-\frac{1}{9}$	$-\frac{1}{18}$				
2	10						$-\frac{1}{10}$	$-\frac{1}{15}$		
3	9				$\frac{1}{24}$	$\frac{1}{48}$				$\frac{1}{16}$
3	10						$\frac{3}{80}$	$\frac{1}{40}$	$\frac{1}{16}$	
4	9						$\frac{9}{80}$	$\frac{3}{40}$	$-\frac{1}{48}$	
4	10				$\frac{1}{8}$	$\frac{1}{16}$				$-\frac{1}{48}$
5	9	$\frac{1}{9}$	$-\frac{1}{24}$				$\frac{5}{144}$		$\frac{1}{48}$	
5	10			$-\frac{1}{8}$		$\frac{1}{16}$				$\frac{1}{48}$
6	9	$\frac{1}{18}$	$-\frac{1}{48}$				$-\frac{1}{90}$	$-\frac{3}{80}$	$-\frac{1}{24}$	
6	10			$-\frac{1}{16}$	$-\frac{1}{16}$					$-\frac{1}{24}$
7	9			$-\frac{9}{80}$	$-\frac{5}{144}$	$\frac{1}{90}$				$\frac{3}{40}$
7	10	$\frac{1}{10}$	$-\frac{3}{80}$					$-\frac{1}{48}$	$\frac{3}{40}$	
8	9			$-\frac{3}{40}$		$\frac{3}{80}$				$-\frac{9}{80}$
8	10	$\frac{1}{15}$	$-\frac{1}{40}$				$\frac{1}{48}$		$-\frac{9}{80}$	
9	9	$-\frac{1}{6}$		$\frac{1}{48}$	$-\frac{1}{48}$	$\frac{1}{24}$				
9	10		$-\frac{1}{16}$				$-\frac{3}{40}$	$\frac{9}{80}$		
10	10	$-\frac{1}{6}$		$\frac{1}{48}$	$-\frac{1}{48}$	$\frac{1}{24}$				

Appendix 4. Mixing *via* Final-State Interactions

We consider a general term in the Born series. Let V_0 correspond to the isospin-conserving potential and V_x , for example, correspond to a rank-1 isospin-violating potential. The Born

series consists of a sum of powers of GV where G is the Green function and V is the interaction potential. We have assumed that the Green function is proportional to the identity operator in isospin space; i.e. all purely kinematical isospin violation such as mass splittings have been incorporated into the potential V . So far as the isospin structure is concerned, then, we may ignore G . Terms in the Born series to first order in the isospin-violating interactions but to all orders in the isospin-conserving interactions may then be grouped as

$$V_0^m V_x V_0^n + V_0^n V_x V_0^m.$$

By the multiplication table (Table 5a) it is seen that all powers of V_0 may be reduced to a sum of $\theta_1 = \mathbf{1}$ and θ_5 . The isospin structure of the above terms in the Born series may then be reduced to a sum of terms proportional to V_x , $\{V_x, V_0\}$, and $V_0 V_x V_0$. The second of these is seen from the multiplication table to be a linear combination of all three of the $I = 1$ operators. To see that this is also true for the last term, we use the identity

$$\theta_i \theta_j \theta_i = \sum_{\ell} d_{ij\ell} \theta_{\ell}$$

where

$$d_{ij\ell} = \sum_k (2C_{ijk} C_{ik\ell} - C_{iik} C_{kj\ell}).$$

Combining this equation with Table 5 we find

$$\theta_5 \theta_2 \theta_5 = -\frac{1}{18} \theta_2 + \frac{1}{6^{3/2}} \theta_3 - \sqrt{\frac{5}{324}} \theta_7$$

$$\theta_5 \theta_3 \theta_5 = \frac{1}{6^{3/2}} \theta_2 + \frac{1}{12} \theta_3 + \sqrt{\frac{5}{216}} \theta_7$$

$$\theta_5 \theta_7 \theta_5 = -\sqrt{\frac{5}{324}} \theta_2 + \sqrt{\frac{5}{216}} \theta_3 + \frac{2}{9} \theta_7.$$

Thus all three of the rank 1 tensor operators are involved due to the FISI.

As promised in Sec.V we now determine the isospin composition of the $\eta\pi^0$ mixing amplitude. In the 6×6 matrix representation the operator is

$$\theta_{\eta\pi} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & & & & & \\ & 0 & -\sqrt{2} & & & \\ & -\sqrt{2} & 2 & & & \\ & & & 0 & & \\ & & & & 0 & \sqrt{2} \\ & & & & \sqrt{2} & -2 \end{pmatrix}$$

as may be found by direct evaluation of the Clebsch-Gordan coefficients given in Sec.V. The following traces are easily computed:

$$Tr(\theta_2^T \theta_{\eta\pi}) = \sqrt{\frac{8}{9}}, \quad Tr(\theta_7^T \theta_{\eta\pi}) = \sqrt{\frac{40}{9}},$$

and the other traces vanish. Thus we have

$$\theta_{\eta\pi} = \sqrt{\frac{8}{9}}\theta_2 + \sqrt{\frac{40}{9}}\theta_7.$$

[1] If the multiplets are identical as in NN scattering there are still 4 independent channels needed in the isospin formalism. For pp and nn elastic scattering the particle identity allows us to restrict the scattering angle to the range $0 \leq \theta \leq \pi/2$. Were we to treat p and n as independent particles the process $pn \rightarrow pn$ requires a knowledge of the reaction for the full range of scattering angles $0 \leq \theta \leq \pi$. In the isospin formalism, however, we treat n and p on the same footing and continue to limit θ to the forward hemisphere. The scattering of the p into the backward hemisphere is described by the process $pn \rightarrow np$ where now θ (where $0 \leq \theta \leq \pi/2$) describes the angle between the incident proton and the outgoing neutron (in the barycentric frame). Thus all four channels pp , pn , np , and nn are required in this formalism.

[2] E. M. Henley and G. A. Miller, in **Mesons in Nuclei**, ed. M. Rho and D. H. Wilkinson (North-Holland, Amsterdam, 1979), p. 405

[3] An example taken from Henley and Miller is $(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \mathbf{L}$, where \mathbf{L} is the orbital angular momentum of the nucleon-nucleon system.

[4] A second choice of operators is $\mathbf{1}$, t_0 , and t_0^2 . In terms of the original set of operators t_0^2 is $\sqrt{\frac{2}{3}}(\mathbf{t} \cdot \mathbf{t})_0^2 + \frac{2}{3}\mathbf{1}$. In the charge basis, constructed from the eigenstates of t_0 , these operators are diagonal with diagonal elements $\mathbf{1} = (1, 1, 1)$, $t_0 = (1, 0, -1)$, $t_0^2 = (1, 0, 1)$, and $(\mathbf{t} \cdot \mathbf{t})_0^2 = (\sqrt{\frac{1}{6}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{6}})$.

[5] That is, $\sum_{ab;cd} \langle ab|\theta_i|cd \rangle \langle ab|\theta_j|cd \rangle = \delta_{ij}$ or

$$\sum_i \langle ab|\theta_i|cd \rangle \langle ab|\theta_i|cd \rangle = \delta_{RR'}$$

where $R = (a, b; c, d)$, etc. The former equation may be written more compactly as $Tr(\theta_i^T \theta_j) = \delta_{ij}$, where Tr is the sum over all charge states or over all isospin components and T means transpose.

[6] For example, we could replace operators 4, 6, 7, and 8 in Table 2 by $\theta_4 = t_0^2$, $\theta_6 = t_0\tau_0$,

$\theta_7 = t_0(\mathbf{t} \cdot \boldsymbol{\tau})_0^0$, and $\theta_8 = t_0^2\tau_0$. With these replacements the set of operators has the advantage that all but two of θ_i (i.e., θ_5 and θ_7) are diagonal in charge space. Charge exchange comes solely from these two terms. The disadvantage of this set is that the new operators are of mixed isospin, although they remain even or odd under charge symmetry depending on whether the term has an even or odd number of factors of the isospin components t_α and (or) τ_β . In the body of this article θ_i will refer only to the operators defined in Table 2.

[7] The reader wishing to adopt the older convention such that θ_1 is the identity and θ_5 is $\boldsymbol{\tau} \cdot \mathbf{t}$ need only change Tables 3 and 4 as follows: the θ_1 column is multiplied by 6 (or divided by 6 for use as an inverse), the θ_5 column is multiplied by -12 (or divided by -12 for use as an inverse).

[8] See, for example, P. Langacker and D. A. Sparrow, *Phys. Rev. C* **25**, (1982) 1194.

[9] E. Pedroni *et al.*, *Nucl. Phys.* **A300**, (1978) 321.

[10] G. R. Smith *et al.*, *Phys. Rev. C* **38**, (1988) 240.

[11] D. H. Fitzgerald *et al.*, *Phys. Rev. C* **34**, (1985) 619.

[12] P. B. Siegel and W. R. Gibbs, *Phys. Rev. C* **33**, (1986) 1407.

[13] J. A. Wightman, *et al.*, *Phys. Rev. D* **38**, (1988) 3365.

[14] P. B. Siegel and W. R. Gibbs, *Phys. Rev. C* **36**, (1987) 2473.