Generalizing vector calculus

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In almost all mathematical treatments of vector calculus an assumption is made that the basis set is orthogonal in nature. But what if it isn't? This paper addresses the topic of performing vector analysis for any coordinate system without prior orthogonalization . In these papers Einstein's summation convention as well as covariant & contravariant notation will not be used. Moreover the metric for a given geometry will be expressed in terms of “scale factors”.

$$h_{\alpha\beta}^2 \equiv g_{\alpha\beta}$$

In this manner the general Pythagorean theorem would have the form

$$\mathrm{ds}^2 = \sum_{\alpha,\beta} \mathrm{h}_{\alpha\beta}^2 \cdot \mathrm{dx}_\alpha \cdot \mathrm{dx}_\beta$$

For example the scale factors for the cylindrical polar coordinates (r,θ,z) would be (1,r,1).

The dot product

Usually the dot product between to vectors **A** and **B** is expressed as

$$\vec{A} \cdot \vec{B} = \sum_\alpha A_\alpha \cdot B_\alpha$$

However this assumes an orthogonal basis. But what if one has a non-orthogonal basis? To answer this question, consider the vector defined below

$$\vec{dr} = \sum_\alpha h_\alpha \cdot dx_\alpha \cdot \hat{x}_\alpha$$

Using the shorthand notation that h_α is the same as $h_{\alpha\alpha}$. The dot product of this vector with itself (assuming an orthogonal basis) is

$$dr^2 = \sum_\alpha h_\alpha^2 \cdot dx_\alpha^2 = \sum_\alpha g_\alpha \cdot dx_\alpha^2$$

Which is of course is the invariant infinitesimal distance or interval in space ds^2. For a non-orthogonal basis we want the same meaning, but now we need the general equation

$$ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta} dx_\alpha dx_\beta = \sum_{\alpha,\beta} h_{\alpha\beta}^2 dx_\alpha dx_\beta$$

As a proposed equation for the dot product, assume that the general dot product has the form

$$\vec{A} \cdot \vec{B} = \sum_{\alpha,\beta} M_{\alpha\beta} \cdot (A_\alpha B_\beta)$$

$M_{\alpha\beta}$ is to be determined. Doing the dot product of dr with itself as done earlier yields

$$dr^2 = \sum_{\alpha,\beta} M_{\alpha\beta} (h_\alpha h_\beta)(dx_\alpha dx_\beta)$$

Making this equal ds^2 means $M_{\alpha\beta}$ must be

$$M_{\alpha\beta}=\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta}$$

The general dot product therefore is

$$\vec{A}\cdot\vec{B}=\sum_{\alpha,\beta}\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta}\cdot(A_\alpha B_\beta)$$

It will be observed that in the case of a completely orthogonal basis this reverts back to the usual form.

Vector calculus

Having expanded the dot product to cover non-orthogonal coordinate systems, the next task is to apply the results to vector calculus.

General gradient

Since a dot product is not involved in gradients, the gradient for a non-orthogonal space will then be the same as for any curvilinear space.

$$\vec{\square}\phi=\sum_\alpha\frac{\hat{x}_\alpha}{h_\alpha}\cdot\partial_\alpha\phi$$

Where the ∂_α signifies the partial derivative with respect to x_α.

General divergence

The divergence in an othogonal 3-D curvilinear coordinate system can be expressed as follows[1]

$$\vec{\nabla}\cdot\vec{v}=\frac{1}{\Pi}\sum_j\partial_j(\frac{\Pi\cdot v_j}{h_j})$$

With Π the product of all h_α. This can also be expressed as the dot product between the vector **v** and a del operator with components

$$\nabla_j^{op}\equiv\frac{1}{\Pi}\cdot\partial_j\frac{\Pi}{h_j}$$

Expanding the dot product to allow for non-orthogonal coordinate systems using the above gives

$$\vec{\square}\cdot\vec{v}=\sum_{\alpha,\beta}\frac{h_{\alpha\beta}^2}{\Pi h_\alpha h_\beta}\cdot\partial_\alpha(\frac{\Pi v_\beta}{h_\alpha})$$

Laplacian

The Laplacian is defined as the divergence of the gradient of some function ϕ.

$$\Box^2 \phi = \sum_{\alpha,\beta} \frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta} \cdot \partial_\alpha \left[\frac{\Pi}{h_\alpha} \cdot (\vec{\nabla} \phi)_\beta \right]$$

Or after simplifying,

$$\Box^2 \phi = \sum_{\alpha,\beta} \frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta} \cdot \partial_\alpha \left(\frac{\Pi}{h_\alpha h_\beta} \cdot \partial_\beta \phi \right)$$

Generalizing to N space

Expanding the results above to any N-space will be simply done by changing the notation, replacing the del operator with □.

$$\Box^2 \phi = \sum_{\alpha,\beta} \frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta} \cdot \partial_\alpha \left(\frac{\Pi}{h_\alpha h_\beta} \cdot \partial_\beta \phi \right)$$

The reason for the choice of symbol can be seen by applying this to the case of the Minkowski space where the Lorentz transformation holds. In this space let the coordinates be (x,y,z,t) and h = (1,1,1,ic). Applying the generalized definition of the Laplacian gives

$$\Box_L^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi - \frac{1}{c^2} \partial_t^2 \phi$$

Which is the d' Alembertian operator for Cartesian coordinates[2] operating on ϕ. However the Laplacian above has been derived for any N dimensional geometry.

References

1. George Arfken and Hans Weber *Mathematical Methods for Physicists* 4th ed. p. 106
2. David Griffiths Introduction to Electrodynamics 2nd ed. p. 318