Generalized Gravity

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I. The Klein-Gordon equation

The Klein-Gordon wave equation in relativistic quantum mechanics can be expressed as follows

$$\nabla^2 \Psi - \frac{1}{c^2} \cdot \partial_t^2 \cdot \Psi = \left(\frac{E_0}{c\hbar}\right)^2 \cdot \Psi$$

E_0 is the rest energy and ∂_t is defined as the partial derivative with respect to t. This has a general solution as follows

$$\Psi(\vec{r}, t) = \int a(\vec{p}) \cdot \mathrm{e}^{i \cdot \frac{(\vec{p} \cdot \vec{r} - E \cdot t)}{\hbar}} \cdot d^3 p + \int b(\vec{p}) \cdot \mathrm{e}^{i \cdot \frac{(\vec{p} \cdot \vec{r} + E \cdot t)}{\hbar}} \cdot d^3 p$$

With

$$E \equiv \sqrt{c^2 \cdot |\vec{p}|^2 + (E_0)^2}$$

The first integration can be interpreted as belonging to states of positive energy E and the second states of negative E. Moreover the **p** can represent particle momentum.

The main objections to the Klein-Gordon equation are three-fold

1. For an initial wave function Ψ(**r**,t=0), one must also have some extra initial condition to know how the wave packet develops over time, unlike the Schrödinger equation.
2. The Ψ(**r**,t) obtained cannot be a probability. It is argued that if the solution is normalized at a given time it must remain normalized.
3. The solutions to do not involve spin and so it can only be useful for spin zero particles.

These objections will be addressed one at a time.

The need for only one initial condition

The generalized solution at t = 0 is

$$\Psi(\vec{r}, t=0) = \int [a(\vec{p}) + b(\vec{p})] \cdot e^{i \cdot \frac{\vec{p} \cdot \vec{r}}{\hbar}} \cdot d^3 p$$

The sum of a(**p**) and b(**p**) must therefore be the Fourier transform of the initial wave function with **k** = **p**/ħ. As it stands, there must be an initial condition to solve for both functions. One way to bypass this objection can is to impose the restriction that a particle can have positive or negative energy, but cannot be in a superposition of both. Under this assumption the nonzero function a(**p**) or b(**p**) would be the same as the Fourier transform of the initial wave function. Later we will return to the question of superposition of positive and negative energy particles.

The probability normalization

If one integrates the modulus squared of the general solution to the Klein Gordon equation above over all space one gets

$$\int_{all\ space} |\Psi(\vec{r}, t)|^2 d^3 r = h \cdot \int (|a(\vec{p})|^2 + |b(\vec{p})|^2) d^3 p + 2h \cdot \Re[\int a(\vec{p}) \cdot \overline{b(\vec{p})} \cdot e^{\frac{2i \cdot E}{\hbar} \cdot t} d^3 p]$$

The line over the function b(**p**) signifies the complex conjugate. It will be noted that under the above restriction the time dependent portion vanishes and the integral becomes constant.

Spin and the Klein-Gordon equation

One argument against the Klein-Gordon equation is that, as opposed to the Dirac equation, spin is not included in its solutions. It must first be noted that one can always multiply the scalars for the solution with a matrix giving the various possible spin states, but it must be noted that such things are irrelevant in the equation itself. Secondly, the Dirac equation is conceptually this equation applied to specific boundary equations. Therefore if there are any deficiencies then it is Dirac that must be called into question.

Particles of negative energy

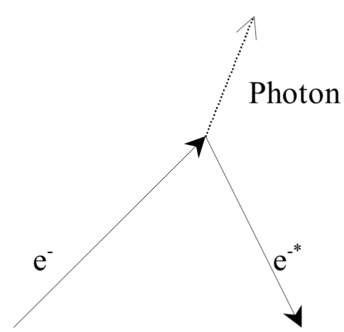
The general solution involves solutions that can be interpreted to be positive and negative energy energy states. So how does a particle of negative energy differ from a particle of positive energy? Comparing the general solution to the Klein-Gordon equation above for the case of positive vs. negative energy one is struck by one major difference, namely time is inverted (t → -t). This can be interpreted to mean that in the relativistic limit the negative energy particle behaves the same as a particle with positive energy that is moving *backwards* in time. It is desirable to express these negative energy particles in terms of positive energy particles so we are on familiar ground, so the distinction between negative and positive energy particles will then be made as follows

Particles of positive energy move "forward" in time while particles of negative energy move backwards, in the relativistic limit.

Additionally, in the case of a negative energy particle the momentum and velocity are anti-parallel in the relativistic limit because of the relation involving velocity, relativistic momentum and energy

$$\vec{u} = c^2 \cdot \frac{\vec{p}}{E}$$

To further exam the matter in depth, consider a positive energy electron giving off a photon and producing a negative energy electron (represented by e^{-*}) as shown below



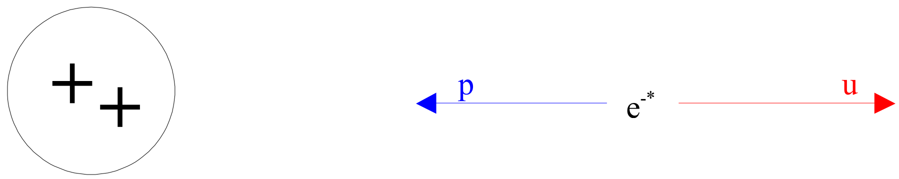
The downward direction of e^{-*} depicts its traveling backwards in time. Clearly there is no difficulty in conserving the appropriate quantities. Describing this process from a purely chronological standpoint means it can be written as follows.

$$e^- + e^{-*} \rightarrow \text{photon}$$

Problems appear to arise from this latter depiction. To begin with, it is not apparent from the latter depiction how energy is conserved. The diagram clearly suggests that the photon energy must have at least twice the rest energy of an electron, yet the latter depiction does not appear to offer this. Moreover charge is certainly not conserved. To correct for these problems, imagine replacing the e^{-*} with a positive energy pseudo-particle that has a positive charge (to conserve charge) called e^+.

$$e^- + e^+ \rightarrow \text{photon}$$

Because of this reaction, this pseudo-particle is called an "anti-particle." Although the anti-particle needs to have an opposite charge above, it can be explained in the following manner. Suppose an e^{-*} is at rest near a large positive charge as shown below



Because of the attractive force the momentum of e^{-*} must be towards the positive charge, however because the electron has a negative energy the velocity u must be anti-parallel to momentum, it will be repelled. But suppose the particle is now depicted as a positive energy anti-particle? Now momentum and velocity must be in the same direction, both away from the positive charge (the electron must still be repelled). The only way to make a consistent picture is to have the anti-particle have an opposite charge.

To summarize what things can be concluded about negative energy particles

- Negative energy particles can be viewed as positive energy particles moving backwards in time.
- The momentum and velocity of a negative energy particles are anti-parallel.
- A negative energy particles can be recast as a positive energy anti-particle with the following changes
    1. The particle moves forward in time
    2. The momentum vector (even the 4 vector) reverses direction
    3. Charge reverses sign

Superposition of positive and negative energy states

It was assumed above that a particle can have either a positive or negative energy but cannot be a superposition of the two. But what if there were such a superposition of states for some particle X?

$$|X\ mix\rangle = \alpha\cdot|X\rangle + \beta\cdot|X^*\rangle$$

Now the wave is a mixture of waves moving forward and backwards in time, the resulting interference will vary the probability that the particle be found somewhere in space. One way to justify this can be seen after rewriting the negative energy state as the anti-particle state.

$$|X\ mix\rangle = \alpha\cdot|X\rangle + \beta\cdot|anti\text{-}X\rangle$$

Since this state is a mixture of matter and anti-matter the reaction

$$X + anti\text{-}X \leftrightarrow photon$$

can occur. With this reaction a possibility the probability of finding the particle somewhere would naturally vary over time.

II. Non-orthogonal math

In almost all mathematical treatments of vector calculus an assumption is made that the basis set is orthogonal in nature. But what if it isn't? This next section takes a detour to address the topic of performing vector analysis for any coordinate system without prior orthogonalization. Hereafter Einstein's summation convention as well as covariant & contra-variant notation will not be used. Moreover the metric for a given geometry will be expressed in terms of “scale factors” $h_{\alpha\beta}$.

$$h_{\alpha\beta}^2 \equiv g_{\alpha\beta}$$

For example the scale factors for the cylindrical polar coordinates (r,θ,z) would be (1,r,1).

The dot product

The first item of business is to explore the dot product. Usually the dot product between to vectors **A** and **B** is expressed as

$$\vec{A}\cdot\vec{B}=\sum_{\alpha} A_{\alpha}\cdot B_{\alpha}$$

However this assumes an orthogonal basis. But what if one has a non-orthogonal basis? To answer this question, consider the vector defined below

$$\vec{dr}=\sum_{\alpha} h_{\alpha}\cdot dx_{\alpha}\cdot\hat{x}_{\alpha}$$

Using the shorthand notation that h_α is the same as $h_{\alpha\alpha}$. The dot product of this vector with itself (assuming an orthogonal basis) is

$$dr^2=\sum_{\alpha} h_{\alpha}^2\cdot dx_{\alpha}^2=\sum_{\alpha} g_{\alpha}\cdot dx_{\alpha}^2$$

Which is of course is the invariant infinitesimal distance or interval in space ds^2. For a non-orthogonal basis we want the same meaning, but now we need the general equation

$$ds^2=\sum_{\alpha,\beta} g_{\alpha\beta}\, dx_{\alpha}\, dx_{\beta}=\sum_{\alpha,\beta} h_{\alpha\beta}^2\, dx_{\alpha}\, dx_{\beta}$$

As a proposed equation for the dot product, assume that the general dot product has the form

$$\vec{A}\cdot\vec{B}=\sum_{\alpha,\beta} M_{\alpha\beta}\cdot(A_{\alpha}B_{\beta})$$

$M_{\alpha\beta}$ is to be determined. Doing the dot product of d**r** with itself as done earlier yields

$$dr^2=\sum_{\alpha,\beta} M_{\alpha\beta}(h_{\alpha}h_{\beta})(dx_{\alpha}dx_{\beta})$$

Making this equal ds^2 means $M_{\alpha\beta}$ must be

$$M_{\alpha\beta}=\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta}$$

The general dot product therefore is

$$\vec{A}\cdot\vec{B}=\sum_{\alpha,\beta}\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta}\cdot(A_\alpha B_\beta)$$

It will be observed that in the case of a completely orthogonal basis this reverts back to the usual form.

General gradient

Having expanded the dot product to cover non-orthogonal coordinate systems, the next task is to apply the results to vector calculus, beginning with the gradient.

Since a dot product is not involved in gradients, the gradient for a non-orthogonal space will then be the same as for any curvilinear space.

$$\vec{\square}\,\phi=\sum_\alpha\frac{\hat{x}_\alpha}{h_\alpha}\cdot\partial_\alpha\phi$$

Where the ∂_α signifies the partial derivative with respect to x_α.

General divergence

A 3-D curvilinear coordinate system can be expressed as follows[1]

$$\vec{\nabla}\cdot\vec{v}=\frac{1}{\Pi}\sum_j\partial_j\left(\frac{\Pi\cdot v_j}{h_j}\right)$$

With Π the product of all h_j. This can also be expressed as the dot product between the vector **v** and a del operator with components

$$\nabla_j^{op}\equiv\frac{1}{\Pi}\cdot\partial_j\frac{\Pi}{h_j}$$

Expanding the dot product to allow for non-orthogonal coordinate systems using the above gives

$$\vec{\square}\cdot\vec{\mathrm{v}}=\sum_{\alpha,\beta}\frac{\mathrm{h}_{\alpha\beta}^2}{\Pi\,\mathrm{h}_\alpha\mathrm{h}_\beta}\cdot\partial_\alpha\left(\frac{\Pi\,\mathrm{v}_\beta}{\mathrm{h}_\alpha}\right)$$

Laplacian

The Laplacian is defined as the divergence of the gradient of some function ϕ.

$$\square^2 \phi = \sum_{\alpha,\beta} \frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta} \cdot \partial_\alpha [\frac{\Pi}{h_\alpha} \cdot (\vec{\nabla} \phi)_\beta]$$

This can be simplified to

$$\square^2 \phi = \sum_{\alpha,\beta} \frac{h_{\alpha\beta}^2}{\Pi\, h_\alpha h_\beta} \cdot \partial_\alpha (\frac{\Pi}{h_\alpha h_\beta} \cdot \partial_\beta \phi)$$

The reason for the choice of symbols can be seen by applying this to the case of the Minkowski space where the Lorentz transformation holds. In this space let the coordinates be (x,y,z,t) and h = (1,1,1,ic). Applying the generalized definition of the Laplacian gives

$$\square_L^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi - \frac{1}{c^2} \partial_t^2 \phi$$

Which is the d' Alembert operator for Cartesian coordinates[2] operating on ϕ. However the Laplacian above has been derived for any N dimensional geometry.

III. Generalizing the Klein-Gordon equation

The Klein-Gordon equation can be expressed as follows

$$\nabla^2 \Psi - \frac{1}{c^2} \partial_t^2 \Psi = (\frac{E_0}{c\hbar})^2 \Psi$$

Before generalizing this equation to fit all cases it is important to show that in the appropriate case this becomes the Schroedinger equation. Having done this focus will then be turned to generalizing the Klein-Gordon equation beginning with a four-dimensional space then generalized to include higher dimensions. Before this can be done another detour must be taken into operators.

Momentum operators & relativistic quantities

In special relativity the momentum 4-vector can be expressed as **P** = $\{P_x, P_y, P_z, iE_D/c\}$. E_D is defined as the dynamical energy, or the sum of the rest and kinetic energy. The imaginary number i is used so the ordinary dot product of P with itself takes a similar form to the mass-shell relation. In fact one can also define **P** differently depending on whether one is dealing on a quantum mechanical operator or in the relativistic limit.

Quantum	$P_{\alpha\, op} \equiv \frac{\hbar}{i\, h_\alpha} \cdot \partial_\alpha$
Relativistic limit	$P_\alpha \equiv \{mL_\alpha\} \cdot \partial_t x_\alpha$

Where m is the relativistic mass and L the scale factors for a free space, free of any forces. For the case of the time component in special relativity L_t = ic. In this case we get $E_D = i\hbar \cdot \partial_t$ for the 1st case and $P_0 = i \cdot mc = iE_D/c$ for the 2nd. These are identical to the energy operator for the total energy of a particle as seen in Schrödinger's equation and special relativity respectively.

Connecting the Klein-Gordon equation to the Schrödinger Equation

This section will prove that the Klein-Gordon equation above becomes the Schrödinger equation for a free particle under the condition that the dynamical energy is approximately equal to the rest energy. For now I will assume a 4-dimensional Minkowski space.

The Klein-Gordon equation above can be reworked as follows

$$-\left(\frac{\hbar}{2m}\right)^2 \cdot \nabla^2 \Psi + \left(\frac{\hbar^2}{2E_0}\right) \cdot \partial_t^2 \Psi + \left(\frac{E_0}{2}\right) \Psi + i\hbar \cdot \partial_t \Psi = i\hbar \cdot \partial_t \Psi$$

It was shown above that under the conditions of special relativity $E_D = i\hbar \cdot \partial_t$, so in the classical limit ($E_D \approx E_0$)

$$\partial_t \Psi = \left(\frac{1}{i\hbar}\right) \cdot E_D \Psi \approx \left(\frac{E_0}{i\hbar}\right) \Psi$$

Under this approximation the Klein Gordon equation simplifies to

$$-\left(\frac{\hbar^2}{2m}\right) \cdot \nabla^2 \Psi + E_0 \Psi = i\hbar \cdot \partial_t \Psi$$

This is the Schrödinger equation with the rest energy included. The total energy operator (kinetic plus rest plus potential energy) E naturally comes out to be $E = i\hbar \cdot \partial_t$. It is therefore postulated that the total energy operator E will always be equal to $i\hbar \cdot \partial_t$ so the relativistic quantum mechanical equations go to the Schroedinger equation in the classical limit.

Generalizing the Klein-Gordon Equation

The space associated with special relativity (in Cartesian coordinates) is as follows

L = {1, 1, 1, ic}

The Laplacian for this space again is

$$\Box_L^2 \phi = \partial_x^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi - \frac{1}{c^2} \partial_t^2 \phi$$

Again, the subscript L denotes a special relativistic coordinate system where no forces are present. In this light the Klein-Gordon equation above can be rewritten as

$$\Box_L^2 \Psi = \left(\frac{E_0}{c\hbar}\right)^2 \Psi$$

Which suggests a generalization of the Klein-Gordon equation as follows

If $\Box^2 \Psi = (\frac{E_0}{c\hbar})^2 \Psi$ however additional dimensions are included this expression will not reduce down to the original Klein-Gordon equation for the case of a particle in a free space. To correct for this the generalized Klein-Gordon equation can be modified as follows

$$\Box^2 \Psi + [\nabla_L^2 \Psi - \frac{1}{c^2} \partial_t^2 \Psi] = (\frac{E_0}{c\hbar})^2 \Psi + \Box_L^2 \Psi$$

With the del operator involving only the 3 spatial coordinates and the L denoting that the geometry of a free space is to be used.

The uncertainty principle

The uncertainty principle for x_α and p_β can be obtained for a general space using the momentum operator defined above and the general form of the uncertainty principle[2]

$$|\Delta \vec{r}_\alpha \cdot \Delta \vec{p}_\beta| \geq (\frac{\hbar}{2}) \cdot |< \frac{\delta_{\alpha\beta}}{h_\beta} >|$$

In the special relativistic case above this reduces to the standard uncertainty principle for position, momentum & energy.

Application: Black Holes

The first application of the above sections will be for the case of black holes. In this case $O_L \rightarrow 0$. The scale factors from the Schwartzchild metric are as follows

h_r	$(1 - R_s/r)^{-1/2}$
h_θ	r
h_ϕ	$r \cdot \sin\theta$
h_t	$ic \cdot (1 - R_s/r)^{1/2}$
Π	$ic \cdot r^2 \cdot \sin\theta$

R_s is the Schwartzchild radius. The generalized Klein-Gordon becomes

$$\frac{1}{r^2} \partial(\{r^2 - r \cdot R_S\} \partial_r \Psi) + \{\frac{1}{r\sin\theta}\} \partial_\theta(\sin\theta\, \partial_\theta \Psi) + (\frac{1}{r\sin\theta})^2 \partial_\phi \Psi + \frac{r}{c^2 \cdot (r - R_S)} \partial_t^2 \Psi = (\frac{E_0}{c\hbar})^2 \Psi$$

One important thing to note is the time derivative term will blow up at R_s unless the time derivative goes to zero. Thus a wave packet moving into a black hole must freeze up at the Schwartzchild radius, consistent with general relativistic considerations for objects falling into a Black Hole.

The uncertainty principle between radial momentum and radial distance as well as time dynamical energy is as follows

$$\Delta r\cdot\Delta p_r \geq (\frac{\hbar}{2})\cdot|<(1-\frac{R_s}{r})^{+1/2}>| \qquad \Delta t\cdot\Delta E_D \geq (\frac{\hbar}{2})\cdot|<(1-\frac{R_s}{r})^{-1/2}>|$$

Consider the ramifications of this by considering a wave packet confined in a region that is much smaller than R_s.

- $r \to \infty$: $\Delta r\cdot\Delta p_r \geq (\hbar/2)$ & $\Delta t\cdot\Delta E_D \geq (\hbar/2)$. The uncertainty principle takes on a familiar look.
- $r \to R_s$: $\Delta r\cdot\Delta p_r \geq 0$ & $\Delta t\cdot\Delta E_D \to \infty$. The uncertainty principle for radial momentum & position disappears completely. Also the uncertainty in dynamical energy & time blows up. This means that the time when an event occurs (such as the particle entering the event horizon) is completely uncertain. This is consistent with the understanding that objects falling into a black hole never makes it through relative to an observer watching on.

IV. Generalizing Gravity

The behavior of a particle in a space is usually viewed using one of three models.

1. Objects motion is affected by the force(s) acting on it.
2. The particle acts under a potential energy field.
3. The geometry of the space is such that the particles trajectory is its geodesic through the space.

These views can be unified by rewriting the general Klein-Gordon equation (with its geometry dependence) so it has the form of the Schrödinger equation and obtaining the potential energy by comparison. Bringing everything to the same side of the equation and reversing sides

$$-\nabla_L^2\Psi + \frac{1}{c^2}\partial_t^2\Psi + [\Box_L^2 - \Box^2]\Psi + (\frac{E_0}{c\hbar})^2\Psi + \frac{1}{c^2}\partial_t^2\Psi = 0$$

Multiply all terms by $\hbar^2$ and divide by twice the rest mass.

$$-(\frac{\hbar^2}{2m_0})\nabla_L^2\Psi + (\frac{\hbar^2}{2E_0})\partial_t^2\Psi + [(\frac{\hbar^2}{2m_0})\Box_L^2 - (\frac{\hbar^2}{2m_0})\Box^2]\Psi + (\frac{E_0}{2})\Psi = 0$$

Add $i\hbar\cdot\partial_t\Psi$ to both sides and doing some rearranging

$$-(\frac{\hbar^2}{2m_0})\nabla_L^2\Psi + (\frac{\hbar^2}{2E_0})\partial_t^2\Psi + (\frac{\hbar^2}{2m_0})[\Box_L^2 - \Box^2]\Psi + (\frac{E_0}{2})\Psi + i\hbar\partial_t\Psi = i\hbar\partial_t\Psi$$

The goal is to reduce this to the Schrödinger's equation in the classical quantum limit of

$E_D \approx E_0$. Here $i\hbar\cdot\partial_t$ can be express as the dynamical energy operator as follows

$$i\hbar\partial_t\Psi=(\frac{h_0}{ic})E_{Dop}$$

So in the classical quantum limit, let $i\hbar\cdot\partial_t\Psi \rightarrow (h_0/ic)E_0\Psi$ on the left side of the equation

$$-(\frac{\hbar^2}{2m_0})\nabla_L^2\Psi+\{(\frac{\hbar^2}{2m_0})[\Box_L^2-\Box^2]\Psi+(\frac{\hbar^2}{2E_0})\partial_t^2\Psi\}+(\frac{E_0}{2})\Psi+i\hbar\partial_t\Psi=i\hbar\partial_t\Psi$$

$$-(\frac{\hbar^2}{2m_0})\nabla_L^2\Psi+\{(\frac{\hbar^2}{2m_0})[\Box_L^2-\Box^2]+(\frac{\hbar^2}{2E_0})\partial_t^2+i\hbar\partial_t-(\frac{E_0}{2})\}\Psi+E_0\Psi=i\hbar\partial_t\Psi$$

This is of the form of Schrödinger's equation with the braced portion playing the role of the potential energy, suggesting a potential energy operator as follows

$$U_{op}'=(\frac{\hbar^2}{2m_0})[\Box_L^2-\Box^2]+(\frac{\hbar^2}{2E_0})\partial_t^2+i\hbar\partial_t-(\frac{E_0}{2})$$

To explore this further, consider 3 cases:

<u>A classical, free particle</u>

In the classical quantum limit $E_D \approx E_0$, so let the dynamical operator operating on the wave function produce

$$E_{op\,D}=-(\frac{c\hbar}{h_t})\partial_t\Psi\approx E_0\Psi$$

If the particle is in a dynamical energy eigenstate

$$U_{op}'\Psi=\frac{-E_0}{2}\Psi+E_0\Psi-\frac{E_0}{2}\Psi=0$$

The potential is zero, as one would expect for a free particle.

<u>Relativistic free particle</u>

In general, for a free particle of dynamical energy E_D, the expression $U_{op}'\Psi$ from above may not be zero as it should be. This can be remedied by replacing E_0 by E_D and m_0 by the relativistic mass m. Moreover an operator for E_D can replace the last term.

$$U_{op}''=(\frac{c^2\hbar^2}{2E_D})[\Box_L^2-\Box^2]+(\frac{\hbar^2}{2E_D})\partial_t^2+[i\hbar+\frac{c\hbar}{2h_t}]\partial_t$$

And the case cited earlier still obtains in the classical quantum limit.

<u>General relativistic particle</u>

Unless the wave function is an eigenstate of E_D, this expression for the potential energy is not practical. It is not the purpose at present to try and make it work quantum mechanically in general. Rather, the task will be to take this to the relativistic limit and obtain an expression relating the space-time scale factors, momentum & potential energy.

Limits of energy conservation

The conservation of energy for a particle in a potential energy is expressed as

$$E_D + U = E \quad \text{E is a constant}$$

Earlier it was shown (using the definition for P_t) that:

$$E_D = -imc \cdot h_t$$

So

$$U - imc \cdot h_t = E$$

There are 2 problems

1. The relativistic mass m is defined as $m_0 \cdot \gamma$ via special relativity. But how is γ defined in general relativistic terms?
2. In the case of an object at the even horizon of a Black Hole $h_t \rightarrow 0$ and so U = E regardless of its initial conditions. This makes no sense.

Resolving these issues implies the following solution

The combined potential and dynamical energy can be conserved only when the space metric is such that $h_{\alpha\beta} \approx L_{\alpha\beta}$ for all $h_{\alpha\beta}$. Potential energy can be only defined in this region.

The development of the final relation between the geometry of a region & potential energy must be done with the above restriction in mind.

Relating U and $h_{\alpha\beta}$ -- attempt # 1

Expanding the generalized Laplacian in the expression of *U' '* and ignoring the iħ term (it is negligible in the relativistic limit) gives

$$\mathrm{U_{op}}' = \left(\frac{c^2 \hbar^2}{2 E_D}\right) \sum_{\alpha,\beta} \left[\left(\frac{L_{\alpha\beta}}{L_\alpha L_\beta}\right)^2 - \left(\frac{h_{\alpha\beta}}{h_\alpha h_\beta}\right)^2 \right] \partial_\alpha \partial_\beta + \left(\frac{\hbar^2}{2 E_D}\right) \partial_t^2 + \left[i\hbar + \left(\frac{c\hbar}{2 h_t}\right) \right]$$

Define the operator

$$P_{op\alpha\beta} \equiv -\left(\frac{\hbar^2}{h \alpha \beta}\right) \cdot \partial_\alpha \partial_\beta = P_\alpha P_\beta - \left(\frac{i\hbar \cdot \partial_\alpha h_\beta}{h_\alpha h_\beta}\right) \cdot P_\beta$$

So putting this into the above and ignoring the iħ portions that do not cancel as before gives

$$U_{op} = \left(\frac{1}{2m}\right) \sum_{\alpha,\beta} \left[\left(\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta}\right) - (h_\alpha h_\beta) \cdot \left(\frac{L_{\alpha\beta}}{L_\alpha L_\beta}\right)^2 \right] \cdot P_\alpha P_\beta - \left(\frac{h_t^2}{2 E_D}\right) P_t^2 + \left[\left(\frac{ic}{2}\right) - h_t \right] P_t$$

The relativistic limit

Taking the relativistic limit means replacing operator by physical quantities.

$$U=(\frac{1}{2m})\sum_{\alpha,\beta}[(\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta})-(h_\alpha h_\beta)\cdot(\frac{L_{\alpha\beta}}{L_\alpha L_\beta})^2]\cdot P_\alpha P_\beta-(\frac{h_t^2}{2E_D})P_t^2+[(\frac{ic}{2})-h_t]P_t$$

To clean this up multiply through by m, remembering that P_0 = imc and E_D = mc^2.

$$m\cdot U=\frac{1}{2}\sum_{\alpha,\beta}[(\frac{h_{\alpha\beta}^2}{h_\alpha h_\beta})-(h_\alpha h_\beta)\cdot(\frac{L_{\alpha\beta}}{L_\alpha L_\beta})^2]\cdot P_\alpha P_\beta-[\frac{1}{2}-\frac{h_t}{ic}-\frac{h_t^2}{2c^2}]P_t^2$$

Application: Black Holes

For the case of Black Holes the non-zero values of $h_{\alpha\beta}$ and $L_{\alpha\beta}$ are as follows:

Coordinate	$h_{\alpha\beta}$	$L_{\alpha\beta}$
t	$ic\cdot(1 - R_S/R)^{1/2}$	ic
r	$(1 - R_S/R)^{-1/2}$	1
θ	R	R
ϕ	R·sinθ	R·sinθ

Using these in the above equation gives

$$U=(\frac{m\cdot v_r^2}{2})[\frac{R_S}{(R_s-r)}]-(\frac{mc^2}{2})(1+\frac{R_S}{2r}-\sqrt{1-\frac{R_S}{r}})$$

Approximating this for the case r >> R_S (The only case where U can really be used as a potential energy)

$$U\approx\frac{-GMn}{r}\cdot[1+\beta_r^2]$$

With M and m the relativistic masses of the object creating the gravitational field and being acted upon respectively. The β_r is the fractional radial component of velocity compared with the speed of light. U is the Newton's law of gravitation multiplied by a factor of $[1 + \beta_r^2]$. In the classical limit this result is the Newton's law of gravitation, as one would expect. But what if one is talking about photons moving in a purely radial direction? Experimental work done on the gravitational redshift would be most inconsistent with this result. The equations used must be fixed.

Fixing the errors

Revisiting the Schrödinger's equation

In recasting the generalized Klein-Gordon equation as the Schrödinger equation it was naturally assumed that the Laplacian in this equation used the L scale factors. But why not have a Laplacian using the h scale factors? In the world of our experience the two scale factors are all but identical.

Postulate: *The Laplacian in the Schrödinger's equation uses the h scale factor.*

A new relation for potential energy

Changing the Laplacian in the wave equation gives

$$-(\frac{\hbar^2}{2m_0})\nabla^2\Psi+\{\frac{\hbar^2}{2m_0}[\Box_L^2-\Box^2+\nabla^2-\nabla_L^2]+(\frac{\hbar^2}{2E_0})\partial_t^2+i\hbar\partial_t-\frac{E_0}{2}\}\Psi+E_0\Psi=i\hbar\partial_t\Psi$$

Following the same steps as above to obtain a relation between U, momentum & the scale factor yield a result with the same form as above except that the double sum is restricted such that the terms in the sum cannot both belong to the 3 spacial coordinates. This will be denoted by a new choice of summation variables ν and η.

$$m\cdot U=\frac{1}{2}\sum_{\eta,\nu}[(\frac{h_{\eta\nu}^2}{h_\eta h_\nu})-(h_\eta h_\nu)\cdot(\frac{L_{\eta\nu}}{L_\eta L_\nu})^2]\cdot P_\eta P_\nu-[\frac{1}{2}-(\frac{h_t}{ic})-(\frac{h_t^2}{2c^2})]P_t^2$$

Newton's law of gravitation (potential energy format) is the natural consequence. Also of note is that the P_t^2 portion becomes $(1-h_t/ic)\cdot P_t^2$.

<u>Expanding the outcome</u>

The resulting equation is does not depend on the nature of the forces acting on an object.

For any fundamental force or potential energy, the root cause is the geometry of a region being distorted from a Minkowski geometry $L_{\alpha\beta}$.

When does U = 0?

One last comment must be made about potential energy. Classically one is free to add a constant onto an expression for potential energy without changing any of the physics of the situation. Yet for the theory of generalized gravity to work as given above, this ambiguity must be removed. It is postulated therefore that for a given metric caused by some source the following applies

Whenever a source distorts space from its special-relativistic nature, any place which is indistinguishable from a space with the source removed (for a given observer) has U = 0.

V. Rotating frames of reference

It can be shown using Lagrangian mechanics that an object in a rotating frame behaves as if its potential energy is due to a centripetal and Coriolis "force."

$$U_{cent}(\vec{r}) = \frac{-m\omega^2}{2} \cdot r_{\perp}^2 \qquad U_{cor}(\vec{r}, \vec{v}) = 2m\omega(\vec{v} \times \vec{r})_{\omega}$$

Where $r_{\perp}$ is the component of **r** perpendicular to the angular momentum vector ω. Suppose we choose as a coordinate system a cylindrical coordinate system (r,θ,z) with ω along the z-axis. Under these conditions the potential energy terms above becomes

$$U_{cent}(\vec{r}) = \frac{-m \cdot (r\omega)^2}{2} \qquad U_{cor}(\vec{r}, \vec{v}) = 2m\omega \cdot r \cdot v_{\theta} = 2\omega \cdot r \cdot P_{\theta}$$

The relativistic mass times the potential energy for this case (remembering that P_t = imc) gives

$$m \cdot U_{rot} = \left(\frac{2\omega r}{ic}\right) \cdot (P_{\theta} P_t) - \frac{1}{2} \cdot \left(\frac{r\omega}{c}\right)^2 \cdot P_t^2$$

The U_{rot} is the combined potential energy from the centripetal and Coriolis forces. An observer in such a rotating frame could account for the behavior of objects under the above apparent potential energy by the geometry of the space-time of the region. The next task therefore is to bring in the theory of generalized gravity to obtain the geometry of space-time in a rotating frame. To do so, rules for using the theory of generalized gravity must be devised.

Symmetry and scale factors

If the physical situation, i.e. the source of a force on objects, has any symmetry then that symmetry should be manifested in the geometry of the region. This suggests the follows rules:

1. If the source of a potential energy is symmetric about some x_α (i.e. symmetric under the transformation $x_\alpha \rightarrow - x_\alpha$ or $x_\alpha \rightarrow x_\alpha + \delta x_\alpha$), then the scale factors must be independent of x_α.
2. There must be a consistency between observers. For example any 2 points along a trajectory of a beam of light will have ds = 0.
3. If there is no are no other restrictions, $h_j \rightarrow L_j$.

The last restraint must be employed with care.

The scale factors

For the case of a rotating frame with constant ω, there is no dependence on θ, t or z (assuming the axis of rotation is along the z-axis). The scale factors therefore must depend only on r. Moreover the theory of generalized gravity dictates that, because of the Coriolis force potential energy, there must be a non-zero $h_{t\theta}$ and $h_{\theta t}$ term that should be equal to each other. The scale factor can now be known to have the form

$$ds^2=(h_r\cdot dr)^2+(h_z\cdot dz)^2+(h_\theta\cdot d\theta)^2+(2h_{t\theta}^2)\cdot(dt\cdot d\theta)+(h_t\cdot dt)^2$$

Solving for h_θ, h_t and $h_{t\theta}$

The equations for for h_θ, h_t and $h_{t\theta}$ are below, obtained from the generalized gravity equations for h_t and $h_{t\theta}$ respectively.

$$2-\frac{2h_t}{ic}=(\frac{r\omega}{c})^2$$

$$h_{t\theta}^2=(\frac{2\omega r}{ic})\cdot h_t\cdot h_\theta$$

Borrowing from the Schwartzchild metric for time, postulate that h_t has the general form

$$h_t=ic\cdot\sqrt{(1+A\cdot(r\omega)^2)}$$

Remembering that an object can be said to conserve energy only when $h_\alpha \approx L_\alpha$, this must occur in this case for small r. We will approximate $h_t \approx ic\cdot(1 + (A/2)\cdot(r\omega)^2)$ for the calculations.

The equation for h_t now reads

$$2-2\cdot(1+\frac{A}{2})\cdot(r\omega)^2=(\frac{r\omega}{c})^2$$

$$A=\frac{-1}{c^2}$$

The general time scale factor is

$$h_t=ic\sqrt{1-(\frac{r\omega}{c})^2}=ic\sqrt{1-(\frac{r}{r'})^2}\quad (r'\equiv\frac{c}{\omega})$$

Next, substitute h_t into the equation for $h_{t\theta}^2$. Moreover use the approximation for h_t appropriate to the limit $r << r'$.

$$h_{t\theta}^2=(\frac{2\omega r}{ic})\cdot(1-\frac{1}{2}(\frac{r}{r'})^2)\cdot(ich_\theta)$$

For the case of $r << r'$, $h_\theta = L_\theta = r$.

$$h_{t\theta}^2=2\omega r^2$$

This would be true at least for $r << r'$. Expanding to larger r requires one to consider speeds at which a beam of light travels if it does so in the x_k direction. Insisting that $ds^2 = 0$ for any two points along the path light takes gives the constraint.

$$c^2\cdot(\frac{h_t}{ic})^2 = h_{kt}^2\cdot(\frac{c_k'}{|L_k|})+(\frac{h_k c_k'}{|L_k|})^2$$

c_k' is the velocity if light in the x_k direction. For some region, the number of values c_k' indicates something about h_t, h_{kt} and h_k.

- *Light can be at rest in a given region*: $h_t \rightarrow 0$.
- *$c_k' = 0$ for select k*: h_{kt} and/or $h_k \rightarrow \infty$ or if (because of the symmetry of the space) this is not possible then $h_t \rightarrow 0$ and other h_{kt} and/or $h_k \rightarrow \infty$.
- *One nonzero value of c_k' exists*: The quadratic term must vanish, so $h_k \rightarrow 0$.
- *Two nonzero values of c_k' exist*: Neither h_{kt} nor h_k must vanish, or go to ∞ if $h_t \rightarrow 0$.
- *c_k' is zero & one nonzero value*: $h_t \rightarrow 0$ and $h_{kt} \neq 0$ & $h_k \neq 0$.

For the rotating case, as $r \rightarrow r'$, c_θ' must have two options. It can be zero (a non-rotating observer shoots a light beam in the direction of rotation) and some other nonzero value. The above rules indicate that in this limit $h_t \rightarrow 0$ (consistent with the above expression for h_t) and $h_\theta \neq 0$. Moreover we must insist that for $r \ll r'$ $c_\theta' \approx \pm c$, so $h_{\theta t} \rightarrow 0$. The given value for $h_{\theta t}$ bears out the second condition. The first condition can also be satisfied if we assume $h_\theta = L_\theta = r$ for all r.

Solving for h_z and h_r

How about the remaining scale factors, h_z and h_r? The symmetry rule #3 above suggests

$$h_z = L_z = 1$$

To determine h_r imagine the following: An observer is at rest in a rotating frame a distance r from its axis. Another observer is in an inertial frame and determines the first observer is a distance r_I from the axis and is moving at a speed $v = r_i\cdot\omega$. For the case where $r_i = r'$, $v = c$. Any 2 events occurring at the rotating observer will, by the estimation of the inertial frame observer, have $\Delta s = 0$. Our rule #2 suggests that for the rotating observer Δs must still be zero. So relative to this observer

$$0 = -c^2\cdot(1-(\frac{r}{r'}))\cdot\delta t_r^2$$

For this to be true for all δt_r, r must also be r′, so the radial distances must be the same for both rotating and inertial observers.

$$h_r = 1$$

So the space-time interval for a rotating frame is

$$ds^2 = dr^2 + dz^2 + (r\,d\theta)^2 + (2\omega r^2)\cdot(dt\,d\theta) - (\frac{c}{\gamma_r})\cdot dt^2 \qquad \gamma_r \equiv \sqrt{1-(\frac{r}{r'})^2}$$

Geodesics

Suppose an object is in the space-time geometry as given above, what would its path be assuming it is a geodesic? In general relativity a geodesic trajectory is described by

$$\frac{d^2 x_\gamma}{d\lambda^2} = -\sum_{\alpha,\beta} \Gamma_\gamma^{\alpha\beta} \cdot \left(\frac{dx_\alpha}{d\lambda}\right) \cdot \left(\frac{dx_\beta}{d\lambda}\right)$$

The Γ is called the Christoffel Symbol and λ some parameter. The x coordinates span all coordinates and include the scale factors. The sum eliminates any duplicates in indices. The terminology of general relativity will be altered according to the following

Rewriting format of general relativistic equations using scale factors

- Let the x_α in general relativity be written as $L_\alpha \cdot x_\alpha$.
- Let all derivatives of coordinates be rewritten as $L_\alpha \cdot dx_\alpha/d\lambda$. Likewise for all higher order derivatives.
- Do not bother with contra vs. covariant quantities, but use complex numbers in the scale factors instead. This allows on to not need to have any a priori knowledge of a new geometry one is exploring.

With this in mind, the geodesic equation now is

$$L_\gamma \cdot \frac{d^2 x_\gamma}{d\lambda^2} = -\sum_{\alpha,\beta} \left[\Gamma_{\alpha\beta\gamma} \cdot \left(L_\alpha \frac{dx_\alpha}{d\lambda}\right) \cdot \left(L_\beta \frac{dx_\beta}{d\lambda}\right)\right]$$

And the Christoffel Symbol is now

$$\Gamma_{\alpha\beta\gamma} \equiv L_\alpha \cdot \partial_\alpha \left[\frac{h_{\beta\gamma}^2}{2|L_\beta||L_\gamma|}\right] + L_\beta \cdot \partial_\beta \left[\frac{h_{\alpha\gamma}^2}{2|L_\alpha||L_\gamma|}\right] - L_\gamma \cdot \partial_\gamma \left[\frac{h_{\alpha\beta}^2}{2|L_\alpha||L_\beta|}\right]$$

For the case of the rotating frame, the possible nonzero values of Γ are (for r <<r′)

$$\Gamma_{ttr} = \frac{-r}{r'^2}$$

$$\Gamma_{t\theta r} = \frac{-1}{r'}$$

$$\Gamma_{trt} = \frac{r}{r'^2}$$

$$\Gamma_{tr\theta} = \frac{1}{r'}$$

$$\Gamma_{\theta rt} = \frac{1}{r'}$$

In the classical limit, v/c << 1 and r <<r′, a geodesic trajectory obeys the following equations

$$\frac{d^2 r}{dt^2}=\frac{rc^2}{r'^2}+[\frac{c^2}{r'}]\cdot\beta_\theta=\omega^2\cdot r+\omega\cdot v_\theta$$

$$r\cdot\frac{d^2\theta}{dt^2}=\omega\cdot v_r$$

$$c\cdot\frac{d^2 t'}{dt^2}=-(\omega\beta_r\beta_\theta)-\omega^2\cdot(r\beta_r)\rightarrow 0$$

Critique of geodesic

- First term of r acceleration equation is the centrifugal force term, the second term is part of the Coriolis force. It is needed so an object at rest in an inertial frame will move in a circular motion about the z axis with velocity $v_\theta = -\omega\cdot r$.
- The second equation says that an object "falling" in the r direction will be deflected in the –θ direction. This is consistent with the Coriolis potential that favors motion in this direction (all assuming ω is positive).
- The last equation is important in the classical limit for time to be the same for all observers.

The proposed scale factors accurately describe the motion of particles in a rotating frame.

Transformations

Creating a transformation to a rotating frame from an inertial frame (where the rotating axis is at rest) means the following constraints.

- $\Delta\theta_r = 0$ if $\Delta\theta_i = \omega\cdot\Delta t$.
- The inertial observer, relative to the rotating frame, must rotate at angular frequency –ω. This is to satisfy the obvious motion of stars relative to the rotating Earth frame.

Satisfying these constraints means assuming a form for the transformation

$$\theta_r=a\cdot(\theta_i-\omega t_i)\quad r_r=r_i\quad z_r=z_i\quad t_r=at_i+(\frac{r}{c})(d\cdot\theta_i)$$

The variables a & d are unitless quantities. Using these to satisfy the condition that ds^2 for 2 nearby events must be the same for both observers (via numerical methods) gives

$$a=1\quad d=0$$

so

$$\theta_r=\theta_i-\omega t_i\quad r_r=r_i\quad z_r=z_i\quad t_r=t_i$$

A rather Galilean looking transformation!

Relativistic quantum wave equation
Applying the rotating frame scale factors to quantum mechanics means the wave equation becomes.

$$\frac{-\hbar^2}{2\,\mathrm{m}_0}\{\frac{\gamma_\mathrm{r}}{\mathrm{r}}\partial_\mathrm{r}(\frac{\mathrm{r}}{\gamma_\mathrm{r}}\partial_\mathrm{r}\Psi)+\partial_z^2\Psi+\frac{1}{\mathrm{r}^2}\partial_\phi^2\}-\frac{\omega\,\mathrm{r}\,\gamma_\mathrm{r}}{\mathrm{E}_0}\tilde{\mathrm{p}}_\phi\,\tilde{\mathrm{E}}\,\Psi=\frac{\gamma_\mathrm{r}^2}{2\,\mathrm{E}_0}\tilde{\mathrm{E}}^2\Psi-\frac{\mathrm{E}_0}{2}\Psi$$

The tildes signify momentum and total energy (relativistic and potential) operators.

VI. Electromagnetism

A particle of charge q moving in an electromagnetic field can be described by motion under the vector potential **A** & scalar potential V as follows

$$U_{EM}=q\cdot V-(\frac{q}{c})\cdot(\vec{A}\cdot\vec{v})$$

Since electromagnetic forces are a fundamental force, the theory of generalized gravity dictates that this potential energy can be described in general relativistic terms, i.e. the geometry of the universe give rise to electromagnetic forces. The particle mass m times this potential energy gives

$$m\,U_{EM}=q(\frac{p_0}{ic})\cdot V-(\frac{q}{c})\cdot(\vec{A}\cdot\vec{p})$$

The theory of generalized gravity states that this must depend on the product of components of momentum. The solution to this is to create a 5th dimension whose component is associated with charge.

Postulate: *Beyond space and time, a 5th dimension gives rise to electromagnetic forces and charge*

The beauty of this hypothesis is that conservation of charge is automatically guaranteed as a consequence of momentum conservation.

Momentum and velocity in the 5th dimension
The general equation for momentum for this 5th dimension is

$$P_Q=m\cdot L_Q\cdot\frac{dx_Q}{dt}=m\cdot u_Q \qquad with\, u_Q\equiv L_Q\cdot\frac{dx_Q}{dt}$$

In order for the momentum to be proportional to q, u_Q must be proportional to q/m. Getting the units of u_Q right while using constants associated with electromagnetic fields leads to the following hypothesis

$$P_Q=(\alpha\cdot\mu_0)\cdot q$$

μ_0 is the permitivity of free space and α is a constant having units of current. For simplicity, this 5th dimension will simply be referred to as "Q."

Quantizing charge

A consequence of introducing Qto explain electromagnetic force is that constraints can be imposed on q that explain the fact that electric charge comes in units of the electron charge e. If Q is not from -∞ to +∞ but instead is periodic, repeating after a period of l_Q, then the quantum eigenstates for q that maintains single-valuedness would be

$$\Psi_Q = A \cdot e^{\frac{2\pi \cdot n x_Q}{l_Q}}$$

Since charge is quantized, the charge on an object would be q = n·e. Applying the Q momentum operator to the eigenstate and applying the quantization of charge gives

$$\left(\frac{\hbar}{i L_Q}\right) \cdot \partial_Q \Psi_Q = (\alpha \mu_0) \cdot (ne) \Psi_Q \qquad L_Q \cdot l_Q = \frac{h}{(\mu_0 \alpha e)}$$

This gives the circumference around Q. If Q has this circumference charge will be quantized in units of e.

Off diagonal values of the scale factors

The theory of generalized gravity gives the equations

$$h_{QQ} = L_Q \qquad h_{Qt}^2 = \frac{2 L_Q}{\mu_0 \alpha} \cdot V \qquad h_{Qk}^2 = \frac{2 L_Q}{c \mu_0 \alpha} \cdot A_k$$

All other coordinates equal zero. Solving the scale factors remains to be solved once α and L_Q are known.

A few words must be said about V and **A** with regards to gauge invariance. Physically the dynamics of a charged object remains the same when V → V + constant or A → A + grad(f) for some function f. However no such ambiguity must exist in the scale factors. We may impose two constraints on the scale factors

1. The geometry of the universe must approach an orthogonal basis for regions far removed from any charge or currents.
2. Since electric and magnetic fields propagate at the speed of light, the scale factors that give rise to them will also propagate at c.

Given the two constraints, the Lorentz gauge seems to be the best gauge to use in expressing the scale factors in terms of the charge and current density ρ and **J**. From electrodynamics the potentials in a Lorentz gauge[3] are

$$V(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\cdot\int d^3r'\frac{\rho(\vec{r}',\tau)}{|\vec{r}-\vec{r}'|} \qquad \vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\cdot\int d^3r'\frac{\vec{J}(\vec{r}',\tau)}{|\vec{r}-\vec{r}'|}$$

The τ is the retarded time $\tau \equiv |\mathbf{r}-\mathbf{r}'|/c$.

Electromagnetism and Quantum Mechanics

Applying what is known about the scale factors and momentum to the Klein-Gordon wave equation gives

$$\frac{-\hbar^2}{2m_0}\nabla^2\Psi+\frac{qV(\vec{r},t)}{E_0}\tilde{E}\Psi-\sum_k\frac{qA(\vec{r},t)_k}{E_0}\tilde{p}_k\Psi=\frac{1}{2E_0}\tilde{E}^2\Psi-\frac{E_0}{2}\Psi$$

The tildes signify operators and the E operator is just $i\hbar\partial_t$. The derivation assumed $L_Q = 1$ (as will be shown below). For energy eigenstates with $E = E_0+\varepsilon$ this becomes

$$\frac{-\hbar^2}{2m_0}\nabla^2\Psi+qV(\vec{r},t)\Psi-\sum_k\frac{qA(\vec{r},t)_k}{E_0}\tilde{p}_k\Psi+[qV(\vec{r},t)\frac{\epsilon}{E_0}-\frac{\epsilon^2}{2E_0}]\Psi=\epsilon\Psi$$

This is the Schrödinger equation with a relativistic correction in brackets.

L_Q and α

So what of the unsolved variables L_Q, l_Q and α? They can be obtained by considering the geodesics for this space. It can be shown that a particle of charge q moving in the vicinity of a point charge will have as its geodesic a the path described by the coulomb force law (using the geodesic equation above and the scale factors also described above) given the following condition

$$\alpha\cdot L_Q=|L_Q|$$

Considering this as well as all done above let the following be proposed.

$$\alpha=1$$
$$L_Q=i\cdot|L_Q|$$
$$h_{Qt}^2=\frac{2i\cdot|L_Q|}{\mu_0}V(\vec{r})$$
$$h_{Qk}^2=\frac{2i\cdot|L_Q|}{c\mu_0}A_k(\vec{r})$$
$$l_Q=\frac{h}{i\cdot\mu_0 e\cdot|L_Q|}$$

A correction needs to be made because the x_Q coordinate is imaginary as is evident from the last equation. This can be easily corrected by dividing the coordinate x_Q by i. Insuring that the space time interval remains the same despite the change means the new scale factors and l_Q would be

$$L_Q = |L_Q|$$
$$h_{Qt}^2 = \frac{2|L_Q|}{\mu_0} V(\vec{r})$$
$$h_{Qk}^2 = \frac{2 \cdot |L_Q|}{c\mu_0} A_k(\vec{r})$$
$$l_Q = \frac{h}{\mu_0 e \cdot |L_Q|}$$

And the momentum is

$$P_Q = \mu_0 \cdot q$$

Lastly, borrowing the de Broglie wavelength $\lambda_Q = h/P_Q$ and insisting that this must equal l_Q for the case for a particle of elementary charge e gives $|L_Q| = 1$ from above. So

$$h_{Qt}^2 = \frac{2}{\mu_0} \cdot V(\vec{r})$$
$$h_{Qk}^2 = \frac{2}{c\mu_0} \cdot A_k(\vec{r})$$
$$l_Q = \frac{h}{\mu_0 e} = 4.55\,\text{nm}$$

Implications for anti-matter

Earlier it was shown that one could convert a negative energy particle to anti-particle of positive energy by reversing the sign on the momentum 4-vector as well as its charge. In light (no pun intended) of the above conclusions one can simplify this rule as follows:

One can switch between describing a positive and negative energy particle by reversing the sign on the N-dimensional momentum vector and switch from particle to anti-particle.

This may seem on face value to like a restatement of an earlier conclusion. Yet if we expand momentum to a 5-vector to include Q, then this rule automatically covers the switch in charge. Moreover, this rule can serve as a guide to know what quantities have opposite signs between a particle and its anti-particle companion.

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