OPTIMAL SEQUENTIAL DETECTION WITH COMPRESSIVE SENSING

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ABSTRACT

We propose a new framework that combines stochastic optimization and compressive sensing tools to recover the support of a portion of an unknown vector and acquire a utility which decreases linearly with the number of measurement, and increases with the reward associated to a given state of each component. We model a utility function to strive the trade-off between number of measurements and the several benefits these can bring such as: a static non-adaptive approach that leverages on compressed measurement but fails if the spectrum is not sufficiently sparse [1–3], and an active approach which instead finds applications in a wide variety of dynamical sparse systems, in particular spectrum sensing, for which the trade-off is typically between two types of strategies: a static non-adaptive approach that leverages on compressed measurement but fails if the spectrum is not sufficiently sparse [1–3], and an active approach which instead typically only consider one component at a time and it is not able to exploit multiple entries at the same time if empty [4–6]. Recently, there have been numerous results on sequential adaptive strategies for measurement acquisition and the several benefits these can bring in terms of reducing the required number of measurements and being able to cope with lower SNR in the signal reconstruction [7–9]. The common goal in these works is the recovery of the full support of a given vector. However, it is desirable to have enough time to exploit the knowledge acquired during the sensing phase before a transition of the system occurs. In this work we model, with a time-dependent utility function, the trade-off between the desire of recovering a large number of components and the number of measurements needed. The more measurements the decision-maker needs to take, the less time will be available to exploit the knowledge acquired, i.e. a decision on the size of the sub-vector aimed to recover is in place. Our main contribution is the optimization of such decision: this work extends the model proposed in [10] by considering a sequential acquisition of measurement and a trade-off with the remaining time instead of having a predetermined number of measurements available. In order to model our utility function, we will make approximations on the compressive sensing performances; this however does not prevent our selection algorithm to be combined with different aforementioned recovery techniques since, as long as their performances are close enough to our model, we can claim the optimality of our strategy would be preserved. We also emphasized that our model is applicable not only when the utility comes from finding empty entries (e.g. spectrum sensing) but also when we are interested in finding the occupied ones (e.g. radar). The paper is organized as follows: Section 2 is dedicated to the model description and the assumptions on the recovery performance, in Section 3 we study the optimal policy and introduce our Algorithm 1 for a double greedy maximization. Numerical results to sustain our claims are presented in Section 4, where we also introduce our Sequential Lasso Recovery Algorithm. Section 5 concludes our work and presents some promising directions for future research.

1.1. Notation

We use bold lower-case to represent vectors, bold upper-case for matrices and calligraphic letters to indicate sets. For vectors, we use the same letter upper-case to indicate the || · ||_1 norm (i.e. \( \Omega = ||\omega||_1 \)) and with the notation \( s_A \) we select the entries \( i \in A \) of vector \( s \). For any set function \( f(A) \) we define the marginal increment for adding element \( a \), as \( \partial_s f(A) = f(A + a) - f(A) \). The submodularity property for a set function \( f(A) \) that will be used in our proofs is

\[
f(A + a) + f(A + b) \geq f(A + a + b) + f(A).
\]

2. MODEL DESCRIPTION

We consider the situation of an agent “playing a game” where she has \( K \) constants of time available, some for sensing and the rest for exploitation of a set \( N = \{1, 2, \ldots, N\} \) of entries. The agent accrues a reward that is a function of an underlying state vector

\[
s \triangleq [s_1, \ldots, s_N] \in \{0, 1\}^N
\]

where the entries \( s_i \in \{0, 1\} \). We consider the \( s_i \) as indicators of good/bad (0/1) state of a resource, or viceversa depending on the application. For instance, the “idle” or “busy” state of a sub-band in a spectrum sensing framework, where the player has to explore/sense the channels before transmitting in the sub-channels found idle. The player acquires information about the entries via linear measurement of an unknown vector \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T \):

\[
y = B\alpha + w
\]

where \( w \) is the measurement noise. We consider the entries \( \alpha_i \) being non-negative, such that the state variable \( s_i = 1 \) when \( \alpha_i > 0 \) and \( 0 \) when \( \alpha_i = 0 \). We assume the states \( s_i \) are mutually independent Bernoulli random variables with known prior probabilities given by a vector \( \omega = [\omega_1, \omega_2, \ldots, \omega_N] \), where \( \omega_i = P(s_i = 0) \).
2.1. Property of the sequential sensor

**Definition 1** A sensing action $A$ restricts the tests and the consequent reward to a certain subset of the entries $\alpha_A = (\alpha_{a_1}, \ldots, \alpha_{a_{|A|}})$.

From previous definitions, we have that for any subset $A \subseteq \mathcal{N}$, the sparsity of $\alpha_A$ is the discrete random variable $S_A = ||\alpha_A||_0$, whose pmf depends on $\omega_A$. In each test, the player can mix multiple entries of $\alpha_A$ to build a code (compressive sensing tests, or CS tests for short) and learn faster if $S_A$ is small. To simplify the study of the optimal strategy we enforce the following

**Assumption 1** Considering any set $A$ such that $|A| \leq \min(K, N)$, perfect recovery of $\alpha_A$ from the observed values requires a minimum number of tests equal to:

$$K_A \triangleq \min\{c_1 \cdot S_A + c_2, |A|\} \quad c_1 > 1, c_2 \geq 0.$$  \hfill (4)

This assumption is inspired by the sufficient conditions for the unique recovery of sparse solution $\alpha_A$ of the so called LASSO [11]:

$$\min_{\alpha_A} \|\alpha_A\|_1 \text{ subject to } \|y - B\alpha_A\|^2 \leq \epsilon$$ \hfill (5)

with $B$ chosen in a random ensemble. We want to highlight that our framework is in the linear regime for the sparsity of the signal $S_A$, since each entry $s_i$ is a Bernoulli independent random variable, and asymptotically, as will be discussed in the remainder of this work for the Kolomogorov’s strong law we have $S_A \sim |A|$. Our assumption in (4) is therefore compatible with the results in the compressive sensing literature that requires the number of measurements to scale as $O(K \log \frac{N}{|A|})$ to be able to solve via $\ell_1$-minimization, whether we are considering the noisy or the noiseless scenario [11] (and the constant $c_2$ is negligible asymptotically). In the noiseless case, one could in principle have perfect recovery with only $(2S_A + 1)$ measurements but at the expense of solving an $\ell_0$-minimization, which under certain conditions can be relaxed as $\ell_1$-minimization [12]. Note that $|A| \leq N$ and that, if $K \leq N$, then in all cases the player is unable to observe all channels one by one, it can only observe at most $K$ directly. If $K > N$ then the player has the option of examining each entry. To focus on the tradeoff between exploration and exploitation, we also introduce:

**Assumption 2** $K \geq N$ and, in the absence of noise, it is always possible to identify the entire support uniquely by performing more tests.

Another direct consequence of Assumption 1 is that the number of tests $K_A$ for each sensing phase (in the CS case) is a random variable that depends on the set $A$. This is the main difference with respect to [10] where the number of observations is fixed and it is obtained through a filter-bank with multiple sensors.

2.2. Reward

The total reward for the player is proportional to the time left for exploitation $K - K_A$.

**Assumption 3** The reward per unit of time left is a set function $r(A) = r_A(s_A)$ which is normalized ($r(0) = 0$), modular ($\partial_a (r(A)) = r(\alpha)$) and non-negative ($r(a) \geq 0$). In particular we limit to the case where $r_i(0) \cdot r_i(1) = 0$ and $r_i(0) + r_i(1) > 0$.

Assumption 3 indicates that each resource gives a reward only in one of the two possible states (i.e. when $s_i = 0$ or $s_i = 1$) and this will depend on the specific application for our framework.

The immediate reward for the agent selecting the sensing action $A$ can be expressed as

$$R(A) = (K - K_A) \sum_{i \in A} r_i(s_i).$$ \hfill (6)

Let $\pi$ denote the function (policy) that maps the state vector $\omega$ onto a sensing action, i.e. $A = \pi(\omega)$. The policy should optimize the expected reward

$$\pi^* = \arg \max_{\pi} \mathbb{E}[R(\pi(\omega))].$$ \hfill (7)

We then examine the optimum policy.

3. Optimum Policy

The main concern is to understand if there is a way of reducing the combinatorial complexity of having to shift through all possible actions, i.e. through the set $2^\mathcal{N}$ of the parts of $\{1, \ldots, N\}$. For instance, a greedy maximization of $R(A)$ would start with $i = 0$ and $A_0 = \emptyset$ and progressively compose the next set $A_{i+1}$ adding another sub-channel to observe as follows

$$A_{i+1} = A_i + \arg \max_{a \in 2^{\mathcal{N} - A_i}} \partial_a R(A_i).$$ \hfill (8)

By defining the function $\hat{r}_i(\omega) = \omega r_i(0) + (1 - \omega)r_i(1)$, the critical value $\kappa_{|A|} = \frac{|A|}{K_A} + c_2$ and the event $\mathcal{A} \triangleq \{S_A \leq |\kappa_{|A|}|\}$ we can introduce the following Lemma

**Lemma 1** The expected immediate reward can be expressed as

$$\mathbb{E}[R(A)] = R^{DI}(A) + R^{CS}(A)$$ \hfill (9)

where

$$R^{DI}(A) \triangleq \mathbb{E}[(K - A)r_A(s_A)] = (K - |A|) \sum_{i \in A} \hat{r}_i(\omega_i)$$ \hfill (10)

$$R^{CS}(A) \triangleq c_1 \mathbb{E}[(\kappa_{|A|} - S_A)r_A(s_A)|\mathcal{A}] P(\mathcal{A})$$ \hfill (11)

**Proof** By law of total expectation and simple algebraic manipulations of (6). We then introduce the following Lemma

**Lemma 2** $R^{DI}(A)$ is a non-monotone, non-negative sub-modular function of $A$ and a greedy maximization procedure, that stops as soon as the increment is negative achieves the optimal value.

**Proof** To prove the submodularity of $R^{DI}(A)$ we show that the property in (1) leads to $\hat{r}_i(\omega_i) + \hat{r}_j(\omega_j) \geq 0$ which is true by assumption on the reward function $\hat{r}_i(\omega_i)$ and this proves the sub-modularity of $R^{DI}(A)$. The remainder of the proof uses standard optimization tools and is omitted for brevity.

Since $R^{CS}(A) \geq 0$, it follows that adopting a compressive sensing strategy gives higher reward than $R^{DI}(A)$ which correspond to the expected reward given by inspecting only one channel at a time in each slot. Unfortunately, the function $R^{CS}(A)$ is non sub-modular and its maximization is in general NP-hard. However, if we look at the asymptotic behavior of the reward, we can describe a procedure that guarantees a constant factor approximation of the optimal policy and this is the aim of the next section.
### 3.1. Asymptotic analysis

If we consider the regime for $K, N >> 1$ it is immediate to see that $|A|$ will also be large. In particular, we can show

**Lemma 3**

$$S_A = \sum_{i \in A} s_i \Rightarrow \sum_{i \in A} (1 - \omega_i) = |A| - \Omega_A \text{ for } |A| \to \infty \quad (12)$$

which implies $\Pr(S_A = |A| - \Omega_A) \to 1$.

**Proof** By Kolmogorov’s strong law, since $\forall i \in A, \Var(s_i) < \frac{1}{4}$ and $\sum_{i \in A} Var(s_i) < \frac{2}{\pi^2} < \infty$.

This implies that for any given set $A$ we have only two possibilities for the asymptotic expected reward $\mathbb{E}[R(A)]$:

$$\mathbb{E}[R(A)] = \begin{cases} R^{DI}(A) & \text{if } \Omega_A < \frac{(c_1 - 1)|A| + c_2}{c_1} \\ R^\infty(A) & \text{if } \Omega_A \geq \frac{(c_1 - 1)|A| + c_2}{c_1} \end{cases} \quad (13)$$

where

$$R^\infty(A) \triangleq R^{DI}(A) \left(1 + \frac{c_1 \Omega_A + (1 - c_1)|A| - c_2}{K - |A|}\right) \quad (14)$$

The expected reward $\mathbb{E}[R(A)]$ for large $|A|$ can then be expressed as:

$$R(A) = \max \left\{ R^{DI}(A), R^\infty(A) \right\} \quad (15)$$

**Lemma 4** $R^\infty(A)$ is a non-monotone submodular function.

**Proof** By replacing in the definition in (14) we can show that proving the submodularity of $R^\infty$ using (1), is equivalent to prove

$$c_1 \left[ r_i(\omega_i)(1 - \omega_i) + r_i(\omega_i)(1 - \omega_i) \right] > 0$$

which is true since $c_1 > 0$, $r_i(\omega) > 0$ and $\omega_i < 1 \forall i \in N$, and this proves the submodularity of $R^\infty(A)$.

In general, the maximum of two submodular functions is not submodular. Nonetheless, we can introduce the next assumption on the belief vector to guarantee that we can achieve a constant factor approximation of the optimal myopic policy.

**Assumption 4** The belief vector $\omega$ is such that

$$\Omega_N = \sum_{i \in N} \omega_i \geq \frac{c_1 N - K + c_2}{c_1} \quad (16)$$

**Lemma 5** Under Assumption 4, Algorithm 1 guarantees a $\gamma$-approximation of the optimal myopic policy, where

$$\gamma = \max \left\{ 1, \min \left\{ \frac{R^{DI}(A_1)}{\max \left\{ R^\infty(A_2), R^\infty(A_3) \right\}} \right\}_3 \right\} \quad (17)$$

**Proof** See Appendix A.

We then support our claims with numerical simulation in the next section.

### 4. SIMULATION RESULTS

We implemented Algorithm 1 for the selection of the sensing action $A$ and then Algorithm 2 for the sequential recovery. We highlight that in terms of recovery, our approach is suboptimal since it does not exploit the statistical knowledge of the entries, which is only used in the selection of $A$. This choice is motivated by a relatively low complexity compared to other possible approaches that exploit partial knowledge of the support [14–18], since we need to run our algorithm at every acquisition of a new block of $c_1$ measurements. The sensing matrix $B$ is selected via a random permutation of the rows of a $|A| \times |A|$ DFT coefficient matrix [11]. For the noiseless case the threshold $\epsilon$ has been set equal to $0$, while for the noisy scenario we consider a power-limited noise $w_i^2 \leq w_{max}^2$ and set $\epsilon = K\omega w_{max}^2$ at each iteration (for different values of $K\omega$). To study the behavior for increasing $N$, we limit to the case of a fixed ratio $\frac{K}{N} = 1.2$. For each channel $i \in N$ we generated the parameters of our problem as follows: $\omega_i \sim \mathcal{U}([0.6, 0.9])$, $r_i \sim \mathcal{U}([3, 20])$, $\alpha_{i,\text{dB}} \sim \mathcal{U}([3, 20])$. For the noiseless case we set $c_1 = 2$ and $c_2 = 1$. In Fig. 1 we can see how our Sequential Lasso Algorithm perfectly matches Assumption 1, which is indicated as “Black Box” in our simulation. We can also notice the function $R^\infty(A)$ as approximation of $R^A$ for the maximization in Algorithm 1, and therefore the selection of $A$ represents in this case a lower bound of the actual utility. We also simulated the case where the reward is given by the channels found busy, i.e. $r_i(1) > 0$ and $r_i(0) = 0$. Our method is effective also for this case and in Fig. 2 we can see how the approximation $R^\infty(A)$ is now an upper bound for $R(A)$.
For the noisy case, we consider i.i.d Gaussian noise, rescaled in order to have $w_i^2 \leq w_{i,\text{max}}^2$ and choose $w_{i,\text{max}} = \alpha_{i,\text{min}}$. It is sufficient to increase the factor $c_1$ from 2 to 3 to reach the performances of Assumption 1 and this is shown in Figure 3.

We also computed the optimal strategy by exhaustive search for $N$ up to 18 and in Fig.4 we can see how our decision basically matches the optimal one, in terms of expected utility.

5. CONCLUSIONS AND FUTURE WORK

We characterize the optimal strategy and proposed a constant factor approximation algorithm for the selection of the best subset of entries of an unknown vector to recover and accrue a time-dependent utility. A promising path for future work is to combine the recovery phase with the exploitation phase, by starting to accrue a certain utility while still exploring other channels: new results on expander graphs for the design of sparse measurement matrices could be investigated.

A. PROOF OF LEMMA 5

We have that $\mathbb{E}[R(A)] = \max \{ R^{D1}(A), R^\infty(A) \}$ therefore to maximize the myopic policy it is equivalent to find

$$\mathbb{E}[R]^{OPT} = \max_A \mathbb{E}[R(A)] = \max \left\{ \max_A R^{D1}(A), \max_A R^\infty(A) \right\}$$

$$= \max \{ R^{D1,OPT}, R^{\infty,OPT} \}$$

It turns out that, since the function $R^\infty$ is normalized (i.e. $R^\infty(\emptyset) = 0$) Assumption 4, which implies $R^\infty(N) \geq 0$ is sufficient to prove the constant factor approximation as in [13] for non-negative functions. Let us call $A_2^* = \arg \max_A R^\infty(A)$ and $A_2$ the output of Algorithm 1. We have

$$R^{\infty,OPT} \leq R^\infty(A_2^*) + R^\infty(\emptyset) + R^\infty(N)$$

$$\leq R^\infty(A_2^* \cap A_2) + R^\infty(A_2^* \cap A_3) + R^\infty(N)$$

$$\leq R^\infty(A_2^* \cap A_2) + R^\infty(A_2^* \cup A_2) + R^\infty(A_2^*)$$

$$\leq R^\infty(A_2) + R^\infty(A_2) + R^\infty(A_2^*)$$

where (a) follows from Assumption 4, (b) and (c) for submodularity of $R^\infty$ and (d) can be proved by using submodularity and the greedy search property of Algorithm 1 (for details see [13]). It follows that

$$\max \{ R^\infty(A_2), R^\infty(A_2^*) \} \geq \frac{1}{3} R^{\infty,OPT}$$

and the factor $\gamma$ in (17) follows from

$$R^{D1,OPT} \geq \frac{R^{D1,OPT}}{3 \max \{ R^\infty(A_2), R^\infty(A_2^*) \}} R^{\infty,OPT}$$

where it is clear that for $\gamma = 3$, we have $R^{D1,OPT} = R^{OPT}$ and this proves the Lemma.
B. REFERENCES


