2. Variational Framework to Compute OT in $\mathbb{R}^3$

Given a Riemannian manifold $X$ with measure $\mu$ and metric $g$, OT is a measure-preserving mapping $\varphi: X \to X$ inducing the minimum transportation cost which is called the $\rho$-Wasserstein distance.

$$\varphi(X, X) = \arg\min \int c(x, \varphi(x))d\mu(x)$$

$$\rho(\mu, \nu) \triangleq \inf \int \delta^2(x, y) d\mu(x) d\nu(y)$$

Where $c(x, y) = \|x - y\|_2^2$ is the squared Euclidean distance.

We introduce a height vector $(h) \in \mathbb{R}^l$ & approach OT via power Voronoi diagrams.

$$V_i = \{ x \in X | \|x - h_i\|_2 \leq \|x - h_j\|_2 \} \quad \forall j \neq i$$

With the squared radius $r_i = 2h_i - |p_i|$.

Total mass of each cell $\omega_i(h) = \sum_{j=1}^{l} \alpha_i(j) \delta(x_j - h_i)$.

Support $\Omega = \{ x \in X | \mu(x) > 0 \} = \bigcup_i V_i(h)$.

According to [Brenier], the gradient map $\nabla V: V(h) \to \mu$ is the OT map.

Transportation cost $C \triangleq \int_{X} \|
abla T(x)|^2 \mu(x)x$.

Algorithm: Discrete optimal transportation

1. Begin
2. $h = (0, 0, ..., 0)$
3. Repeat
4. Compute $V$ with current $(V, P)$.
5. Compute mass $\omega(h)$ of each cell.
6. Compute gradient $\nabla V(h)$.
7. Compute Hessian $H$.
8. $H \rightarrow \lambda H$ for each cell.
9. Until $|\nabla T(x)| < \epsilon$
10. Return $\nabla T(h)$
11. End

References


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