Note on Matrix Algebra

Definition 1:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

A is called a m×n matrix. (m = # of rows ; n = # of column.)

Definition 2:

Let A be an m×n matrix. The transpose of A is denoted by A^t (or A′), which is a n×m matrix; and it is obtained by the following procedure.

1st column of A → 1st row of A^t

2nd column of A → 2st column of A^t ... etc.

[EXAMPLE]

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 6 & 3 & 3 \end{pmatrix}_{2\times 3} ; A^t = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ 4 & 3 \end{pmatrix}_{3\times 2} .$$

Definition 3:

Let A be a m×n matrix. If m = n, A is called a square matrix.

[EXAMPLE]

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}.$$

Definition 4:

Let A be an m×n matrix. If all the $a_{ij} = 0$, then A is called a zero matrix.

[EXAMPLE]

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A is not a zero matrix but B is.

Definition 5:

Let A be a square matrix. A is call an identity matrix if all the diagonal entries are one and all the off-diagonals are zero.

[EXAMPLE]

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 6:

Let A be a square matrix. A is called symmetric if and only if $A = A^t$ (or A').

[EXAMPLE]

$$A=\begin{pmatrix}1 & 3 & 4\\ 3 & 2 & 1\\ 4 & 1 & 1\end{pmatrix}\rightarrow A^t=\begin{pmatrix}1 & 3 & 4\\ 3 & 2 & 1\\ 4 & 1 & 1\end{pmatrix}.$$

Note:

For any matrix A, A^tA is always symmetric.

Definition 7:

Let A and B are m× n matrices. A + B is obtained by adding corresponding entries of A and B.

[EXAMPLE]

$$\begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}; \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & -3 \end{pmatrix}.$$

Definition 8:

Let A be a m×n matrix and c be a scalar (real number). Then, cA is obtained by multiplying all the entries of A by c.

[EXAMPLE]

$$6 \times \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 24 \\ 18 & 30 \end{pmatrix}.$$

Definition 9:

Let $A_1 = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \end{pmatrix}$ and $B_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix}$.

Then, $A_1B_1 = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1p}b_{p1}$.

[EXAMPLE]

$$A_1 = \begin{pmatrix} 4 & 1 & 2 \end{pmatrix};\ B_1 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

$A_1B_1 = 1\times4 + 2\times1 + 3\times2 = 12$.

Definition 10:

Let A and B are m×p and p×n matrices, respectively.

Let:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix};$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \dots & a_{pn} \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & \dots & B_n \end{pmatrix}.$$

Then, $AB = \begin{pmatrix} A_1B_1 & A_1B_2 & \dots & A_1B_n \\ A_2B_1 & A_2B_2 & \dots & A_2B_n \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \dots & A_mB_n \end{pmatrix}_{m\times n}$.

[EXAMPLE]

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}; B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1\times 4 + 3\times 2 + 5\times 1 & 1\times 3 + 3\times 1 + 5\times 0 \\ 2\times 4 + 4\times 2 + 6\times 1 & 2\times 3 + 4\times 1 + 6\times 1 \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 22 & 10 \end{pmatrix}.$$

Definition 11:

Let A and B are n×n matrices. If AB = I_n or BA = I_n, then B is called the inverse of A, and is denoted by A^{-1}.

[EXAMPLE]

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \rightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

[EXAMPLE]

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}.$$

(Check this by yourself.)

Theorem 1:

Let A be a m×n matrix. Then, $I_m A = A I_n = A$.

Theorem 2:

Let A and B are m×p and p×n matrices. Then,

$$(AB)^t = B^t A^t.$$

[EXAMPLE]

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}; B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[Check Theorem 2, using this example.]

Theorem 3:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

[EXAMPLE]

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \to A^{-1} = \frac{1}{3 \times 1 - 1 \times 2}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$