TOPIC III
LINEAR ALGEBRA

[1] Linear Equations

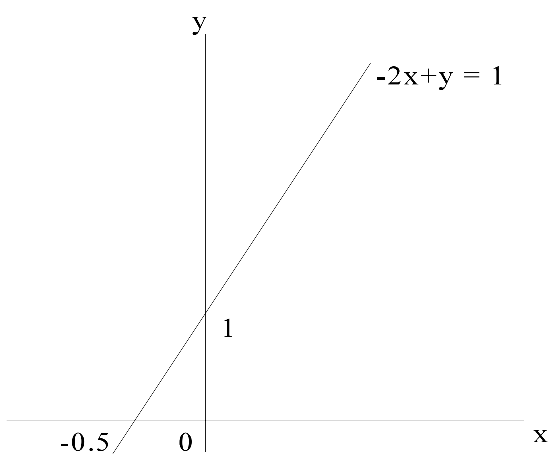
(1) Case of Two Endogenous Variables

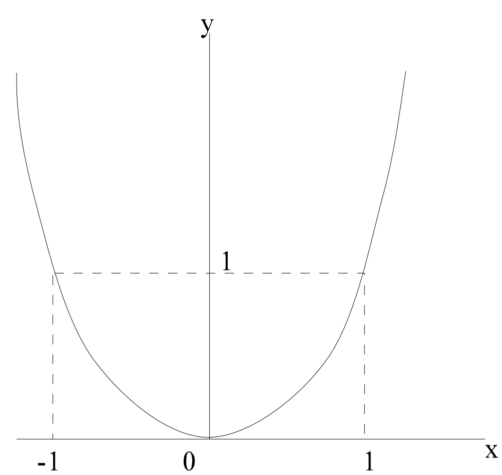
1) Linear vs. Nonlinear Equations

- Linear equation: $ax + by = c$, where a, b and c are constants.
- Nonlinear equation: $ax^2 + by = c$.
- Can express a linear equation by a line in a graph.

EX 1: $-2x + y = 1$

EX 2: $-x^2 + y = 0$ (nonlinear equation)





2) System of Linear Equations

- m equations:

$$a_1x + b_2y = c_1$$
$$a_2x + b_2y = c_2$$
$$\vdots$$
$$a_mx + b_my = c_m$$

- If $(\bar{x}, \bar{y})$ satisfies all of the equations, it is called a solution.
- 3 possible cases:
  - Unique solution
  - No solution
  - Infinitely many solutions

EX 1: Case of no solution (inconsistent equations)

1) $x - y = 1$;
2) $x - y = 2$ $\Rightarrow$ inconsistent. $\Rightarrow$ No solution.

EX 2: Case of infinitely many solutions

(two equations and one redundant equation)

1) $x - y = 1$;
2) $2x - 2y = 2$ $\Rightarrow (\bar{x}, \bar{y}) = (1,0), (2,1), \ldots$

EX 3: Case of unique solution

(two equations and no inconsistent and no redundant equations)

1) $x - y = 1$;
2) $x + y = 1$ $\Rightarrow (\bar{x}, \bar{y}) = (1,0)$ (unique)

(2) Extension to Systems of Multiple Endogenous Variables

- System of m equations and n endogenous variables:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

→ variables: $x_1, \ldots, x_n$

constants: a_{ij}, b_i ($i = 1,\ldots,m$; $j = 1,\ldots n$)

- If $(\bar{x}_1, \ldots, \bar{x}_n)$ satisfies all of the equations, it is called a solution.
- 3 possible cases.
- Is there any systematic way to solve the system?

[3] Matrix and Matrix Operations

Definition:

- A matrix, A, is a rectangular array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n}$$

  where i denotes row and j denotes columns.

- A is called a m × n matrix. (m = # of rows ; n = # of columns.)

EX:

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$$, $[2\ 1\ 0\ \text{-}3]_{1\times 4}$, $[4]_{1\times 1}$ (scalar).

Definition:

If m = n for a m × n matrix A, A is called a square matrix.

EX:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

Definition:

Let A be a m × n matrix. The transpose of A is denoted by A^t (or A'), which is a n × m matrix; and it is obtained by the following procedure.

- 1st column of A ⇒ 1st row of A^t
- 2nd column of A ⇒ 2st column of A^t ... etc.

EX:

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 6 & 3 & 3 \end{bmatrix} ; \; A' = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}.$$

Definition:

Let A be a square matrix. A is called symmetric if and only if $A = A^t$ (or A').

EX:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} = A^t .$$

Note:

For any matrix A, A^tA is always symmetric.

Definition:

Let $A = [a_{ij}]$ be a m×n matrix. If all of the $a_{ij} = 0$, then A is call a zero matrix.

EX:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \;;\; B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \;\Rightarrow\; \text{A is not a zero matrix, but B is.}$$

Definition:

Let A and B are m × n matrices. A + B is obtained by adding corresponding entries of A and B.

EX:

$$\begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \;;\; \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -2 & -3 \end{bmatrix}.$$

Definition:

Let A be a m × n matrix and c be a scalar (real number). Then, cA is obtained by multiplying all the entries of A by c.

EX:

$$6 \times \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 18 & 30 \end{bmatrix}.$$

Definition:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \end{bmatrix} ;\ B_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{bmatrix} \Rightarrow A_1B_1 = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1p}b_{p1}.$$

EX:

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} ;\ B_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} . \quad \rightarrow \ A_1B_1 = 1 \times 4 + 2 \times 1 + 3 \times 2 = 12$$

Definition:

Let A and B are m × p and p × n matrices, respectively.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} ; B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2b} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 & \cdots & A_1B_n \\ A_2B_1 & A_2B_2 & \cdots & A_2B_n \\ \vdots & \vdots & & \vdots \\ A_mB_1 & A_mB_2 & \vdots & A_mB_n \end{bmatrix}_{m \times n}$$

EX 1:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \; ; \; B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\Rightarrow \quad AB = \begin{bmatrix} 1\times4+3\times2+5\times1 & 1\times3+3\times1+5\times0 \\ 2\times4+4\times2+6\times1 & 2\times3+4\times1+6\times0 \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 22 & 10 \end{bmatrix}.$$

EX 2:

- $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

  $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

  $\vdots$

  $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$

- Define:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \; ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Observe that

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_m \end{bmatrix} = b.$$

- Thus, we can express the above system of equations by:

  $Ax = b.$

EX 3:

$x_1 + 2x_2 = 1;\ x_2 = 0$

→ $1 \cdot x_1 + 2 \cdot x_2 = 1$

$0 \cdot x_1 + 1 \cdot x_2 = 0$

→ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

<u>Some Caution in Matrix Operations</u>:

(1) AB may not equal BA.

EX: $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \Rightarrow$ Can show $AB \neq BA$.

(2) Suppose that AB = AC. It does not mean that B = C.

(3) AB = 0 (zero matrix) does not mean that A = 0 or B = 0.

EX: $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$; $D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ Can show AB = AC and AD = $0_{2\times2}$ (zero matrix)

Definition:

Let A be a square matrix. A is call an identity matrix if all of the diagonal entries are one and all of the off-diagonals are zero.

EX:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note:

For $A_{m\times n}$, $I_m A_{m\times n} = A_{m\times n}$ and $A_{m\times n} I_n = A_{m\times n}$.

Definition:

For $A_{n\times n}$ and $B_{n\times n}$, B is the inverse of A iff $AB = I_n$ or $BA = I_n$.

EX:

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} ; B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow B = A^{-1}$$

Note:

The inverse is unique, if it exists.

Theorem:

$$A_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ If ad = bc, no inverse.

Theorem:

Suppose that $A_{n\times n}$ and $B_{n\times n}$ are invertible. Then, $(AB)^{-1} = B^{-1}A^{-1}$.

EX 1:

$A^n = A \cdot A \cdot \cdots A$, n times. $\Rightarrow (A^n)^{-1} = A^{-1} \cdots\cdot A^{-1} = (A^{-1})^n$.

EX 2:

- A system of linear equations is given by

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Assume m = n, and A is invertible. Then,

$$A^{-1}Ax = A^{-1}b \Rightarrow I\,x = A^{-1}b \Rightarrow \bar{x} = \begin{bmatrix} \bar{x}_1 \\ : \\ \bar{x}_n \end{bmatrix} = A^{-1}b \text{ (solution)}.$$

[4] Determinant

Definition:

$$\text{Let } A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \text{ Then, } |A| \equiv \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

EX:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \Rightarrow \det(A) = 8 - 3 = 5$$

Definition:

$$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \Rightarrow \begin{matrix} a_{11} & a_{12} & a_{13} & \vdots & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & \vdots & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & \vdots & a_{31} & a_{32} & a_{33} \end{matrix}$$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 3 & 4 \end{matrix} \Rightarrow \det(A) = 20 + 2 + 36 - 3 - 32 - 15 = 58 - 50 = 8.$$

Definition:

Let $A_{n\times n} = [a_{ij}]$. Then,

- Minor of a_{ij} ($\equiv M_{ij}$) = $(n-1) \times (n-1)$ matrix excluding i^{th} row and j^{th} column of $A_{n\times n}$
- Cofactor of a_{ij} ($\equiv |C_{ij}|$) = $(-1)^{i+j}|M_{ij}|$

Definition: (Laplace expansion)

- $\det(A) = \sum_{j=1}^{n} a_{ij}|C_{ij}|$ for any given i $= a_{i1}|C_{i1}| + ... + a_{in}|C_{in}|$.

  $= \sum_{i=1}^{n} a_{ij}|C_{ij}|$ for any given j $= a_{1j}|C_{1j}| + ... + a_{nj}|C_{nj}|$.

- This holds for general n.

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 3 & 4 \end{bmatrix}$$

- Choose 1^{st} row:

$$a_{11} = 1: \quad |M_{11}| = \begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix} = 17 \Rightarrow |C_{11}| = (-1)^{1+1}|M_{11}| = 17$$

$$a_{12} = 2: \quad |M_{12}| = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15 \Rightarrow |C_{12}| = (-1)^{1+2} \bullet 15 = -15$$

$a_{13} = 3: \; |M_{13}| = \begin{vmatrix} 4 & 5 \\ 1 & 3 \end{vmatrix} = 7 \Rightarrow |C_{13}| = (-1)^{1+3} \bullet 7 = 7$

$\Rightarrow \det(A) = a_{11}|c_{11}| + a_{12}|c_{12}| + a_{13}|c_{13}| = 1 \times 17 + 2 \times (-15) + 3 \times 7 = 8$

- Choose the 2nd row and do Laplace expansion (Do it by yourself).

EX:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 3 & 1 & 4 \end{bmatrix} \Rightarrow \det(A) = 3(-1)^{3+1} \begin{bmatrix} 2 & 1 & 2 \\ 6 & 1 & 1 \\ 3 & 1 & 4 \end{bmatrix}.$$

Theorem:

Consider $A_{n\times n}$. If all of the entries in the i^{th} row of A are zero, det(A) = 0.

Definition: (Triangular matrices)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \text{ upper t.m; } B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \text{ lower t.m.}$$

EX: (not triangular)

$$B = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 6 & 4 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem:

Let $A_{n\times n}$ be a triangular matrix. Then, det(A) = product of diagonals.

EX:

$\det(A) = 1 \times 4 \times 6 = 24; \det(B) = 1 \times 3 \times 6 = 18$

Theorem:

(a) If B is a matrix that results when a single row (column) of A is multiplied by a constant k, then det(B) = k•det(A).

EX:

$$A = \begin{bmatrix} 1 \quad 2 \quad 3 \quad 4 \\ \text{something} \end{bmatrix}; B = \begin{bmatrix} 2 \quad 4 \quad 6 \quad 8 \\ \text{same as A} \end{bmatrix} \Rightarrow \det(B) = 2\times\det(A).$$

- Same results when $A = \begin{bmatrix} 1 & \\ 2 & \\ 3 & \text{something} \\ 4 & \end{bmatrix}$ & $B = \begin{bmatrix} 2 & \\ 4 & \\ 6 & \text{same as A} \\ 8 & \end{bmatrix}$.

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}; B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\rightarrow \det(B) = 2 \bullet \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = 2 \bullet 2 \bullet \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 4 \bullet \det(A).$$

(b) If B results when a multiple of one row (column) of A is added to another row (column), det(B) = det(A).

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ & \text{something} & & \end{bmatrix}; B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ \text{same} & \text{as} & \text{A} & \end{bmatrix} \Rightarrow \det(B) = \det(A)$$

- Same results when $A = \begin{bmatrix} 1 & 2 & \\ 2 & 3 & \\ 3 & 4 & \cdots \\ 4 & 5 & \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & \\ 2 & 1 & \\ 3 & 1 & \cdots \\ 4 & 1 & \end{bmatrix}$.

(c) If B results when two rows (columns) of A are interchanged,

det(B) = -det(A)

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ & \text{same} & \end{bmatrix}; B = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ & \text{same} & \end{bmatrix} \Rightarrow \det(B) = -\det(A)$$

- Same results if $A = \begin{bmatrix} 1 & 4 & \\ 2 & 5 & \text{same} \\ 3 & 6 & \end{bmatrix}$ & $B = \begin{bmatrix} 4 & 1 & \\ 5 & 2 & \text{same} \\ 6 & 3 & \end{bmatrix}$.

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow \quad \det(A) = \det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Note:

If two rows or two columns are the same, then det = 0.

EX 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow \det(A) = 1 + 8 - 3 - 8 = -2$$

(1) $A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$. $\Rightarrow$ 1^{st} row of $A_1 = 4 \times 1^{st}$ row of A

$\Rightarrow \det(A_1) = 4 \times \det(A) = -8$ [by (a)]

(2) $A_2 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow 2^{nd}$ row of $A_2 = 2^{nd}$ row of A - 2 × 1^{st} row of A

$\Rightarrow \det(A_2) = \det(A) = -2$ [by (b)]

(3) $A_3 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow 1^{st}$ and 2^{nd} rows of A interchanged.

$\Rightarrow \det(A_3) = -\det(A) = 2$ [by (c)]

EX 2:

$$A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 6 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \det(A) = -\det\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (-)(-)\det\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (+)1\bullet 1\bullet 3\bullet 1 = 3.$$

EX 3:

$$B = \begin{bmatrix} 0 & 0 & 2 & 2 & 2 \\ 3 & -3 & 3 & 3 & 3 \\ -4 & 4 & 4 & 4 & 4 \\ 4 & 3 & 1 & 9 & 2 \\ 1 & 2 & 1 & 3 & 1 \end{bmatrix} \Rightarrow \det\begin{bmatrix} 0 & 0 & 2 & 2 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 4 & 3 & 1 & 9 & 2 \\ 1 & 2 & 1 & 3 & 1 \end{bmatrix} = \det\begin{bmatrix} 0 & 0 & 2 & 2 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & 9 & 2 \\ 1 & 2 & 1 & 3 & 1 \end{bmatrix} = 0.$$

[2nd row - (3/2)•1st row] [3rd row + (4/3)•2nd row]

[3rd row - 2•1st row]

Definition:

Let $A_{m\times n} = [a_{ij}]$. Then, A^t (transpose of A) = $[a_{ji}]_{n\times m}$

EX:

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}.$$

Theorem:

(a) $(A^t)^t = A$

(b) $(A+B)^t = A^t + B^t$

(c) $(AB)^t = B^tA^t$

(d) $\det(A^t) = \det(A)$.

Theorem:

Let A and B be n×n matrices. Then, $\det(AB) = \det(A) \bullet \det(B)$.

EX:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix};\ B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix} \quad \Rightarrow |A| = 3 - 2 = 1;\ |B| = -8 - 15 = -23$$

$$\Rightarrow |A||B| = -23.$$

$$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix} \quad \Rightarrow |AB| = 28 - 51 = -23!!!$$

Note:

$\det(A+B) \neq \det(A) + \det(B)$.

Theorem:

$A_{n\times n}$ invertible iff $\det(A) \neq 0$.

Theorem:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

EX:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow \det(A) = 0 \Rightarrow A \text{ is not invertible.}$$

Notation:

When det(A) = 0, we say A is singular.

When det(A) $\neq$ 0, we say A is nonsingular.

[5] Inverse

Some useful facts for A^{-1}:

(1) $$A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 & \dots & 0 \\ 0 & 1/a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1/a_{nn} \end{bmatrix}$$

(2) $$A = \begin{bmatrix} B_{p\times p} & O_{p\times q} \\ O_{q\times p} & C_{q\times q} \end{bmatrix}_{(p+q)\times(p+q)} \Rightarrow A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$$

EX1:

$$A = \begin{bmatrix} 5 & 7 & 0 \\ 7 & 10 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix}^{-1} = \begin{bmatrix} 10 & -7 \\ -7 & 5 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 10 & -7 & 0 \\ -7 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

EX2:

$$\begin{bmatrix} 5 & 7 & 0 & 0 \\ 7 & 10 & 0 & 0 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 7 & 10 \end{bmatrix} = A. \quad A^{-1} = ?$$

Definition:

Let $A = [a_{ij}]_{n\times n}$;

$$C = \text{matrix of cofactors} = \begin{bmatrix} |C_{11}| & \cdots & |C_{1n}| \\ |C_{21}| & \cdots & |C_{2n}| \\ \vdots & & \vdots \\ |C_{n1}| & \cdots & |C_{nn}| \end{bmatrix}; \text{ and } \operatorname{adj}(A) = C^t.$$

Theorem:

Suppose that A is invertible. Then,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Note:

A^{-1} exists iff $\det(A) \neq 0$.

EX:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$|C_{11}| = (-1)^{1+1}\begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12;\ |C_{12}| = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6;\ |C_{13}| = (-1)^{1+3}\begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -16.$$

$$|C_{21}| = (-1)^{2+1}\begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = 4;\ |C_{22}| = 2;\ |C_{23}| = 16;$$

$|C_{31}| = 12$; $|C_{32}| = -10$; $|C_{33}| = 16$.

$$\Rightarrow C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix} \Rightarrow C^t = adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$\Rightarrow det(A) = 12 + 4 + 12 + 36 = 64$$

$$\Rightarrow A^{-1} = \frac{1}{64} \times \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

[4] Cramer's Rule

- Consider Ax = b where $A_{n\times n}$.
    - n equations and n unknowns:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- How can we find solutions, $\bar{x}_1 \cdots \bar{x}_n$?
    - Assume that A is invertible.
    - Ax = b

$$\Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow I_n x = A^{-1}b \Rightarrow \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} = A^{-1}b$$

- Cramer's Rule:

  Consider Ax = b, where $A_{n\times n}$. Define

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}. \text{ Then, } \bar{x}_j = \frac{\det(A_j)}{\det(A)}.$$

EX:

$$x_1 \qquad + 2x_3 = 6$$

$$-3x_1 + 4x_2 + 6x_3 = 30$$

$$-x_1 - 2x_2 + 3x_3 = 8$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} : b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix} : x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$\Rightarrow A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}; A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

$$\Rightarrow \bar{x}_1 = |A_1|/|A|;\ \bar{x}_2 = |A_1|/|A|;\ \text{and}\ \bar{x}_3 = |A_1|/|A|.$$

EX:

$$Q_d = a + bp\ ;\ \ Q_s = c + dp$$

$\Rightarrow$ Then, $\qquad\qquad\qquad \Rightarrow$

$$\bar{Q} = a + b\bar{P} \qquad\qquad \bar{Q} - b\bar{P} = a$$

$$\bar{Q} = c + d\bar{P} \qquad\qquad \bar{Q} - d\bar{P} = c$$

$$\Rightarrow \quad \underset{A}{\begin{bmatrix} 1 & -b \\ 1 & -d \end{bmatrix}} \underset{x}{\begin{bmatrix} \bar{Q} \\ \bar{P} \end{bmatrix}} = \underset{b}{\begin{bmatrix} a \\ c \end{bmatrix}}$$

$$\Rightarrow \quad A_1 = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \Rightarrow \det(A_1) = -ad + bc; \; A_2 = \begin{bmatrix} 1 & a \\ 1 & c \end{bmatrix} \Rightarrow \det(A_2) = c - a$$

$$\Rightarrow \quad \det(A) = -d + b$$

$$\Rightarrow \bar{Q} = \frac{\det(A_1)}{\det(A)} = \frac{-ad + bc}{-d + b} = \frac{ad - bc}{d - b}; \quad \bar{P} = \frac{\det(A_2)}{\det(A)} = \frac{c - a}{-d + b} = \frac{a - c}{d - b}$$

EX:

$$Y = C + I_0 + G_0$$

$$C = a + bY$$

⇒ Then,

$$Y \quad - C = I_0 + G_0$$

$$-bY + C = a$$

$$\Rightarrow \underset{A}{\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} \bar{Y} \\ \bar{C} \end{bmatrix}} = \underset{b}{\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}}$$

$$\Rightarrow \det(A) = 1 - b$$

$$\Rightarrow \det(A_1) = \begin{bmatrix} I_0 + G_0 & -1 \\ a & 1 \end{bmatrix} = I_0 + G_0 + a; \; \det(A_2) = \begin{bmatrix} 1 & I_0 + G_0 \\ -b & a \end{bmatrix} = a + b(I_0 + G_0)$$

$$\Rightarrow \bar{Y} = \frac{I_0 + G_0 + a}{1 - b}; \quad \bar{C} = \frac{a + b(I_0 + G_0)}{1 - b}.$$

- Solution Outcomes for a Linear-Equation System:

$Ax = b$ where $A_{n\times n}$.

- When $b \neq 0$:
  - If $|A| \neq 0$, there is a unique solution $\bar{x} \neq 0$.

    $\rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  - If $|A| = 0$, there is an infinite number of solutions or no solution.

    $\rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow$ no solution ($\bar{x}_1 = |A_1|/|A| = 1/0$)

    $\rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ infinitely many solutions.

($\bar{x}_1 = |A_1|/|A| = 0/0$)

- When $b = 0$:
  - If $|A| \neq 0$, there is a unique solution $\bar{x} = 0$.

    $\rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. ($\bar{x}_1 = 0/1$)

  - If $|A| = 0$, there is an infinite number of solutions.

    $\rightarrow$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ infinitely many solutions. ($\bar{x}_1 = 0/0$)

[5] Leontief Input-Output Model

(1) An Example of Simple Economy:

1) Three industries (industry 1, 2 and 3; say, steel, autos and food)

2) An industry's products are used for production of the industry itself and others:

- The products of industries 1, 2 and 3 are used for industry 1.

⇒ a_{11} = $ worth of ind. 1's product required for $1 worth of ind. 1's product.

($a_{11} = 0.2 \Rightarrow$ $.2 worth of steel is needed to produce $1 worth of steel.)

⇒ a_{21} = $ worth of ind. 2's product required for $1 worth of ind. 1's product

($a_{21} = 0.4 \Rightarrow$ $.4 worth of autos is needed to produce $1 worth of steel.)

⇒ a_{31} = $ worth of ind. 3's product required for $1 worth of ind. 1's product

($a_{31} = 0.1 \Rightarrow$ $.1 worth of food is needed to produce $1 worth of steel.)

- The products of industries 1, 2 and 3 are used for industry 2.

⇒ a_{12} = $ worth of ind. 1's product required for $1 worth of ind. 2's product

($a_{12} = 0.3 \Rightarrow$ $.3 worth of steel is needed to produce $1 worth of autos.)

⇒ a_{22} = $ worth of ind. 2's product required for $1 worth of ind. 2's product

($a_{22} = 0.1 \Rightarrow$ $.1 worth of autos is needed to produce $1 worth of autos.)

⇒ a_{32} = $ worth of ind. 3's product required for $1 worth of ind. 2's product

($a_{32} = 0.3 \Rightarrow$ $.3 worth of food is needed to produce $1 worth of autos.)

- The products of industries 1, 2 and 3 are used for industry 3.

⇒ $a_{13} = 0.2 \Rightarrow$ $.2 worth of steel is needed to produce $1 worth of food.

⇒ $a_{23} = 0.2 \Rightarrow$ $.2 worth of autos is needed to produce $1 worth of food.

⇒ $a_{33} = 0.2 \Rightarrow$ $.2 worth of food is needed to produce $1 worth of food.

3) Final consumption (noninput demand):

- d_1 = $ worth of households' consumption of steel.
- d_2 = $ worth of households' consumption of autos.
- d_3 = $ worth of households' consumption of food.

4) The sum of input values need to produce $1 worth of industry j's (j = 1, 2, 3) products should not be greater than 1:

$$a_{11}+a_{21}+a_{31} = \sum_{i=1}^{3} a_{i1} < 1 \;;\; a_{12}+a_{22}+a_{32} = \sum_{i=1}^{3} a_{i2} < 1 \;;\; a_{13}+a_{23}+a_{33} = \sum_{i=1}^{3} a_{i3} < 1.$$

5) Let x_j be the total value of industry j's products. Then,

- $x_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + d_1$:

  $\Rightarrow$ $a_{11}x_1$ = total value of ind. 1's products used in ind. 1.

  $\Rightarrow$ $a_{12}x_2$ = total value of ind. 1's products used in ind. 2.

- $x_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + d_2$;
- $x_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + d_3$.

6) Let:

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$, which is called "input coefficient matrix".

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};\; d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

7) Then,

$Ix = Ax + d.$

→ $(I-A)x = d$, where I-A is called "technology matrix."

→ If we assume (I-A) is invertible and d is given,

$\bar{x} = (I-A)^{-1}d.$

→ In our example, if $d = (10,5,6)'$, we can obtain

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = (I-A)^{-1}d = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}.$$

(2) Generalization:

1) Suppose that there are n industries. Then we can define $A_{n\times n}$, $x_{n\times 1}$ and $d_{n\times 1}$.

2) Then, we have $(I-A)x = b$.

3) If (I-A) is invertible and b is given, we can solve for $\bar{x}$.