[6] Vector

Definition:

Vector is a n×1 or 1×n matrix.

Note:

- Vector is usually expressed by the form of (1,1) or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- (1,1) ⇒ row vector; $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ⇒ column vector.
- When people talk about vectors, they are usually column vectors.
- Vector arithmetics follow matrix arithmetics.

EX:

$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$; $u = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

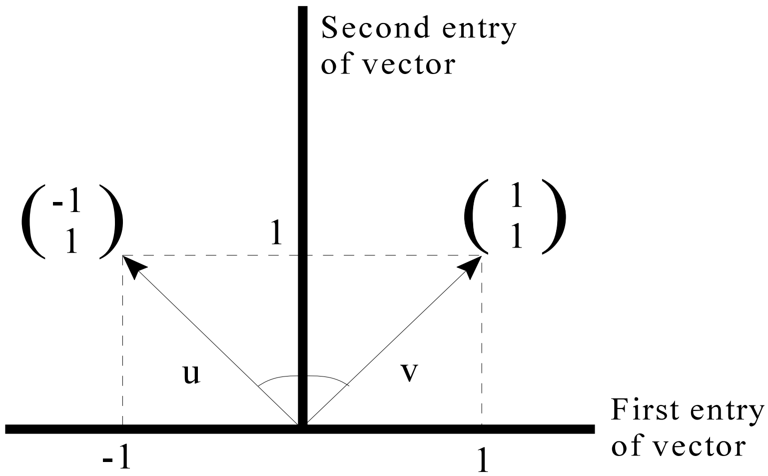
- $v + u = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$; $av = \begin{pmatrix} ax_1 \\ ax_2 \end{pmatrix}$; $v'u = (x_1,x_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1y_1 + x_2y_2$

Definition:

Two vectors u and v are called orthogonal iff $u'v = 0$.

EX:

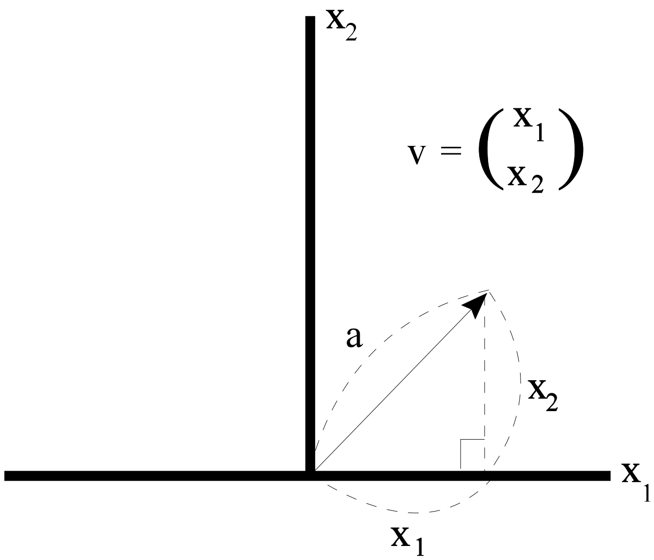
$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } u = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow u'v = 0.$$



Definition:

Norm of a vector v ≡ $\|v\|$= Length of v

EX:



- $a^2 = x_1^2 + x_2^2 \Rightarrow a = \sqrt{x_1^2 + x_2^2}$ (Pythagoras' Theorem)

- $\|v\| = \sqrt{x_1^2 + x_2^2}$.

- In general,

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow \|v\| = \sqrt{\Sigma_i x_i^2}.$$

- Classic mistake:

$$\sqrt{\Sigma_i x_i^2} = \Sigma_i x_i \text{ (please say no!!)}$$

EX:

$$v = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \Rightarrow \|v\| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}.$$

Definition: (Distance between two vectors)

Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $u = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. Then, $d(v,u) = \|v-u\| = \sqrt{\Sigma_i (x_i - y_i)^2}$.

$[= \Sigma_i(x_i - y_i)$ (No!!!)]

Definition:

Let $v_1, ..., v_r \in \mathbb{R}^n$. Suppose that $\exists$real #s $a_1, ..., a_r \ni w = a_1v_1 + ... + a_rv_r$. Then, w is called a linear combination of $v_1 \cdots v_r$.

EX:

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix};\ v_2 = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix};\ w = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}.$$ Show that w is a linear combination.

<proof>

- Have to show that $\exists a_1$ and a_2 such that $a_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + a_2\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}$.

- Set $\begin{pmatrix} a_1 \\ 2a_1 \\ -a_1 \end{pmatrix} + \begin{pmatrix} 6a_2 \\ 4a_2 \\ 2a_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}$.

$\Rightarrow$ ① $a_1 + 6a_2 = 9$

② $2a_1 + 4a_2 = 2$

③ $-a_1 + 2a_2 = 7$

From ① and ②,

④ $a_1 = -3, a_2 = 2$

We can show ④ satisfies ③.

So, solutions are $a_1 = -3, a_2 = 2$.

- $w = -3v_1 + 2v_2 \Rightarrow w$ is a linear combination of v_1 and v_2.

EX:

Is $\xi = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}$ a linear combination of v_1 and v_2?

- Set $\underset{\xi}{\begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}} = a_1 \underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} + a_2 \underset{v_2}{\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}}$

- ① $a_1 + 6a_2 = 4$

  ② $2a_1 + 4a_2 = -1$

  ③ $-a_1 + 2a_2 = 8$

- From ① and ②,

  ④ $a_1 = -\frac{22}{8}$; $a_2 = \frac{9}{8}$.

- But, this solution does not satisfy ③.
- No solution for a_1 and a_2.
- ξ is not a linear combination of v_1 and v_2.

Definition:

$v_1 \cdots v_r$ are linearly independent,

iff $a_1 = \cdots = a_r = 0$ whenever $a_1v_1 + \cdots + a_rv_r = 0$ [zero vector].

EX1:

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Show that e_1 and e_2 are lin. indep.

<proof>

- Start by assuming $a_1e_1 + a_2e_2 = 0_{2\times 1}$:

$$a_1e_1 + a_2e_2 = a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$\Rightarrow a_1 = a_2 = 0.$

- e_1 and e_2 are linearly independent.

EX2:

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- Start by assuming $a_1v_1 + a_2v_2 = 0_{2\times 1}$:

$$\begin{pmatrix} a_1 + a_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a_2 = 0, a_1 = 0.$

- v_1 and v_2 are linearly independent.

EX3:

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

- $a_1v_1 + a_2v_2 = \begin{pmatrix} a_1 + 2a_2 \\ a_1 + 2a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- A solution is $a_1 = 2$, $a_2 = -1$ (Not only $a_1 = 0$, $a_2 = 0$)
- v_1 and v_2 are linearly dependent.

Implication of linear dependence:

- Suppose that $a_1v_1 + \cdot\cdot\cdot + a_rv_r = 0$ and $a_1 \neq 0$. Then,

$$v_1 = \left(-\frac{a_2}{a_1}\right)v_2 + \cdots + \left(-\frac{a_r}{a_1}\right)v_r.$$

  $\Rightarrow v_1$ is a linear combination of $v_2 \cdots v_r$.
- So, $v_1 \cdots v_r$ are linearly independent iff no one is a lin. comb. of others.

Definition:

Let $v_1 \cdots v_r \in \mathbb{R}^m$. Suppose that for any $u \in \mathbb{R}^m$, $\exists a_1 \cdots a_r$ such that,

$$u = a_1v_1 + \cdots + a_rv_r.$$

Then, we say $\{v_1, \cdots v_r\}$ spans $\mathbb{R}^m$.

EX1:

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Show that e_1 and e_2 span $\mathbb{R}^2$.

<proof>

- Let $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. It is enough to show that

$\exists a_1, a_2 \in \mathbb{R} \Rightarrow u = a_1 e_1 + a_2 e_2$.

- Note that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\Rightarrow e_1$ and e_2 span $\mathbb{R}^2$

EX2:

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Show that v_1 and v_2 span $\mathbb{R}^2$.

<proof>

- Have to show that $\exists a_1, a_2 \in \mathbb{R}$ such that $a_1 v_1 + a_2 v_2 = u$, for all $u \in \mathbb{R}^2$.

- Note that $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$.

$$= (x_1 - x_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Thus, v_1 and v_2 span $\mathbb{R}^2$.

EX3:

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Do they span $\mathbb{R}^2$?

<proof> Do it by yourself.

Note:

- In EX1) and EX2), two vectors are lin. indep.
- In EX3), two vectors are lin. dep.

Theorem:

$v_1 \cdots v_m, v_{m+1} \cdots v_n$ $(m < n) \in \mathbb{R}^m$. Then, at best m vectors are linearly independent.

Theorem:

Let $v_1, \ldots, v_m \in \mathbb{R}^m$. $v_1, \ldots, v_m$ are linearly independent iff $v_1, \ldots, v_m$ span $\mathbb{R}^m$.

Back to Matrix:

- $$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

- $A = [A_1, A_2, \ldots, A_n]$.

Definition:

Let $A_{m\times n} = [A_1, A_2, \ldots, A_n]$. Then, rank(A) ≡ # of linearly indep. A_j's.

Theorem:

$\mathrm{rank}(A) \leq n$, $\mathrm{rank}(A) \leq m$.

<proof>

- rank(A) $\leq$ n $\Rightarrow$ obvious.
- Note that $A_1, A_2, A_3 \cdots A_n \in \mathbb{R}^m$

  $\Rightarrow$ only m vectors can be linearly independent.

  $\Rightarrow$ rank(A) $\leq$ m.

EX1: $A_{3\times 2}, A_{1\times 10}, A_{20\times 20}$

EX2:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; A_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; A_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \text{rank} = 2.$$

EX3:

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; A_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \text{rank} = 1$$

EX4:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = 1$$

EX5:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank} = 0$$

EX6:

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} = A$$

- rank(A) ≤ 2
- Drop the last column of A and consider A_1 and A_2:

$$\Rightarrow A_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix};\ A_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\Rightarrow a_1A_1 + a_2A_2 = \begin{pmatrix} a_1 + 3a_2 \\ 2a_1 + 4a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow a_1 = a_2 = 0.$$

- rank(A) = 2.

Back to the System of Equations:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} ①$$

$$\Rightarrow Ax = b \qquad ②$$

- Let A_j = j^{th} column of A:

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

$$\Rightarrow x_1A_1 + x_2A_2 + \cdots + x_nA_n = b \qquad ③$$

- When does a solution $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)$ exist?

  [Answer] If b is a linear combination of $A_1 \cdots A_n$.

  [Note that $b \in \mathbb{R}^m$, $A_j \in \mathbb{R}^m$]

Theorem:

The system of linear equations Ax = b has a (not necessarily unique) solution if b is a linear combination of $A_1, ..., A_n$.

Theorem:

If rank(A) = m, Ax = b has a solution (unique or infinitely many) for all b.

<proof>

- If rank(A) = m, m columns of of $A_1 \cdots A_n$ are linearly independent.

  ⇒ These m independent columns will span $\mathbb{R}^m$.

  ⇒ Thus, $\{A_1, \cdots A_n\}$ spans $\mathbb{R}^m$.

  ⇒ For every $b \in \mathbb{R}^m$, b is a linear combination of A_j's.

  ⇒ Ax = b has a solution for every b.

Implication:

If rank(A) = # of equations (or # of rows of A), Ax = b has at least a solution.

Theorem:

If rank(A) < m, there may be no solution.

<proof>

- If rank(A) < m, $A_1 \cdots A_n$ fail to span $\mathbb{R}^m$.

- For some $b \in \mathbb{R}^m$, there may be no solution.

EX1: [Case of no solution]

$x_1 + x_2 = 1$ ①

$x_1 + x_2 = 2$ ②

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- A_1 and A_2 are linearly dependent.
- rank(A) = 1 < 2 = n.

EX2: [Case of infinitely many solutions]

$x_1 + x_2 = 2$

$x_1 + x_2 = 2$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

- Rank(A) = 1 < 2

Consider cases where m = n (A is square):

- If rank(A) = n, Ax = b has a solution for any b.

  Furthermore, the solution $\bar{x} = A^{-1}b$ is unique.
- If rank(A) < n, then, rank(A) < m:
  - There may not be a solution.
  - If any, infinitely many.

Theorem:

Following statements are equivalent for $A_{n \times n}$:

(a) A is invertible.

(b) $\det(A) \neq 0$.

(c) $Ax = b$ has unique solution for every b.

(d) $\text{rank}(A) = n$

(e) All of the columns of A are linearly independent.