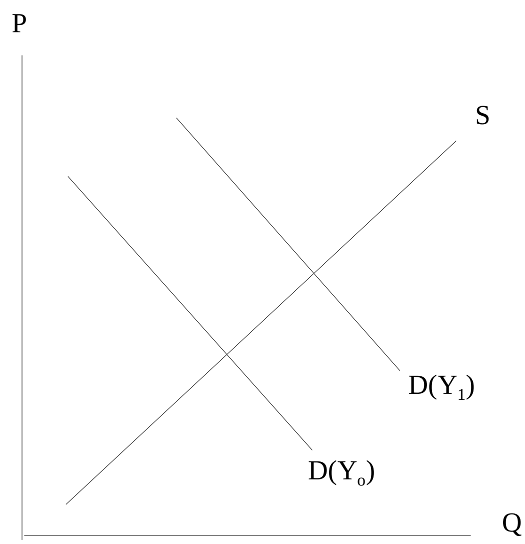
TOPIC V
COMPARATIVE STATICS

[1] Motivation



- Here, $D(Y_o)$ denotes the demand curve when $Y = Y_o$. Similarly, $D(Y_1)$ denotes the demand curve when $Y = Y_1$.
- Observe that as Y changes, $\overline{P}$ and $\overline{Q}$ change.
- In economic models, equilibrium values of endogenous variables change whenever values of exogenous variables changes.
- **Comparatice statics** are motivated to find how much equilibrium values changes when exogenous variables change.
- **Comparative statics** are rates of changes in equilibrium values of endogenous variables in response to changes in exogenous variables.

EX 1) $Q_d = a + bP + eY$, $a > 0$, $b < 0$, $e > 0$

$Q_s = c + dP$, $c < 0$, $d > 0$.

$Q_d = Q_s$

- For this model, let $\overline{Q}$ and $\overline{P}$ be equilibrium price and quantity. We assume that Y is exogenous.
- Then, we have:

$$\overline{Q} = a + b\overline{P} + eY$$

$$\overline{Q} = c + d\overline{P}.$$

- Solving these equations, we have:

$$\overline{Q} = \frac{bc - ad}{d - b} + \frac{de}{d - b}Y \;;\; \overline{P} = \frac{-c + a}{d - b} + \frac{e}{d - b}Y \;.$$

- Then,

$$\frac{\partial \overline{Q}}{\partial Y} = \frac{de}{d - b} = \frac{(+)(+)}{(+) - (-)} = \frac{+}{+} > 0,$$

$$\frac{\partial \overline{P}}{\partial Y} = \frac{e}{d - b} = \frac{+}{(+) - (-)} = \frac{+}{+} > 0$$

Note:

1) Solving an economic model, we can obtain equilibrium quantities of endogenous variables, which are functions of exogenous variables. Once the solutions are obtained, it is easy to find comparative statics.
2) An important question is whether it is possible to obtain comparative statics without solving the model explicitly. Well, the answer is yes.

[2] Explicit function Vs. Implicit function

Definition:

- Explicit functions are of the form:

  $y = f(x)$.

- Consider a equation of the form:

  $F(y,x) = 0$ (Here, y is endo. and x is exo.)

  ⇒ This type of equation (or relation between y and x) sometimes implies an implicit function $y = f(x)$.

- The function $y = f(x)$ implied by the relation $F(y,x) = 0$ is called an implicit function.

EX 1) Explicit functions

$y = 2x^2 + x + 1$,

$y = e^x + \ln(x) + x^3$.

EX 2) Implicit functions

$x - y - 1 = 0$ ⇒ $y = x - 1$ (implicit f^n)

$x^2 + y^2 - 3 = 0$ ($y \geq 0$, $-\sqrt{3} \leq x \leq \sqrt{3}$) ⇒ $y = \sqrt{3 - x^2}$ (implicit f^n)

$xy + x + y + 1 = 0$ ($x \neq -1$) ⇒ $y = -1$ (implicit f^n)

EX 3) National Income Model

$Y = C + I_o + G_o$; $C = a + bY$.

$\Rightarrow\ \bar{Y} - \bar{C} - I_o - G_o = 0$

$-b\bar{Y} + \bar{C} - a = 0$

$\Rightarrow$ They are implicit functions.

Structure of Economic Models:

- Economic models in general have the following structure:

  (1-1) $F^1(y_1, \dots, y_n, x_1, \dots, x_m) = 0$;

  (1-2) $F^2(y_1, \dots, y_n, x_1, \dots, x_m) = 0$;

  : : :

  (1-n) $F^n(y_1, \dots, y_n, x_1, \dots, x_m) = 0$,

  where $y_1, \dots, y_n$ are endogenous and $x_1, \dots x_m$ are exogenous.

- If we solve (1-1)-(1-n), we obtain a system of implicit f^n:

  (2-1) $\bar{y}_1 = f^1(x_1, \dots, x_m)$;

  (2-2) $\bar{y}_2 = f^2(x_1, \dots, x_m)$;

  : : :

  (2-n) $\bar{y}_n = f^n(x_1, \dots, x_m)$.

- If we have (2-1) - (2-n), we can easily obtain comparative statics, $\partial\bar{y}_i/\partial x_j$.

- From now on, we consider how to obtain comparative statics from the implicit functions (1-1) - (1-n).

[3] Implicit Function Theorem

- Suppose that you have a relation $F(x,y) = 0$. We now have two questions.
  - (i) Is it possible to convert this function into an explicit form, $y = f(x)$?
  - (ii) Is it possible to find $\partial y/\partial x$?

(1) Implicit Function Theorem: Part I

- $F(y, x_1, ... , x_m) = 0$.
- Consider a point $(y, x_1, ... , x_m) = (y^o, x_1^o, ... , x_m^o)$.
- Let $F_y(y^o,x_1^o, ... , x_m^o) = \partial F/\partial y$ evaluated at $(y^o,x_1^o, ... , x_m^o) \neq 0$.
- Then, in a neighborhood of $(y^o, x_1^o, ... , x_m^o)$, there exists an explicit function $y = f(x_1, ... , x_m)$, and $\partial y/\partial x_j = - F_j/F_y$, where $F_j = \partial F/\partial x_j$.

<Sketch of the Proof>

- Total Differentials with respect to y and x_1:

$$F_y dy + F_1 dx_1 = 0$$

$$\Rightarrow F_y dy = - F_1 dx_1$$

$$\Rightarrow dy = (- F_1/F_y)dx_1$$

$$\Rightarrow dy/dx_1(\text{other things being equal}) = - F_1/F_y .$$

EX) $F(x,y,w) = y^3x^2 + w^3 + yxw - 3 = 0$.

Is an implicit function $y = f(x,w)$ defined at $(x,y,w) = (1,1,1)$? If it is, find $\partial y/\partial x$.

(Answer)

- $F_y = 3y^2x^2 + xw \Rightarrow F_y(1,1,1) = 3 + 1 = 4 \neq 0$.

$\Rightarrow$ $y = f(x,w)$ is defined.

- $\partial y/\partial x = - F_x/F_y = -(2y^3x+yw)/(3y^2x^2+xw)$.

  $\Rightarrow \partial y/\partial x$ at (1,1,1) = -3/4.

(2) Implicit Function Theorem: Part II

- Consider the following n equations with n endogenous variables $(y_1,...,y_n)$ and m exogeneous variables $(x_1,...,x_m)$:

$$F^1(y_1,y_2, ... , y_n, x_1, x_2, ... , x_m) = 0$$

$$F^2(y_1,y_2, ... , y_n, x_1, x_2, ... , x_m) = 0$$

$$\vdots$$

$$F^n(y_1,y_2, ... , y_n, x_1, x_2, ... , x_m) = 0$$

- Suppose that we want to know $\partial y_i/\partial x_j$ (i = 1,2,...,n). Then, write

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \vdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_j} \\ \frac{\partial y_2}{\partial x_j} \\ \vdots \\ \frac{\partial y_n}{\partial x_j} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_j} \\ -\frac{\partial F^2}{\partial x_j} \\ \vdots \\ -\frac{\partial F^n}{\partial x_j} \end{bmatrix} .$$

- Let $J = \begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \vdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix}$

- If $|J| \neq 0$, a unique solution exists.

<Sketch of the Proof>

- Total differentials with respect to $y_1, \ldots, y_n$ and x_1:

$$(\partial F^1/\partial y_1)dy_1 + (\partial F^1/\partial y_2)dy_2 + \ldots + (\partial F^1/\partial y_n)dy_n + (\partial F^1/\partial x_1)dx_1 = 0$$

$$(\partial F^2/\partial y_1)dy_1 + (\partial F^2/\partial y_2)dy_2 + \ldots + (\partial F^2/\partial y_n)dy_n + (\partial F^2/\partial x_1)dx_1 = 0$$

$$\vdots$$

$$(\partial F^n/\partial y_1)dy_1 + (\partial F^n/\partial y_2)dy_2 + \ldots + (\partial F^n/\partial y_n)dy_n + (\partial F^n/\partial x_1)dx_1 = 0$$

- Divide both sides by dx_1:

$$(\partial F^1/\partial y_1)(\partial y_1/\partial x_1)+(\partial F^1/\partial y_2)(\partial y_2/\partial x_1)+\ldots+(\partial F^1/\partial y_n)(\partial y_n/\partial x_1)=-(\partial F^1/\partial x_1)$$

$$(\partial F^2/\partial y_1)(\partial y_1/\partial x_1)+(\partial F^2/\partial y_2)(\partial y_2/\partial x_1)+\ldots+(\partial F^2/\partial y_n)(\partial y_n/\partial x_1)=-(\partial F^2/\partial x_1)$$

$$\vdots$$

$$(\partial F^n/\partial y_1)(\partial y_1/\partial x_1)+(\partial F^n/\partial y_2)(\partial y_2/\partial x_1)+\ldots+(\partial F^n/\partial y_n)(\partial y_n/\partial x_1)=-(\partial F^n/\partial x_1)$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \vdots & \frac{\partial F^n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_j} \\ \frac{\partial y_2}{\partial x_j} \\ \vdots \\ \frac{\partial y_n}{\partial x_j} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial x_j} \\ -\frac{\partial F^2}{\partial x_j} \\ \vdots \\ -\frac{\partial F^n}{\partial x_j} \end{bmatrix}.$$

EX 1) $Y = C + I_o + G_o$

$C = a + bY$ $(a > 0, 0 < b < 1)$

- $\bar{Y} - \bar{C} - I_o - G_o = 0 \ (= F^1)$

$-b\bar{Y} + \bar{C} - a \qquad = 0 \ (= F^2)$

(Y and are endogenous; I_o and G_o are exogeneous)

- Suppose we want to know the effect of I_o on equilibrium Y and C.

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{Y}}{\partial I_o} \\ \frac{\partial \bar{C}}{\partial I_o} \end{bmatrix} = \begin{bmatrix} -(-1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\Rightarrow$ By Cramer's rule,

$$\frac{\partial \bar{Y}}{\partial I_o} = \frac{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{1}{1-b} > 0; \quad \frac{\partial \bar{C}}{\partial I_o} = \frac{\begin{vmatrix} 1 & 1 \\ -b & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{b}{1-b} > 0$$

- Find $\partial \bar{Y}/\partial G_o$ and $\partial \bar{C}/\partial G_o$. Determine their signs. (Do it by yourself)

EX 2) Demand: $Q = D(P,Y)$

Suppy: $Q = S(P,T_p)$

where T_p is the lump-sum tax on profits.

- P and Q : endo; Y and T_p: exo.
- Assume:

$$\frac{\partial D}{\partial P} < 0 \ ; \ \frac{\partial S}{\partial P} > 0 \ ; \ \frac{\partial D}{\partial Y} > 0 \ ; \ \frac{\partial S}{\partial T_p} < 0 \ .$$

$$\Rightarrow \quad \bar{Q} - D(\bar{P},T) = 0 \; (= F^1)$$

$$\Rightarrow \quad \bar{Q} - S(\bar{P},T_p) = 0 \; (= F^2)$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -\frac{\partial D}{\partial \bar{P}} \\ 1 & -\frac{\partial S}{\partial \bar{P}} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{Q}}{\partial Y} \\ \frac{\partial \bar{P}}{\partial Y} \end{bmatrix} = \begin{bmatrix} -\left(-\frac{\partial D}{\partial Y} \right) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial D}{\partial Y} \\ 0 \end{bmatrix}.$$

$$\Rightarrow \quad \frac{\partial \bar{P}}{\partial Y} = \frac{\begin{vmatrix} 1 & \frac{\partial D}{\partial Y} \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\partial D}{\partial \bar{P}} \\ 1 & -\frac{\partial S}{\partial \bar{P}} \end{vmatrix}} = \frac{-\frac{\partial D}{\partial Y}}{\frac{\partial D}{\partial \bar{P}} - \frac{\partial S}{\partial \bar{P}}} = \frac{-(+)}{(-)-(+)} = \frac{(-)}{(-)} > 0 \; .$$

$$\Rightarrow \frac{\partial \bar{Q}}{\partial Y} = \frac{\begin{vmatrix} \frac{\partial D}{\partial Y} & -\frac{\partial D}{\partial \bar{P}} \\ 0 & -\frac{\partial S}{\partial \bar{P}} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\partial D}{\partial \bar{P}} \\ 1 & -\frac{\partial S}{\partial \bar{P}} \end{vmatrix}} = \frac{-\frac{\partial S}{\partial \bar{P}}\frac{\partial D}{\partial Y}}{\frac{\partial D}{\partial \bar{P}} - \frac{\partial S}{\partial \bar{P}}} = \frac{-(+)(+)}{(-)-(+)} = \frac{(-)}{(-)} > 0 \; .$$

- Find $\partial Q/\partial T_p$ and $\partial P/\partial T_p$. Determine their signs. (Do it by yourself.)

EX 3) National income Model

- Product Market: (IS)

  $Y = C + I(i) + G_o$; $C = C(Y)$

  (Here i denotes interest rate)

- Money Market: (LM)

  Demand: $M = L(Y,i)$; Supply: $M = M_o$

- IS curve: $Y = C(Y) + I(i) + G_o$

  LM curve: $M_o = L(Y,i)$

- Endogenous: Y, i (and C); Exogenous: M_o, G_o.

- Assume $\frac{\partial I}{\partial i} < 0$; $0 < \frac{\partial C}{\partial Y} < 1$; $\frac{\partial L}{\partial Y} > 0$; $\frac{\partial L}{\partial i} < 0$.

- We want to know $\partial Y/\partial G_o$, $\partial i/\partial G_o$ and $\partial C/\partial G_o$.

  $\bar{Y} - C(\bar{Y}) - I(\bar{i}) - G_o = 0;$

  $M_o - L(\bar{Y},\bar{i}) = 0$

$$\begin{bmatrix} 1-\frac{\partial C}{\partial \bar{Y}} & -\frac{\partial I}{\partial \bar{i}} \\ -\frac{\partial L}{\partial \bar{Y}} & -\frac{\partial L}{\partial \bar{i}} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{Y}}{\partial G_o} \\ \frac{\partial \bar{i}}{\partial G_o} \end{bmatrix} = \begin{bmatrix} -(-1) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|J| = \begin{vmatrix} 1-\frac{\partial C}{\partial \bar{Y}} & -\frac{\partial I}{\partial \bar{i}} \\ -\frac{\partial L}{\partial \bar{Y}} & -\frac{\partial L}{\partial \bar{i}} \end{vmatrix} = -\left(1-\frac{\partial C}{\partial \bar{Y}}\right)\frac{\partial L}{\partial \bar{i}} - \frac{\partial I}{\partial \bar{i}}\frac{\partial L}{\partial Y} = -(+)(-) - (-)(+) > 0.$$

$$|J_1| = \begin{vmatrix} 1 & -\frac{\partial I}{\partial \bar{i}} \\ 0 & -\frac{\partial L}{\partial \bar{i}} \end{vmatrix} = -\frac{\partial L}{\partial \bar{i}} = -(-) > 0;$$

$$|J_2| = \begin{vmatrix} 1-\frac{\partial C}{\partial \bar{Y}} & 1 \\ -\frac{\partial L}{\partial \bar{Y}} & 0 \end{vmatrix} = \frac{\partial L}{\partial \bar{Y}} = (+) > 0 .$$

$$\frac{\partial \bar{Y}}{\partial G_o} = \frac{|J_1|}{|J|} = \frac{(+)}{(+)} > 0; \quad \frac{\partial \bar{i}}{\partial G_o} = \frac{|J_2|}{|J|} = \frac{(+)}{(+)} > 0$$

- How about $\partial C/\partial G_o$?

$\Rightarrow$ Note that $\bar{C} = C(\bar{Y})$ and $\bar{Y} = Y(M_o, G_o)$. Then, By Chain rule,

$$\frac{\partial \bar{C}}{\partial G_o} = \frac{\partial \bar{C}}{\partial \bar{Y}} \frac{\partial \bar{Y}}{\partial G_o} = (+)(+) > 0$$

- Want to know $\frac{\partial \bar{Y}}{\partial M_o}, \frac{\partial \bar{i}}{\partial M_o}, \frac{\partial \bar{C}}{\partial M_o}$?

$\bar{Y} - C(\bar{Y}) - I(\bar{i}) - G_o = 0$

$M_o - L(\bar{Y}, \bar{i}) = 0$

$$\Rightarrow \begin{bmatrix} 1-\frac{\partial C}{\partial \bar{Y}} & -\frac{\partial I}{\partial \bar{i}} \\ -\frac{\partial L}{\partial \bar{Y}} & -\frac{\partial L}{\partial \bar{i}} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{Y}}{\partial M_o} \\ \frac{\partial \bar{i}}{\partial M_o} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

$$|J| = \begin{vmatrix} 1-\frac{\partial C}{\partial \overline{Y}} & -\frac{\partial I}{\partial \overline{i}} \\ -\frac{\partial L}{\partial \overline{Y}} & -\frac{\partial L}{\partial \overline{i}} \end{vmatrix} = -\left(1-\frac{\partial C}{\partial \overline{Y}}\right)\frac{\partial L}{\partial \overline{i}} - \frac{\partial I}{\partial \overline{i}}\frac{\partial L}{\partial \overline{Y}} = -(+)(-) - (-)(+) > 0.$$

$$|J_1| = \begin{vmatrix} 0 & -\frac{\partial I}{\partial \overline{i}} \\ -1 & -\frac{\partial L}{\partial \overline{i}} \end{vmatrix} = -\frac{\partial I}{\partial \overline{i}} = -(-) > 0$$

$$|J_2| = \begin{vmatrix} 1-\frac{\partial C}{\partial Y} & 0 \\ -\frac{\partial L}{\partial Y} & -1 \end{vmatrix} = -\left(1-\frac{\partial C}{\partial Y}\right) = -(+) < 0 \ .$$

$$\frac{\partial Y}{\partial M} = \frac{|J_1|}{|J|} = \frac{(+)}{(+)} > 0; \quad \frac{\partial i}{\partial M_o} = \frac{|J_2|}{|J|} = \frac{(-)}{(+)} < 0 \ .$$

- Note that $\overline{C} = C(\overline{Y})$ and $\overline{Y} = Y(M_o, G_o)$. Therefore,

$$\frac{\partial \overline{C}}{\partial M_o} = \frac{\partial \overline{C}}{\partial \overline{Y}}\frac{\partial \overline{Y}}{\partial M_o} = (+)(+) > 0 \ .$$