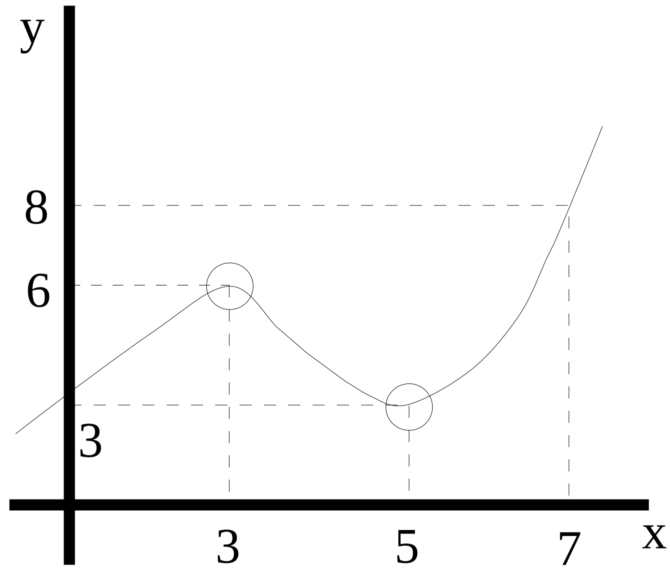
TOPIC VI
UNCONSTRAINED OPTIMIZATION I

[1] Motivation



- Consider Dom (f) = $\{x | 0 \leq x \leq 7\}$.
- Global max: $y_{gmax} = 8$ at $x = 7$
- Global min: $y_{gmin} = 3$ at $x = 5$
- Local max: $y_{lmax} = 6$ at $x = 3$
- Local min: $y_{lmin} = 3$ at $x = 5$

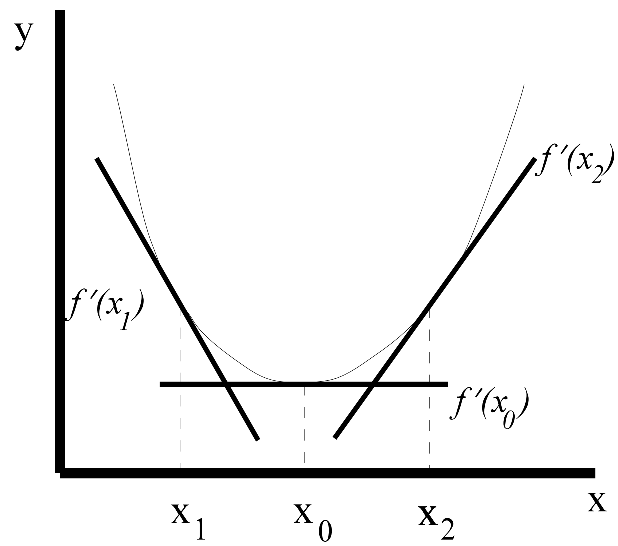
- We wish to find global max or min points.
- When global max or min points happen to be also local max or min, we can find the global points using calculus.

[2] Properties of Local Minima and Maxima

(1) Stationary points

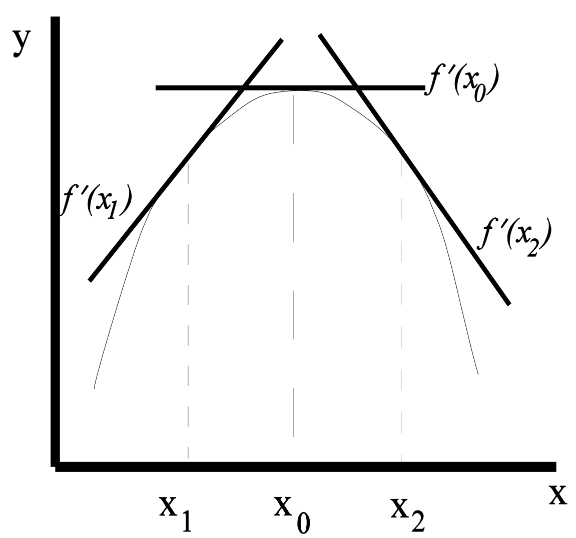
1) Local minimum points

- $f'(x_0) = 0$, and in a neighborhood of x_o, $f'(x)\uparrow$ as $x\uparrow$.



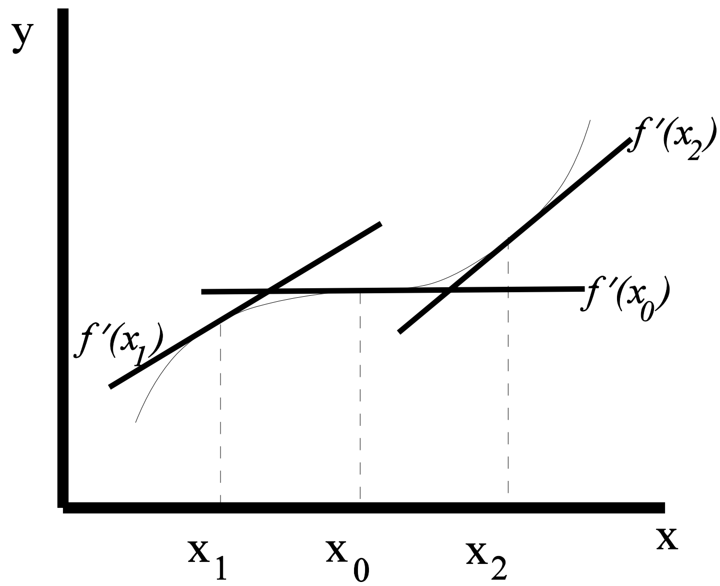
2) Local maximum points.

- $f'(x_0) = 0$, and in a neighborhood of x_o, $f'(x)\downarrow$ as $x\uparrow$.



3) Inflection points

- $f'(x_0) = 0$, $f'(x)\downarrow\rightarrow\uparrow$ as $x\uparrow$



4) Summary:

- If $f'(x_0) = 0$, $(x_0, f(x_0))$ is called stationary point.
- Three possibilities for a stationary point.

① local max.

② local min.

③ inflection point.

[2] Conditions for Local Max. or Min.

(1) First-Order (necessary) Condition:

- Find x_o such that $f'(x_0) = 0$.

- Note:
  - For max at x_0, in the neighborhood of x_o, $f'(x)\downarrow$ as $x\uparrow$
  - For min at x_0, in the neighborhood of x_o, $f'(x)\uparrow$ as $x\uparrow$
  - For inflection point, in the neighborhood of x_o, $f'(x)\downarrow \rightarrow \uparrow$ as $x\uparrow$

(2) Second-order (sufficient) condition:

- Let $f''(x) = \frac{d}{dx} f'(x)$.

- For x_o such that $f'(x_0) = 0$,

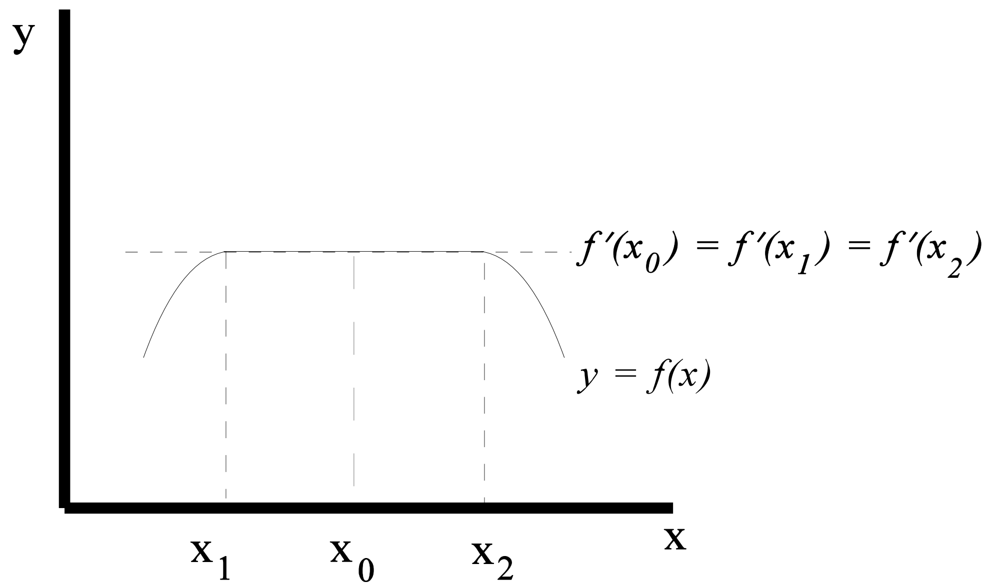
  ① $f''(x_0) < 0 \Rightarrow$ local max.

  ② $f''(x_0) > 0 \Rightarrow$ local min.

  ③ $f''(x_0) = 0 \Rightarrow$ local min or local max or inflection point

- Note
  - If x_0 is inflection point, then $f''(x_0) = 0$.
  - However, $f''(x_0) = 0$ does not mean that x_0 is an inflection point.

EX)



- $f''(x_0) = 0$, but local max.

EX 1)

- Π (profit) = TR - TC = R(Q) - C(Q).
  - FOC for max. Π:
    - $\frac{d\Pi}{dQ} = \frac{dR}{dQ} - \frac{dC}{dQ} = 0 \Rightarrow MR - MC = 0 \Rightarrow$ get Q^*
  - SOC for max. Π:
    - $\frac{d^2\Pi}{dQ^2} = \frac{d^2R}{dQ^2} - \frac{d^2C}{dQ^2} < 0$ at $Q^* \Rightarrow \frac{dMR}{dQ} - \frac{dMC}{dQ} < 0$ at Q^*.

EX 2)

① Demand: Q = 100 - P

② Total Cost: $TC = \frac{1}{3}Q^3 - 7Q^2 + 111Q + 50$

- $P = 100 - Q \quad \Rightarrow TR = P \cdot Q = (100 - Q)Q = 100Q - Q^2$
- $\Pi = TR - TC = (100Q - Q^2) - ⅓Q^3 + 7Q^2 - 111Q - 50$

$$= -⅓Q^3 + 6Q^2 - 11Q - 50.$$

- FOC for profit maximization
  - $\dfrac{d\Pi}{dQ} = -Q^2 + 12Q - 11 = 0$

  $\Rightarrow Q^2 - 12Q + 11 = 0 \Rightarrow (Q - 11)(Q - 1) = 0 \quad \Rightarrow Q^* = 11$ or 1.

- SOC:
  - $\dfrac{d^2\Pi}{dQ^2} = -2Q + 12.$
  - At $Q = 1$, $\dfrac{d^2\Pi}{dQ^2} = 10 > 0$
  - At $Q = 11$, $\dfrac{d^2\Pi}{dQ^2} = -10 < 0$
  - Thus, local profit max. $Q^* = 11$
- $\Pi^* = TR(Q^*) - C(Q^*) = 111⅓.$

EX 3)

- $y = ax^2 + bx + c,\ a \neq 0$

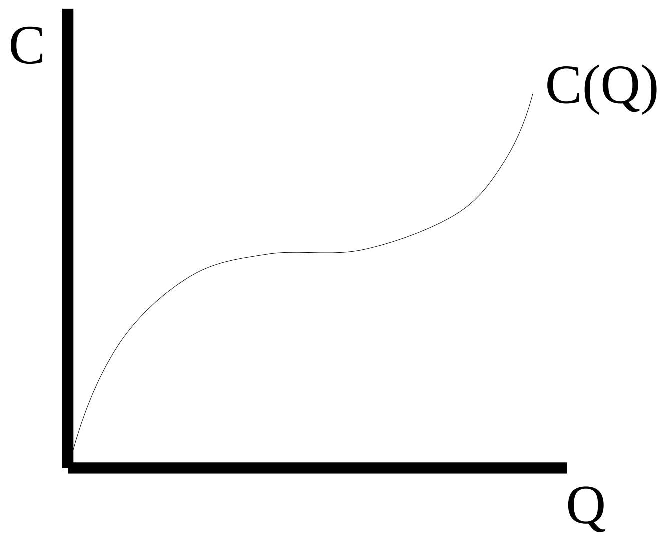
- Find local max. or min.
  - FOC:
    - $y' = 2ax + b \Rightarrow$ Set $y' = 0 \Rightarrow x^* = -\frac{b}{2a}$.
  - SOC:
    - $y'' = 2a$
      - If $a > 0 \Rightarrow y'' > 0$, $y^* = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$ is local min.
      - If $a < 0 \Rightarrow y'' < 0$, y^* is local max.

EX 4)

- $y = f(x) = x^3 - 3x^2 + 2$
  - FOC:
    - Set $f'(x) = 3x^2 - 6x = 0$
      - $x^2 - 2x = 0 \Rightarrow x(x - 2) = 0$
      - $x_1^* = 0,\ x_2^* = 2$
  - SOC:
    - $f''(x) = 6x - 6$
      - $f''(0) = -6 < 0$;
      - $f''(2) = 12 - 6 = 6 > 0$
  - $f(0) = 2$ is local max.;

    $f(2) = 8 - 12 + 2 = -2$ is local min.

EX 5) About cost function



- $C(Q) = aQ^3 + bQ^2 + CQ + d$
- What restrictions should be imposed on this cubic total cost function?

① $C(0) = d \Rightarrow$ fixed cost

- $d \geq 0$

② Marginal cost ≥ 0 for all Q.

- $MC = \dfrac{\partial C}{\partial Q} = 3aQ^2 + 2bQ + C \geq 0$
    - It must be that $MC_{min} \geq 0$
        - Find minimum cost.

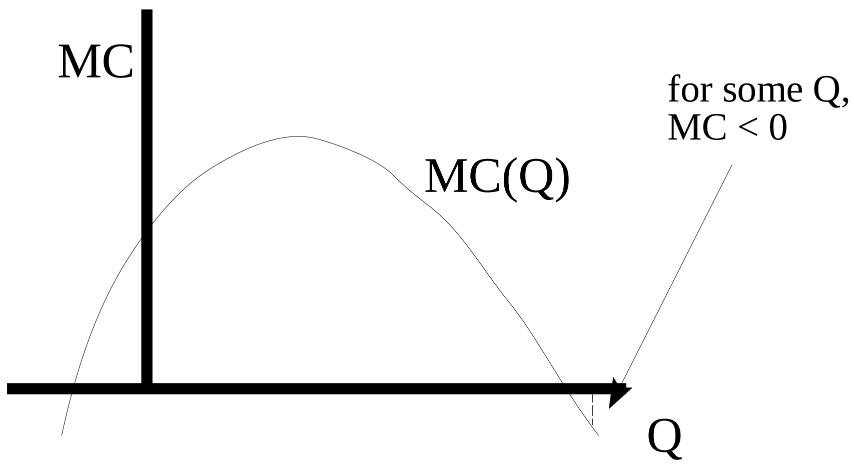
          FOC: $\dfrac{\partial MC}{\partial Q} = 6aQ + 2b = 0$

$$\Rightarrow Q^* = -\frac{b}{3a}.$$

$$\text{SOC: } \frac{\partial^2 MC}{\partial Q^2} = 6a$$

⇒ For Q* to be the cost-minimizing output, a should be positive.

[If $a < 0$, Q^* is a locally cost-maximizing output level.]



$$\text{If } a > 0,\ MC_{min} = 3a\left(-\frac{b}{3a}\right)^2 + 2b\left(-\frac{b}{3a}\right) + c = \frac{3ac - b^2}{3a}$$

- $MC_{min} \geq 0 \Rightarrow 3ac - b^2 \geq 0$.

- In short, 3 restrictions are required.

① $d \geq 0$; ② $a > 0$; ③ $3ac \geq b^2$

[3] Conditions for Global Max. and Min.

Definition:

If $f''(x) > 0$ for all x, f(x) is called convex function.

If $f''(x) < 0$ for all x, f(x) is called concave function.

Theorem:

For a function y = f(x), suppose that there exists x* such that $f'(x^*) = 0$.

If $f''(x) > 0$ for **<u>all x</u>**, then f(x*) is local min. **<u>and</u>** global min.

If $f''(x) < 0$ for **<u>all x</u>**, then f(x*) is local max. **<u>and</u>** global max.

EX 1) $y = f(x) = x^2 - 2x + 2$

- FOC: Set $f'(x) = 2x - 2 = 0 \Rightarrow x - 1 = 0 \Rightarrow x^* = 1$.
- SOC: $f''(x) = 2 > 0$
- x = 1 is the global minimum point.

  y = 1 is the global minimum of y.

EX 2) $y = -x^2 + 2x + 2$.

- FOC: Set $f'(x) = -2x + 2 = 0 \Rightarrow -x + 1 = 0 \Rightarrow x^* = 1$
- SOC: $f''(x) = -2 < 0$.
- x = 1 is the global maximum point.

  y = 3 is the global maximum of y.

[4] Taylor's Series

Short Digression

- $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

EX: $8! = 8 \times 7 \times 6 \times \cdots \times 1$.

$4! = 4 \times 3 \times 2 \times 1 = 24$.

$0! = 1$.

End of Digression

- Taylor Expansion:
  - $y = f(x)$.
  - Choose a point x_0. Then,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n \text{ (Remainder)}$$

  - Generally, $R_n \to 0$ as $n \to \infty$.

EX: $f(x) = ax^2 + bx + c$

- Choose $x_0 = 0$
  - $f'(x) = 2ax + b \Rightarrow f'(0) = b$

- $f''(x) = 2a \qquad \Rightarrow f''(0) = 2a$
- $f'''(x) = f^{(4)}(x) = \cdots = 0$
- $f(0) = c.$
- $f(x_0) + \dfrac{f'(x_0)}{1!}(x - x_0) + \dfrac{f''(x_0)}{2!}(x - x_0)^2 = c + bx + \dfrac{2a}{2}x^2$

  $= c + bx + ax^2 = f(x).$

- Choose $x_0 = 2$:
  - $f'(x) = 2ax + b \quad \Rightarrow f'(2) = 4a + b$
  - $f''(x) = 2a \qquad \Rightarrow f''(2) = 2a$
  - $f(x) = (4a + 2b + c) + (4a + b)(x - 2) + \dfrac{2a}{2!}(x - 2)^2 = ax^2 + bx + c.$

Theorem: (Lagrangean form of Remainder)

$$R_n = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}, \text{ where p is between x and } x_0.$$

Implication:

- $f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \dfrac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n.$

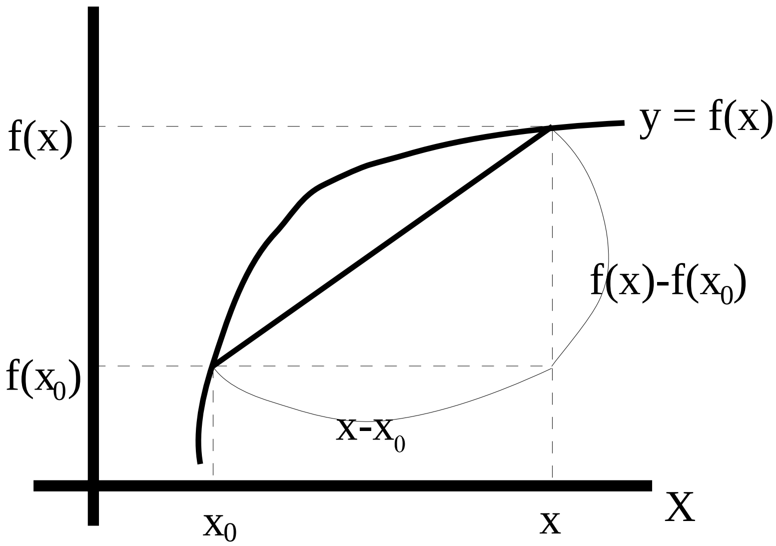
Question:

- Choose $n = 0$; then $f(x) = f(x_0) + f''(p)(x - x_0)$. Does such p exist?

Mean-Value Theorem:

Suppose that a function y = f(x) is differentiable. Then, there exists a value p (between x_0 and x) such that $f'(p) = \dfrac{f(x) - f(x_0)}{x - x_0}$.

<Intuitive Proof>



Corollary 1:

$R_0 = f'(p)(x-x_0)$, where p is between x and x_0.

<Proof>

$$f'(p) = \frac{f(x) - f(x_0)}{x - x_0} \Rightarrow f(x) - f(x_0) = f'(p)(x - x_0)$$

$$\Rightarrow f(x) = f(x_0) + f'(p)(x - x_0) \Rightarrow R_0 = f'(p)(x-x_0).$$

Corollary 2:

$$R_n = \frac{f^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1}.$$

Theorem:

Consider a function y = f(x). Suppose that

$f'(x_0) = 0$; $f''(x_0) = 0$; ... ; $f^{(n)}(x_0) \neq 0$.

Then, the following is true:

- $f(x_0)$ is local max. if n is even and $f^{(n)}(x_0) < 0$.
- $f(x_0)$ is local min. if n is even and $f^{(n)}(x_0) > 0$.
- $f(x_0)$ is inflection point if n is odd and $f^{(n)}(x_0) \neq 0$.

<Sketch of the proof>

- Taylor's expansion around x_0:

$$f(x) \approx f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \Rightarrow f(x)\text{-}f(x_0) \approx \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

- If n is even: $f(x) - f(x_0) \geq 0$ when $f^{(n)}(x_0) > 0 \Rightarrow x_0$ is a local min. point.
  $f(x) - f(x_0) \leq 0$ when $f^{(n)}(x_0) < 0 \Rightarrow x_0$ is a local max. point.
- If n is odd, either $f(x) - f(x_0) \leq 0$ or $f(x) - f(x_0) > 0$.

EX1) $y = f(x) = (7 - x)^3$

- $f'(x) = 3(7 - x)^2(-1) = -3(7 - x)^2$
  - Set $f'(x) = 0 \Rightarrow x_0 = 7$.
  - $f''(x) = -3 \times 2\,(7 - x)(-1) = 6(7 - x) \Rightarrow f''(x_0) = 0$
  - $f'''(x) = -6 < 0$.

- $x = 7$ is an inflection point

EX2) $y = f(x) = (7 - x)^4$

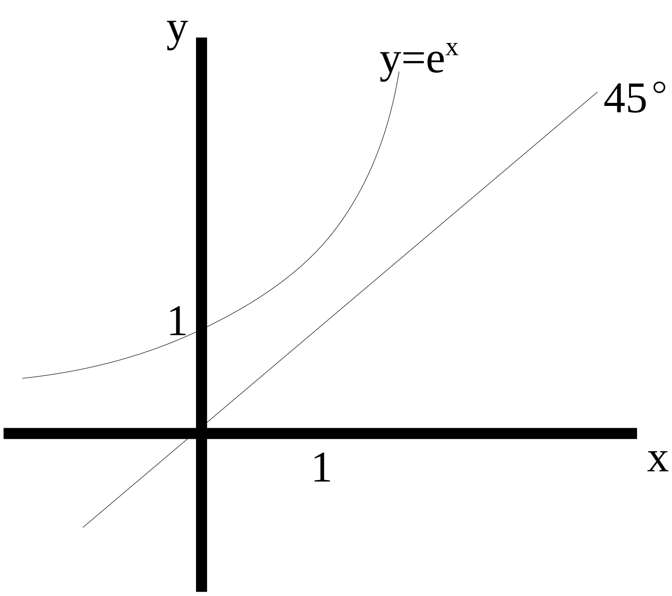
- $f'(x) = -4(7 - x)^3$
  - Set $f'(x) = 0$. $\Rightarrow$ $x_0 = 7$.
  - $f''(x) = 12(7 - x)^2 \Rightarrow f''(7) = 0$
  - $f'''(x) = -24(7 - x) \Rightarrow f'''(7) = 0$
  - $f^{(4)}(x) = 24 > 0$

$\therefore$ $x = 7$ is the local min. point. The local minimum value of $y = 0$.

[5] Exponential Function

(1) Exponential function:

- $y = b^x$, $b > 1$, $x \in R$
- When $x = 0$, $y = b^0 = 1$
- As $x \to \infty$, $y \to \infty$; and as $x \to -\infty$, $y \to 0$
- A popular choice of b = e (natural #) = 2.7182819...



Fact:

$$e = \lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^x = 2.7182819...$$

<Proof> By Taylor expansion using the fact that $de^x/dx = e^x$. See book (pp. 274-276).

Economic meaning of e:

- Suppose that a Bank pays annual interest rate = i. You deposit M_o.
  - Suppose the bank pays interests only once a year:

    $M_1 = M_0 + iM_0 = M_0(1 + i)$
  - Suppose the bank pays interests every 6 months.
    - After 6 months, you have

      $$M_0\left(1 + \frac{i}{2}\right).$$
    - After 1 year,

      $$M_1 = M_0\left(1 + \frac{i}{2}\right)\left(1 + \frac{i}{2}\right) = M_0\left(1 + \frac{i}{2}\right)^2.$$
  - Suppose the bank pays interests every 1 month,

    $$M_1 = M_0\left(1 + \frac{i}{12}\right)^{12}.$$
  - Suppose everyday,

    $$M_1 = M_0\left(1 + \frac{i}{365}\right)^{365}.$$
  - If the interests can be continuously compounded,

    $$M_1 = \lim_{x\to\infty} M_0\left(1 + \frac{i}{x}\right)^x.$$

Implication:

- If i = 1 (100% interest), $M_1 = M_o e$.

  ⇒ Since e = 2.718, 100% compounded interest rate = 171.8% one-shot interest rate.

Theorem:

$$\lim_{x\to\infty}\left(1+\frac{i}{x}\right)^{x} = e^{i}.$$

<Proof>

- Let y = x/i. Then, x = yi.
- Note that as $y \to \infty$, $x \to \infty$.
- $\lim_{x\to\infty}\left(1+\frac{i}{x}\right)^{x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{iy} = \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{i} = e^{i}$

Implication:

- $M_1 = M_0 e^i$

Question:

When interests are continuously compounded, how much money will you have t years later?

Answer:

- $M_2 = M_1 e^i = (M_0 e^i)e^i = M_0 e^{2i}$
- $M_3 = M_0 e^{3i}$

- $M_t = M_0 e^{it}$

Future and present value of an asset:

- Let V_t be the future value of your asset at time t.
- Let A be the current (present) value of your asset.
- $V_t = Ae^{it}$
  - $Ae^{it} e^{-it} = V_t e^{-it}$
  - $A = V_t e^{-it}$

Notation:

$e^{2x^2 + x + 3} = \exp(2x^2 + x + 3)$

$e^{f(x)} = \exp(f(x))$

(2) Logarithm

- $y = b^x$, $b > 1$, $x \in R$, $y \in R_+$

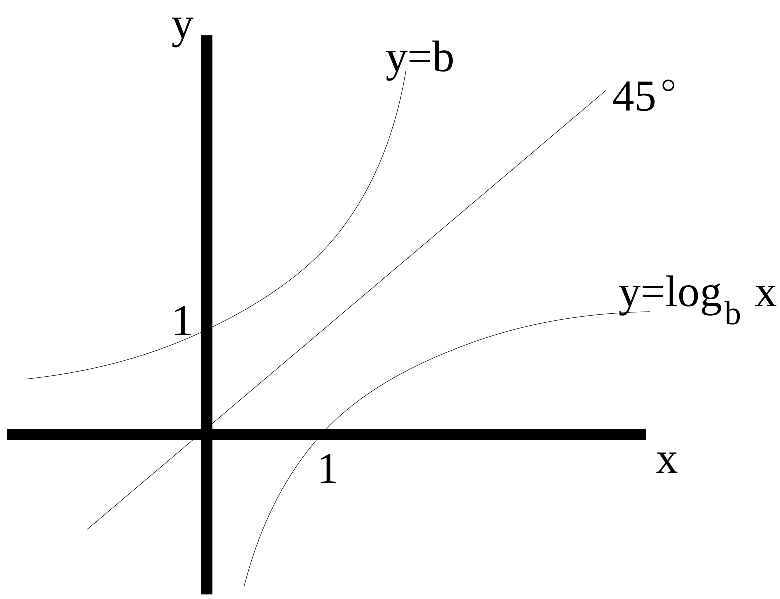
  $\Rightarrow$ monotonically increasing function

  $\Rightarrow$ can define inverse $f^{\underline{n}}$
- $y = b^x \Leftrightarrow x = \log_b y$, $y > 0$

  or

  $x = b^y \Leftrightarrow y = \log_b x$
- Usually, we use $\log_e x \equiv \ln(x)$.

Graphical Relation between exponential and logarithmic functions



For $y = \log_b x$, $x < 1 \Rightarrow y < 0$

$x = 1 \Rightarrow y = 0$

$x > 1 \Rightarrow y > 0$

Rules for logarithmic functions:

Rule I: $b^x = e^{x\ln(b)}$. [$b = e^{\ln(b)}$.]

Rule II: $\log_b b = 1$.

Rule III: $\log_b x^n = n \cdot \log_b x$

Rule IV: $\log_b(uv) = \log_b u + \log_b v$

$\log_b(u/v) = \log_b u - \log_b v$

Rule V: $\log_b x = \dfrac{\ln(x)}{\ln(b)}$

EX 1) $2^x = e^{x\ln(2)}$

EX 2) $\log_{17}17 = 1$

EX 3) $\log_{10}1000 = \log_{10}10^3 = 3\log_{10}10 = 3$

EX 4) $\log_{10}(3 \times 10^3) = \log_{10}3 + 3\log_{10}10 = \log_{10}3 + 3$

EX 5) $\log_{10}e = \dfrac{\log_e e}{\log_e 10} = \dfrac{1}{\ln 10}$

EX 6) $\log_{10}12 = \dfrac{\ln 12}{\ln 10}$

EX 7) Solve $ab^x - c = 0$

- $ab^x = c \Rightarrow \ln(ab^x) = \ln c \Rightarrow \ln a + x \ln b = \ln c$

$$\Rightarrow x = \frac{\ln c - \ln a}{\ln b}$$

EX 8) $V = Ae^{rt}$ $(r = i)$

- $\ln V = \ln A + rt \ln e = \ln A + rt$
- $t = \dfrac{\ln V - \ln A}{r}$

Conversions of base:

1) $a^x = e^{x\ln(a)}$. [Why? $a = e^{\log_e a} = e^{\ln(a)} \Rightarrow a^x = [e^{\ln(a)}]^x = e^{x\ln(a)}$.]

EX 1) $y = 2^t = (e^{\ln 2})^t = e^{t \ln 2}$

EX 2) $y = 3^{2t} = (e^{\ln 3})^{2t} = e^{2t \ln 3}$

2) $\log_b x = \dfrac{\ln x}{\ln b}$.

EX1) $y = \log_2 t = \frac{\ln t}{\ln 2}$.

EX2) $y = 7\log_{10}2t = 7 \times \frac{\ln 2t}{\ln 10} = \frac{7}{\ln 10}\ln 2t = \frac{7}{\ln 10}(\ln 2 + \ln t)$.

Derivatives:

1) $\frac{d}{dt}(e^t) = e^t$

2) $\frac{d}{dt}(\ln t) = \frac{1}{t}$

3) $\frac{d}{dt}(b^t) = b^t \ln b$ [Why?: $b^t = e^{t\ln(b)} \Rightarrow d(b^t)/dt = e^{t\ln(b)} \times \ln(b) = b^t \ln(b)$.]

4) $\frac{d}{dt}(\log_b t) = \frac{1}{t \ln b}$ [Why?: $\log_b t = \ln(t)/\ln(b) \Rightarrow d(\log_b t)/dt = \frac{\frac{1}{t}}{\ln(b)}$.]

5) e^y, where $y = f(t)$:

$$\frac{d}{dt}(e^y) = \frac{d}{dy}(e^y) \times \frac{dy}{dt} = e^y f'(t) = e^{f(t)} f'(t).$$

6) $\ln(y)$, where $y = f(t)$:

$$\frac{d}{dt}(\ln y) = \frac{d}{dy}(\ln y) \times \frac{dy}{dt} = \frac{1}{y} \times \frac{dy}{dt} = \frac{\frac{dy}{dt}}{y}.$$

$$\Rightarrow \frac{\frac{dy}{dt}}{y}$$ is called instantaneous growth rate.

EX 1) $y = 12^{1-t}$

- $y = e^{(1-t)\ln(12)}$.
- Let $z = (1 - t)\ln(12)$
- Then, $y = e^z$, $z = (1 - t)\ln(12)$.
- $\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt} = e^z[-\ln(12)] = -e^{(1-t)\ln(12)}\ln 12 = -12^{(1-t)}[\ln(12)]$.

EX 3) $y = \log_{17} x$

- $y = [\ln(17)]^{-1}\ln(x)$
- $\frac{dy}{dx} = [\ln(7)]^{-1}(1/x) = \frac{1}{x\ln 17}$.

EX 4) $y = \ln(6x^2 + 3x)$

- $z = 6x^2 + 3x$
- $y = \ln(z)$, $z = 6x^2 + 3x$.
- $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{z} \times (12x + 3) = \frac{12x + 3}{(6x^2 + 3x)}$

EX 5) $V = Ae^{rt}$. Derive the instantaneous growth rate of V?

- $\ln V = \ln A + rt$
- growth rate = $\frac{d\ln V}{dt} = r$.

EX 6) $y = x^a e^{kx-c}$.

- $\frac{dy}{dx} = ax^{a-1}e^{kx-c} + x^a e^{kx-c} \cdot k = x^a e^{kx-c}\left(\frac{a}{x} + k\right)$.

About $\frac{d\ln y}{d\ln x}$:

- Let w = ln(y) and z = ln(x)
- $dw = \frac{dw}{dy} \cdot dy \Rightarrow d\ln y = \frac{1}{y}dy$; $dz = \frac{dz}{dx} \cdot dx \Rightarrow d\ln x = \frac{1}{x}dx$.
- $\frac{d\ln y}{d\ln x} = \frac{\frac{1}{y}dy}{\frac{1}{x}dx} = \frac{x}{y} \cdot \frac{dy}{dx} = \varepsilon_{y,x}$.

EX) Demand: $Q = \frac{1}{P}$

- $\frac{dQ}{dP} = -\frac{100}{P^2} \Rightarrow \epsilon_{QP} = \frac{P}{Q} \times \frac{dQ}{dP} = \frac{P}{\left(\frac{100}{P}\right)} \times \left(-\frac{100}{P^2}\right) = -1$

- ln Q = ln 100 - ln p $\Rightarrow \epsilon_{QP} = \frac{d\ln Q}{d\ln p} = -1$ (A lot easier to compute).