TOPIC VII
UNCONSTRAINED OPTIMIZATION II

[1] Relative Maximum and Minimum

(1) Review for $y = f(x)$:

- FOC: $f'(x) = 0 \Rightarrow$ get x^*.
- SOC: $f''(x^*) > 0 \Rightarrow x^*$ is relative min. point.

  $f''(x^*) < 0 \Rightarrow x^*$ is relative max. point.

(2) For $y = f(x_1,x_2,\ldots,x_n)$:

- FOC: $\dfrac{\partial f}{\partial x_1} \equiv f_1 = 0$; $\dfrac{\partial f}{\partial x_2} \equiv f_2 = 0$; ...; $\dfrac{\partial f}{\partial x_n} \equiv f_n = 0$.

- SOC:
  - From $f_1 = \dfrac{\partial f}{\partial x_1}$:
    - $\dfrac{\partial^2 f}{\partial x_1 \partial x_1} = \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial f}{\partial x_1}\right) = f_{11}$;
    - $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial f}{\partial x_1}\right) = f_{12}$;

      :

    - $\dfrac{\partial^2 f}{\partial x_1 \partial x_n} = \dfrac{\partial}{\partial x_n}\left(\dfrac{\partial f}{\partial x_1}\right) = f_{1n}$;

- From $f_2 = \frac{\partial f}{\partial x_2}$:

  - $\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right) = f_{21}$

  - $\frac{\partial^2 f}{\partial x_2 \partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}\right) = f_{22}$

  :

- $H_n = \begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & & \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{pmatrix}$: Hessian matrix.

(3) Young's Theorem:

- $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ ($f_{ij} = f_{ji}$)

Digression on Matrix Theory:

Definition 1:

$A_{n\times n}$ is called symmetric, iff $A^t = A$.

- $[a_{ji}] = [a_{ij}] \Rightarrow a_{ji} = a_{ij}$

EX 1: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

EX 2: H_n

- $f_{ij} = f_{ji} \Rightarrow H_n^t = H_n$.

Definition 2: Quadratic form

- x : n×1 vector = $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; $A_{n\times n}$: symmetric.

- Then, $x'Ax$ is of a quadratic form in n variables, $x_1, \ldots, x_n$.
  - $\underset{1\times n}{X^t} \; \underset{n\times n}{A} \; \underset{n\times 1}{X}$ is a scalar.

EX 1: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$; $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; A: symmetric ; $a_{ij} = a_{ji}$

- $$\begin{aligned} x'Ax = a_{11}x_1^2 &+ 2a_{12}x_1x_2 + 2a_{13}x_1x_3 \\ &+ a_{22}x_2^2 + 2a_{23}x_2x_3 \\ &+ a_{33}x_3^2. \end{aligned}$$

EX 2: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $a_{12} = a_{21}$

- $x'Ax = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

EX 3: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

- $$\begin{aligned} x'Ax &= 2x_1^2 + 2x_1x_2 + 2x_2^2 = 2(x_1^2 + x_1x_2) + 2x_2^2 \\ &= 2\left(x_1^2 + x_1x_2 + \frac{x_2^2}{4} - \frac{x_2^2}{4} \right) + 2x_2^2 \\ &= 2\left(x_1 + \frac{x_2}{2} \right)^2 + \frac{3}{2}x_2^2 > 0, \text{ unless } x_1 = x_2 = 0. \text{ (i.e., } x = 0) \end{aligned}$$

EX 4: $A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

$$x'Ax = -2x_1^2 - 2x_1x_2 - 2x_2^2 = -2\left(x_1 + \frac{x_2}{2}\right)^2 - \frac{3}{2}x_2^2 < 0,$$

unless $x = 0$

Definition 3:

- $A_{n\times n}$ is symmetric.
- If, for any $x \neq 0$ (vector), $x'Ax > 0$, A is called positive definite

  [If, for any $x \neq 0$ (vector), $x'Ax \geq 0$, A is called positive semidefinite.]
- If, for any $x \neq 0$ (vector), $x'Ax < 0$, A is called negative definite.

  [If, for any $x \neq 0$ (vector), $x'Ax \leq 0$, A is called negative semidefinite.]

EX: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ p.d.; $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$ n.d.

Definition 4: Principal Minors

- $A_{n\times n.}$
- $|A_1| = a_{11}$; $|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$; $|A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$; $|A_4| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$.

EX 1: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

- $|A_1| = 2 > 0$; $|A_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0.$

EX 2: $A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

- $|A_1| = -2 < 0$; $|A_2| = (-2) \times (-2) - (-1) \times (-1) = 3 > 0$

End of Digression

Theorem:

- $A_{n \times n}$: symmetric.
- A is negative definite iff $|A_1| < 0$; $|A_2| > 0$; $|A_3| < 0$; $|A_4| > 0$; . . . .
- A: positive definite iff $|A_1| > 0$; $|A_2| > 0$; $|A_3| > 0$; . . . .

Return to Hessian matrix:

SOC:

- For max, H_n should be negative definite.
- For min, H_n should be positive definite.

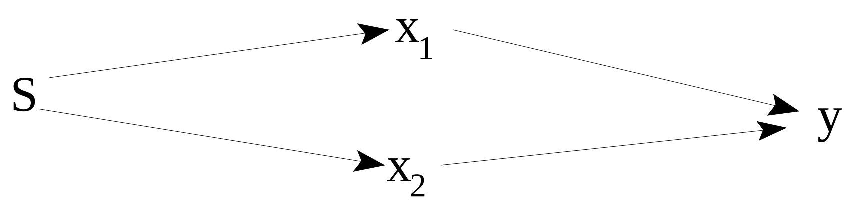
Sum-up:

- FOC: $f_1 = f_2 = f_3 = . . . = f_n = 0$
- SOC:
  - For max, $|H_1| < 0$, $|H_2| > 0$, $|H_3| < 0$, . . .
  - For min, $|H_1| > 0$, $|H_2| > 0$, $|H_3| > 0$, . . .

<Intuitive Proof>

- $y = f(x_1, x_2)$.
- God's world (we don't know):

$$x_1 = x_1(s);\ x_2 = x_2(s);\ y = f(x_1(s), x_2(s))$$



- FOC:

$$\frac{dy}{ds} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{ds} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{ds} \equiv f_1 \frac{dx_1}{ds} + f_2 \frac{dx_2}{ds} = 0 .$$

  - Choose x_1, x_2, such that $f_1 = f_2 = 0$.
  - Then, FOC holds, regardless of $\frac{dx_1}{ds}, \frac{dx_2}{ds}$.

- SOC:

  - $\frac{dy}{ds} = f_1 \frac{dx_1}{ds} + f_2 \frac{dx_2}{ds} = f_1(x_1, x_2) \frac{dx_1}{ds} + f_2(x_1, x_2) \frac{dx_2}{ds}$.

  - $\frac{d^2 y}{ds^2} = f_{11}\left(\frac{dx_1}{ds}\right)^2 + 2f_{12}\left(\frac{dx_1}{ds}\right)\left(\frac{dx_2}{ds}\right) + f_{22}\left(\frac{dx_2}{ds}\right)^2$.

$$= \left(\frac{dx_1}{ds}, \frac{dx_2}{ds}\right) \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{dx_1}{ds} \\ \frac{dx_2}{ds} \end{pmatrix} .$$

- For max y, $\frac{d^2y}{ds^2} < 0$.

  It happens regardless of $\frac{dx_1}{ds}, \frac{dx_2}{ds}$, if H_2 is negative definite.

- For min y, $\frac{d^2y}{ds^2} > 0$.

  It happens regardless of $\frac{dx_1}{ds}, \frac{dx_2}{ds}$, if H_2 is positive definite.

EX 1:

- $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$
- FOC:
  - $\frac{\partial z}{\partial x} = 24x^2 + 2y - 6x = 0$. ①
  - $\frac{\partial z}{\partial y} = 2x + 2y = 0$. ②
  - From ②:

    $y = -x$ ③
  - ③ → ①;

    $24x^2 - 2x - 6x = 0$

    $24x^2 - 8x = 0$

    $3x^2 - x = 0$

    $(3x - 1)x = 0$

- $(x^*, y^*) = (0,0)$ or $(x^*, y^*) = (⅓, -⅓)$.

- SOC:
  - $\dfrac{\partial^2 z}{\partial x \partial x} = 48x - 6$; - $\dfrac{\partial^2 z}{\partial x \partial y} = 2$
  - $\dfrac{\partial^2 z}{\partial y \partial x} = 2$; - $\dfrac{\partial^2 z}{\partial y \partial y} = 2$
  - $H_2 = \begin{pmatrix} 48x - 6 & 2 \\ 2 & 2 \end{pmatrix}$
  - When $(x,y) = (⅓, -⅓)$, $H_2 = \begin{pmatrix} 10 & 2 \\ 2 & 2 \end{pmatrix}$.
    - $|H_1| = 10 > 0$; $|H_2| = 20 - 4 = 16 > 0$.
    - min. point: $(x,y,z) = (⅓, -⅓, 28/29)$
  - When $(x,y) = (0,0)$, $H_2 = \begin{pmatrix} -6 & 2 \\ 2 & 2 \end{pmatrix}$.
    - $|H_1| = -6 < 0$; $|H_2| = -12 - 4 = -16 < 0$
    - Who knows? (saddle point)

EX 2:

- $z = x + 2ey - e^x - e^{2y}$

- FOC:
  - $\frac{\partial z}{\partial x} = 1 - e^x = 0$
  - $\frac{\partial z}{\partial y} = 2e - 2e^{2y} = 0$
  - $(x^*, y^*, z^*) = (0, ½, -1)$.

    (Why? $e - e^{2y} = 0 \rightarrow e = e^{2y} \rightarrow \ln e = \ln e^{2y} \rightarrow 1 = 2y$)

- SOC:
  - $\frac{\partial^2 z}{\partial x \partial x} = -e^x$;
  - $\frac{\partial^2 z}{\partial x \partial y} = 0$
  - $\frac{\partial^2 z}{\partial y \partial y} = -4e^{2y}$.
  - $H_2 = \begin{pmatrix} -e^x & 0 \\ 0 & -4e^{2y} \end{pmatrix}$
  - When x = 0 and y = ½, $H_2 = \begin{pmatrix} -1 & 0 \\ 0 & -4e \end{pmatrix}$.
    - $|H_1| = -1 < 0$; $|H_2| = 4e > 0 \Rightarrow$ local max. (x,y,z) = (0,½,-1).

EX 3:

- $z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_2 + x_1^2 + 2.$

- FOC:
  - $\partial z/\partial x_1 = 4x_1 + x_2 + x_3 = 0$
  - $\partial z/\partial x_2 = x_1 + 8x_2 = 0$
  - $\partial z/\partial x_3 = x_1 + 2x_3 = 0$

$$\Rightarrow \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

  - $x_1^* = x_2^* = x_3^* = 0.$

- SOC:
  - $H_n = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$

  - $|H_1| = 4 > 0;$

    $|H_2| = \begin{vmatrix} 4 & 1 \\ 1 & 8 \end{vmatrix} = 32\text{-}1 = 31 > 0;$

$$|H_3| = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4 \bullet 8 \bullet 2 - 1 \bullet 8 \bullet 1 - 1 \bullet 1 \bullet 2 = 54 > 0.$$

- local min. point: $(x_1,x_2,x_3,z) = (0,0,0,2)$.

EX 4:

- $z = -x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$.

- FOC:
  - $\partial z/\partial x_1 = -3x_1^2 + 3x_3 = 0$ ①
  - $\partial z/\partial x_2 = 2 - 2x_2 = 0$ ②
  - $\partial z/\partial x_3 = 3x_1 - 6x_3 = 0$ ③

    - From ②:

      $x_2^* = 1$ ④
    - 2•① + ③:

      $-6x_1^2 + 3x_1 = 0 \rightarrow -2x_1^2 + x_1 = 0 \rightarrow x_1(-2x_1+1) = 0$

      $x_1^* = 0$ or $x_1^* = 1/2$ ⑤
    - By ③:

      $x_3 = (1/2)x_1$ ⑥

      $x_1^* = 0 \rightarrow x_3^* = 0$; $x_1^* = 1/2 \rightarrow x_3^* = 1/4$.
  - $(x_1^*,x_2^*,x_3^*,z^*) = (0,1,0,1)$ or $(1/2,1,1/4,17/16)$.

SOC:

- $H_3 = \begin{bmatrix} -6x_1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{bmatrix}$.

- When $x_1 = 1$:
  - $|H_1| = 0\ (\geq 0)$; $|H_2| = \begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix} = 0\ (\geq 0)$; $|H_3| = \begin{vmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix} = 18 > 0$.

  - Can't determine local max. or min. (In fact, local minimum.)

- When $x_1 = 1/2$:
  - $H_3 = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{bmatrix}$.

  - $|H_1| = -3$; $|H_2| = \begin{vmatrix} -3 & 0 \\ 0 & -2 \end{vmatrix} = 6$; $|H_3| = \begin{vmatrix} -3 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{vmatrix} = -36 + 18 = -18$.

- max. point: $(x_1,x_2,x_3,z) = (1/2,1,1/4,17/16)$.

[2] Concavity Vs. Convexity

(1) Case for $y = f(x)$:

Definition:

A function $y = f(x)$ is called strictly concave iff, for all $x_1 \neq x_2$ and $0 < \theta < 1$,

$$f(\theta x_1+(1-\theta)x_2) > \theta f(x_1) + (1-\theta)f(x_2) .$$

A function $y = f(x)$ is called strictly convex iff, for all $x_1 \neq x_2$ and $0 < \theta < 1$,

$$f(\theta x_1+(1-\theta)x_2) < \theta f(x_1) + (1-\theta)f(x_2) .$$

Note:

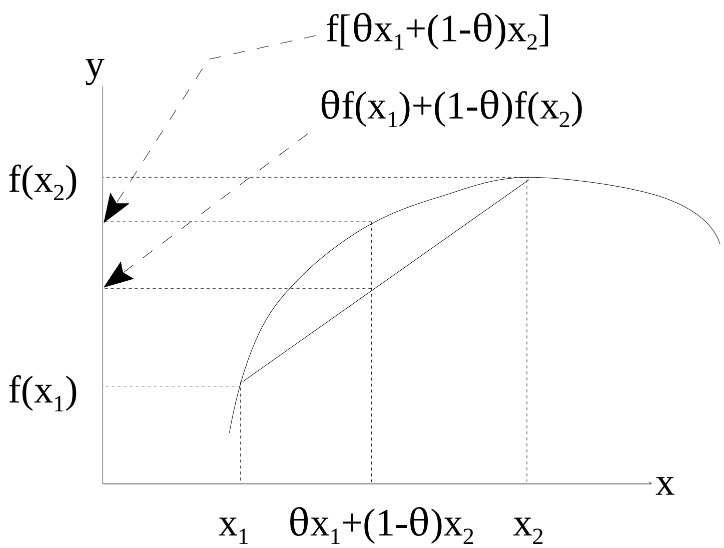
A function $y = f(x)$ is called concave iff, for all $x_1 \neq x_2$ and $0 < \theta < 1$,

$$f(\theta x_1+(1-\theta)x_2) \geq \theta f(x_1) + (1-\theta)f(x_2) .$$

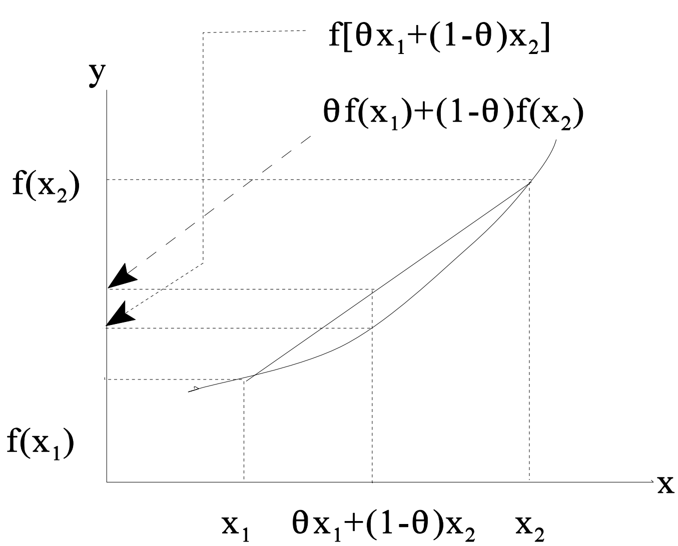
A function $y = f(x)$ is called strictly convex iff, for all $x_1 \neq x_2$ and $0 < \theta < 1$,

$$f(\theta x_1+(1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2) .$$

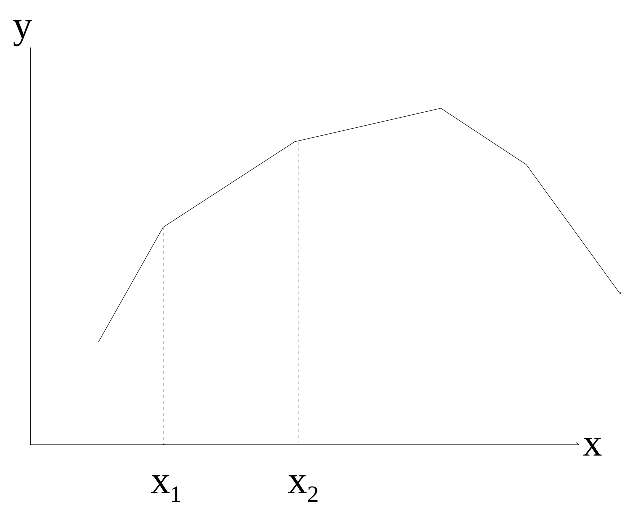
[Strictly Concave Function]



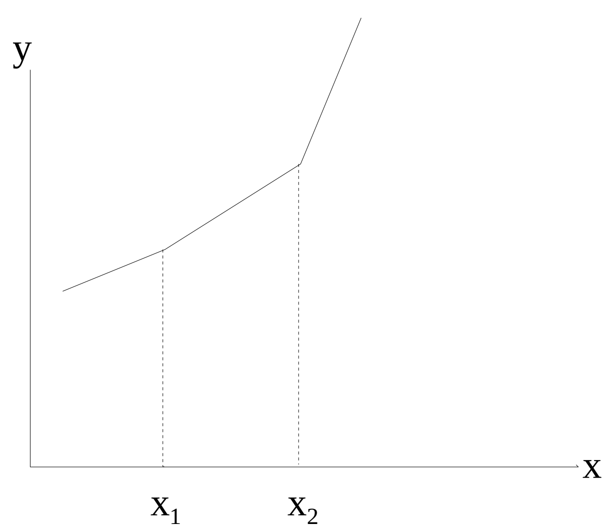
[Strictly Convex Function]



[Concave but Not Strict Concave Function]



[Convex but Not Strict Function]



Theorem:

For all x, if $f''(x) < (>) 0$, f(x) is strictly concave (convex).

For all x, if $f''(x) \leq (\geq) 0$, f(x) is concave (convex).

(2) Case for $y = f(x_1,x_2,x_3, \ldots , x_n)$:

Definition:

- Choose any two different points $u = (u_1,u_2, \ldots , u_n)$ and $v = (v_1,v_2, \ldots , v_n)$.
- Let $y_u = f(u_1,u_2, \ldots , u_n)$;

  $y_v = f(v_1,v_2, \ldots , v_n)$;

  $y_\theta = f(\theta u_1+(1-\theta)v_1,\theta u_2+(1-\theta)v_2, \ldots , \theta u_n+(1-\theta)v_n)$,

  for any $0 < \theta < 1$.
- The function $f(x_1,x_2, \ldots , x_n)$ is strictly concave iff $\theta y_u + (1-\theta)y_v < y_\theta$.

  [$f(x_1, \ldots , x_n)$ is concave iff $\theta y_u + (1-\theta)y_v \leq y_\theta$.]
- The function $f(x_1,x_2, \ldots , x_n)$ is strictly convex iff $\theta y_u + (1-\theta)y_v < y_\theta$.

  [$f(x_1, \ldots , x_n)$ is convex iff $\theta y_u + (1-\theta)y_v \leq y_\theta$.]

Theorem:

- $y = f(x_1, \ldots , x_n)$.
- The function f is strictly concave iff H_n is nd for all points $(x_1, \ldots , x_n)$.

  [The function f is concave iff H_n is nsd for all points $(x_1, \ldots , x_n)$.]
- The function f is strictly convex iff H_n is pd for all points $(x_1, \ldots , x_n)$.

  [The function f is concave iff H_n is psd for all points $(x_1, \ldots , x_n)$.]

EX 1:

- $y = 2x - x^2$.
  - $y' = 2 - 2x \Rightarrow y'' = -2 < 0$.
  - $y = 2x - x^2$ is strictly concave.

EX 2:

- $y = f(x_1,x_2) = x_1^{\;2} + x_2^{\;2}$.
- $f_1 = 2x_1$;

  $f_2 = 2x_2$
- $f_{11} = 2$; • $f_{12} = 0$

  $f_{21} = 0$; • $f_{22} = 2$
- $H_n = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ for all x_1 and x_2.

  $\Rightarrow |H_1| = 2 > 0$; $|H_2| = 4 > 0$

  $\Rightarrow$ f is strictly convex.

Theorem:

- $y = f(x)$.
- Suppose that f is concave (convex). If $f'(x_o) = 0$, then, $y_o = f(x_o)$ is a global maximum (minimum).

Theorem:

- $y = f(x_1,x_2, ... , x_n)$.

- Suppose that f is concave (convex).
- Suppose that at $(x_1^*, \ldots, x_n^*)$,

  $f_1 = 0; f_2 = 0; \ldots ; f_n = 0.$
- Then, $y^* = f(x_1^*, \ldots, x_n^*) =$ is a global maximum (minimum).

EX 1:

- $y = f(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2$.

- FOC:
  - $f_1 = 2x_1 + x_2 = 0$

    $f_2 = x_1 + 2x_2 = 0$

    $$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1^* = x_2^* = 0.$$

- SOC:
  - $f_{11} = 2;$
  - $f_{12} = 1$
  - $f_{21} = 1;$
  - $f_{22} = 2$

  $$\Rightarrow H_n = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

  $\Rightarrow |H_1| = 2 > 0;$ $|H_2| = 3 > 0$, for all x_1 and x_2.

- $y^* = f(x_1^*,x_2^*) = 0$ is a global minimum.

[3] Firm's Behavior

(1) Assumptions:

- Production function: $Q = f(L,K)$.
  - $MP_L = \partial Q/\partial L = f_L > 0$.
  - $MP_K = \partial Q/\partial K = f_K > 0$.
  - $f(L,K)$ is strictly concave:
    - Hessian = $\begin{bmatrix} f_{LL} & f_{LK} \\ f_{KL} & f_{KK} \end{bmatrix}$.
    - $f_{LL} < 0$ (Decreasing marginal product of labor.)
    - $f_{LL}f_{KK} - (f_{LK})^2 > 0$.

  [$f_{LL}f_{KK} - (f_{LK})^2 > 0 \rightarrow f_{LL}f_{KK} > (f_{LK})^2 > 0 \rightarrow f_{KK} < 0$, since $f_{LL} < 0$. Thus, the law of decreasing marginal product also applies to K.]

- The firm is perfectly competitive in both product and input markets.
  - Means that input and output prices are exogenously given.

(2) Profit maximization:

- $TR = pQ = Pf(L,K)$.
- $TC = wL + rK$.
- $\Pi = TR - TC$

  $= pf(L,K) - wL - rK$.

- FOC:
  - $\partial\Pi/\partial L = pf_L - w = 0$
  - $\partial\Pi/\partial K = pf_K - r = 0$.

- SOC:
  - $H_2 = \begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{LK} & pf_{KK} \end{bmatrix}$

    - $|H_1| = pf_{LL} < 0$;
    - $|H_2| = p^2f_{LL}f_{KK} - p^2(f_{LK})^2 = p^2[f_{LL}f_{KK}-(f_{LK})^2] < 0$ for all L and K.
    - SOC for global max. holds.

(3) Implication of FOC:

- $pf_L = w \Rightarrow pMP_L = w \Rightarrow$ Marginal Revenue Product of Labor = wage.
  - Marginal Revenue Product of Labor: $ you can make by hiring an extra unit of labor.
  - wage: $ you should pay to hire an extra unit of labor.
- $pf_K = r \Rightarrow pMP_K = r \Rightarrow$ MRP of Capital = price of capital.

(4) Comparative Statics.

- Let L^* and K^* be the Π-max. input levels.
- Consider FOC:
  - $pf_L(L^*,K^*) - w = 0$
  - $pf_K(L^*,K^*) - r = 0$

- If r, w or p changes, L^* and K^* changes.
  - Endogenous variables: L^*, K^*.
  - Exogenous variables: p, w, and r.
- Wish to find comparative statics.

1) Effect of w:

- $$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{LK} & pf_{KK} \end{bmatrix} \begin{bmatrix} \dfrac{\partial L^*}{\partial w} \\ \dfrac{\partial K^*}{\partial w} \end{bmatrix} = \begin{bmatrix} -(-1) \\ -0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- $$\frac{\partial L^*}{\partial w} = \frac{\begin{vmatrix} 1 & pf_{LK} \\ 0 & pf_{KK} \end{vmatrix}}{|H_2|} = \frac{pf_{KK}}{|H_2|} = \frac{-}{+} < 0.$$

- $$\frac{\partial K^*}{\partial w} = \frac{\begin{vmatrix} pf_{LL} & 1 \\ pf_{LK} & 0 \end{vmatrix}}{|H_2|} = \frac{-pf_{LK}}{|H_2|} = \frac{?}{+} .$$

2) Effect of r:

- $$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{LK} & pf_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial r} \\ \frac{\partial K^*}{\partial r} \end{bmatrix} = \begin{bmatrix} -0 \\ -(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- $$\frac{\partial L^*}{\partial r} = \frac{\begin{vmatrix} 0 & pf_{LK} \\ 1 & pf_{KK} \end{vmatrix}}{|H_2|} = \frac{-pf_{LK}}{|H_2|} = \frac{?}{+}.$$

- $$\frac{\partial K^*}{\partial r} = \frac{\begin{vmatrix} pf_{LL} & 0 \\ pf_{LK} & 1 \end{vmatrix}}{|H_2|} = \frac{pf_{LL}}{|H_2|} = \frac{-}{+} < 0.$$

3) Effect of p:

- $$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{LK} & pf_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial p} \\ \frac{\partial K^*}{\partial p} \end{bmatrix} = = \begin{bmatrix} -f_L \\ -f_K \end{bmatrix}.$$

- $$\frac{\partial L^*}{\partial p} = \frac{\begin{vmatrix} -f_L & pf_{LK} \\ -f_K & pf_{KK} \end{vmatrix}}{|H_2|} = \frac{-pf_L f_{KK} + pf_K f_{LK}}{|H_2|} = \frac{?}{+}.$$

- $$\frac{\partial K^*}{\partial p} = \frac{\begin{vmatrix} pf_{LL} & -f_L \\ pf_{LK} & -f_K \end{vmatrix}}{|H_2|} = \frac{-pf_K f_{LL} + pf_L f_{LK}}{|H_2|} = \frac{?}{+}.$$

- But at least one should be positive. (Why?)
  - Let $L^* = L(p,w,r)$; $K^* = K(p,w,r)$.
  - $Q^* = f(L^*,K^*)$.
  - By Chain rule,

$$\partial Q^*/\partial p = f_L(\partial L^*/\partial p) + f_K(\partial K^*/\partial p)$$

$$= \frac{f_L(-pf_L f_{KK} + pf_K f_{LK}) + f_K(-pf_K f_{LL} + pf_L f_{LK})}{|H_2|}$$

$$= \frac{-pf_{KK} f_L^2 + 2pf_{LK} f_L f_K - pf_{LL} f_K^2}{|H_2|}$$

$$= \frac{-p}{|H_2|}[f_{KK}(f_L)^2 - 2f_{LK}(f_L)(f_K) + f_{LL}(f_K)^2]$$

$$= \frac{-p}{|H_2|}(f_K, -f_L)\begin{bmatrix} f_{LL} & f_{LK} \\ f_{LK} & f_{KK} \end{bmatrix}\begin{bmatrix} f_K \\ -f_L \end{bmatrix} > 0 .$$

  - $\partial Q^*/\partial p = f_L(\partial L^*/\partial p) + f_K(\partial K^*/\partial p) > 0$

    ⇒ Either $\partial L^*/\partial p$ or $\partial K^*/\partial p$ should be negative.

    ⇒ If both are positive, $\partial Q^*/\partial p < 0$, which is a contradiction.

EX 1:

- $Q = AL^{1/3}K^{1/3}$.
- $\Pi = pAL^{1/3}K^{1/3} - wL - rK$.

- FOC:
  - $\partial\Pi/\partial L = (1/3)pAL^{-2/3}K^{1/3} - w = 0$;
  - $\partial\Pi/\partial K = (1/3)pAL^{1/3}K^{-2/3} - r = 0$.

$\Rightarrow (1/3)pAL^{-2/3}K^{1/3} = w$; ①

$(1/3)pAL^{1/3}K^{-2/3} = r$. ②

①÷②:

$$\frac{L^{-2/3}K^{1/3}}{L^{1/3}K^{-2/3}} = \frac{w}{r}$$

$\Rightarrow L^{-1}K = w/r$

$\Rightarrow L = (w/r)L$ ③

③→①:

$(1/3)pAL^{-2/3}[(w/r)L]^{1/3} = w$

$\Rightarrow (1/3)pA(w/r)^{1/3}L^{-1/3} = w$

$\Rightarrow (1/3)pA(w/r)^{1/3} = wL^{1/3}$

$\Rightarrow (1/3)^3p^3A^3(w/r) = w^3L$

$\Rightarrow L^* = (1/27)p^3A^3/(w^2r)$ ④

④→③:

$K^* = (w/r)L^* = (1/27)p^3A^3/(w^2r) = (1/27)p^3A^3/(r^2w)$. ⑤

- SOC:
- $$H_2 = \begin{bmatrix} -\frac{2}{9}pAL^{-5/3}L^{1/3} & \frac{1}{9}pAL^{-2/3}K^{-2/3} \\ \frac{1}{9}pAL^{-2/3}K^{-2/3} & -\frac{2}{9}pAL^{1/3}K^{-5/2} \end{bmatrix}.$$

- $|H_1| = \frac{2}{9}pAL^{-5/3}K^{1/3} < 0$ for all L and K (> 0).

  $$|H_2| = \frac{4}{81}p^2A^2L^{-4/3}K^{-4/3} - \frac{1}{81}p^2A^2L^{-4/3}K^{-4/3}$$

  $$= \frac{3}{81}p^2A^2L^{-4/3}L^{-4/3} > 0 \text{ for all L and K } (>0)$$

- $L^* = (1/27)p^3A^3/(w^2r)$ and $K^* = (1/27)p^3A^3/(r^2w)$ are global Π-max. input levels.