[5] Solving Multiple Linear Equations

- A system of m linear equations and n unknown variables:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

- $Ax = b$, where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{pmatrix} = \begin{pmatrix} a_{\bullet 1} & a_{\bullet 2} & \cdots & a_{\bullet n} \end{pmatrix}$,

  where,

$$a_{i\bullet} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}; a_{\bullet j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

- Question:
  - Let $\overline{x} = (\overline{x}_1, \overline{x}_2, ...\overline{x}_n)^t$ be a solution vector.
  - Solution exists? Is the solution unique? When? Why?

Definition: **Vector**

Any m×1 matrix is called a column vector. Any 1×n matrix is called a row vector. Vectors are normally denoted by lower cases (e.g., x, y, a, b).

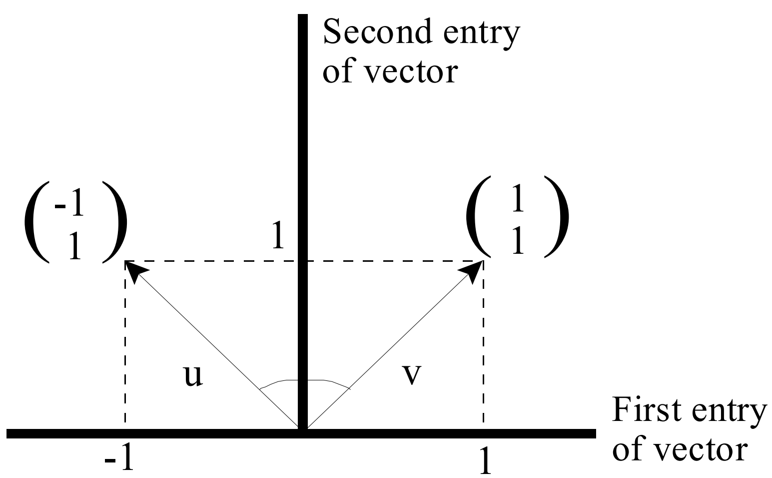
Note:

- When people talk about vectors, they are usually column vectors.

Definition: **Orthogonality**

Two m×1 vectors $u = (u_1, \ldots, u_m)^t$ and $v = (v_1, \ldots, v_m)^t$ are said to be orthogonal iff $u^t v = u_1 v_1 + u_2 v_2 + \ldots + u_m v_m = 0$.

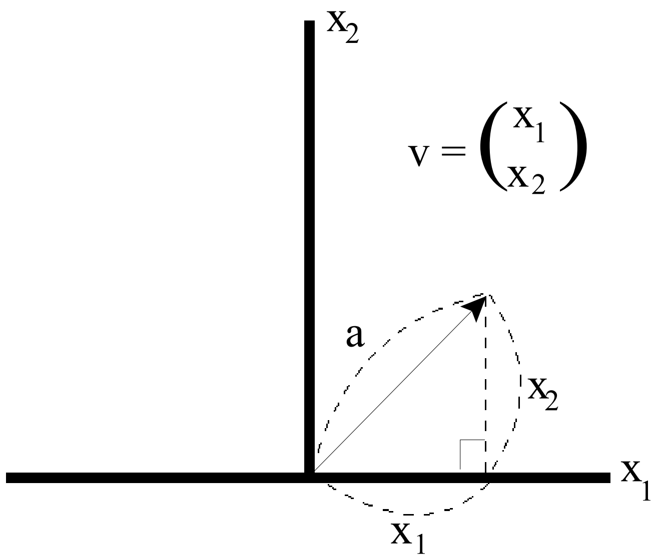
EX: m = 2.



Definition: **Norm (Length) of Vector**

Norm (length) of $v = (v_1, ..., v_m)^t \equiv \| v \| = \sqrt{v_1^2 + ... + v_m^2}$.

EX: $v = (x_1, x_2)^t$.



$a^2 = x_1^2 + x_2^2$ (by Pythagoras' Theorem) $\rightarrow a = \sqrt{x_1^2 + x_2^2}$.

EX: $v = (1,3,4)^t \rightarrow \| v \| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}$.

Definition: **Distance between Two Vectors**

For $u = (u_1, ..., u_m)^t$ and $v = (v_1, ..., v_m)^t$,

$$d(u, v) = \| u - v \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_m - v_m)^2}.$$

Definition: **m-dimensional Euclidean Space**:

$$\mathbb{R}^m = \{(h_1, h_2, ..., h_m)^t \mid h_1, ..., h_m \in \mathbb{R}\}.$$

Definition: **Linear Combination**

Let $b, a_{\bullet 1}, a_{\bullet 2}, ..., a_{\bullet n} \in \mathbb{R}^m$. Suppose that $\exists x_1, x_2, ..., x_n \in \mathbb{R} \ni$ $b = x_1 a_{\bullet 1} + x_2 a_{\bullet 2} + ... + x_n a_{\bullet n}$. Then, b is said to be a linear combination of $a_{\bullet 1}, a_{\bullet 2}, ..., a_{\bullet n}$.

Note:

- $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$
- $Ax = \left(a_{\bullet 1}, a_{\bullet 2}, ..., a_{\bullet n}\right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 a_{\bullet 1} + x_2 a_{\bullet 2} + ... + x_n a_{\bullet n}.$
- $b = x_1 a_{\bullet 1} + x_2 a_{\bullet 2} + ... + x_n a_{\bullet n} = Ax$
- Saying that b is a linear combination of $a_{\bullet 1}, ..., a_{\bullet n}$ is equivalent to saying that $Ax = b$ has a solution.

Definition: **Linear Independence**

The r vectors $a_{\bullet 1},...,a_{\bullet r} \in \mathbb{R}^m$ are said to be linearly independent iff

$\overline{x}_1 = \overline{x}_2 = ... = \overline{x}_r = 0$ whenever $x_1 a_{\bullet 1} + x_2 a_{\bullet 2} + ... + x_r a_{\bullet r} = 0_{m\times 1}$.

Note:

- This means that none of $a_{\bullet 1},...,a_{\bullet r}$ are linear combinations of the others.
- Suppose $\overline{x}_1 \neq 0$ (so the condition for linear independent violated). Then,

$$\overline{x}_1 a_{\bullet 1} = -\overline{x}_2 a_{\bullet 2} - ... - \overline{x}_r a_{\bullet r}.$$

$$a_{\bullet 1} = \frac{-\overline{x}_2}{\overline{x}_1} a_{\bullet 2} + \frac{-\overline{x}_3}{\overline{x}_1} a_{\bullet 3} + ... + \frac{-\overline{x}_r}{\overline{x}_1} a_{\bullet r}.$$

Theorem: **Maximum Number of Linearly Independent Vectors**

1) The maximum number of linearly independent vectors in $\mathbb{R}^m$ is m.
2) The m vectors, $a_{1\bullet},...,a_{m\bullet} \in \mathbb{R}^m$, are linearly independent, iff any $b \in \mathbb{R}^m$ is a linear combination of $a_{\bullet 1}, a_{\bullet 2},...,a_{\bullet m}$. (For this case, we say that $a_{1\bullet},...,a_{\bullet m}$ span $\mathbb{R}^m$.)

Definition: **Rank of Matrix**

Let $A_{m\times n} = (a_{\bullet 1}, a_{\bullet 2},...,a_{\bullet n})$. Suppose that r (≤ n) is the maximum number of the linearly independent columns of A. Then, rank(A) = r.

Theorem:

For $A_{m \times n}$, rank(A) ≤ m and rank(A) ≤ n.

Proof:

It is obvious that $rank(A) \le n$. Observe that all of the columns in A are in $\mathbb{R}^m$. But the maximum number of linearly independent vectors in $\mathbb{R}^m$ is m. Thus, $rank(A) \le m$.

Definition: **Echelon Form**

The echelon form of a matrix A is obtained by applying elementary row and/or column operations to A to reduce A to a matrix B = [b_{ij}] such that $b_{ij} = 0$ for all i > j.

EX:

$$\begin{pmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem:

The rank of a matrix A equals to the number of non-zero rows in its echelon form.

EX 1: $rank\begin{pmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} =, \quad rank\begin{pmatrix} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$

EX 2:

$$A = \begin{pmatrix} 1 & 7 & 5 \\ 3 & 21 & 15 \\ -2 & 8 & 15 \end{pmatrix} \underset{-3\times r1+r2}{\cong} \begin{pmatrix} 1 & 7 & 5 \\ 0 & 0 & 0 \\ -2 & 8 & 15 \end{pmatrix} \underset{r2\leftrightarrow r3}{\cong} \begin{pmatrix} 1 & 7 & 5 \\ -2 & 8 & 15 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{2\times r1+r2}{\cong} \begin{pmatrix} 1 & 7 & 5 \\ 0 & 22 & 25 \\ 0 & 0 & 0 \end{pmatrix}$$

EX 3:

$$A = \begin{pmatrix} 1 & -1 & 1 & 3 \\ 2 & -2 & 2 & 1 \\ -2 & 2 & -2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \underset{\substack{-2\times r1+r2, \\ 2\times r1+r3, \\ -r1+r4}}{\cong} \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 8 \\ 0 & 2 & 0 & -2 \end{pmatrix} \underset{5/8\times r3+r2}{\cong} \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

$$\underset{r2\leftrightarrow r4}{\cong} \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem:

1) For any matrix $B_{m\times n}$, $rank(B) = rank(B^t B) = rank(BB^t)$.

2) A square matrix $A_{n\times n}$ is invertible iff $rank(A) = n$.

Theorem:

Consider a system of linear simultaneous equations, $A_{m\times n}x_{n\times 1} = b_{m\times 1}$, where m and n may be different (m equations and n unknowns).

Assume that $rank(A) = m$. Then, the followings hold for a given b:

1) There is at least one solution for x.

2) If $rank(A) = m < n$, then, there are infinitely many solutions for x.

3) If $rank(A) = m = n$, then, there is one unique solution for x.

Proof:

1) Without loss of generality, assume that $a_{\bullet 1},\dots,a_{\bullet m}$ ($m \le n$) are linearly independent. Since $a_{\bullet 1},\dots,a_{\bullet m}$ span $\mathbb{R}^m$, $\exists x_1,\dots,x_m \in \mathbb{R}^m \ni$ $b = x_1 a_{\bullet 1} + \dots + x_m a_{\bullet m}$. Set $x_{m+1} = x_{m+2} = \dots = x_n = 0$. Then,

$$b = x_1 a_{1\bullet} + \dots + x_m a_{m\bullet} + x_{m+1} a_{m+1\bullet} + \dots + x_n a_{n\bullet}$$

Thus, b is a linear combination of the columns of A.

2) $\exists z_1, \ldots, z_m \in \mathbb{R}^m \ni a_{\bullet m+1} = z_1 a_{\bullet 1} + \ldots + z_m a_{\bullet m}$. Choose an arbitrary $c \in \mathbb{R}^m$ and define

$$x_1^* = x_1 - cz_1; \ldots; x_m^* = x_m - cz_m; x_{m+1}^* = c;$$

$$x_{m+2}^* = \ldots = x_n^* = 0.$$

Then,

$$\begin{aligned} & x_1^* a_{\bullet 1} + \ldots + x_m^* a_{\bullet m} + x_{m+1}^* a_{\bullet m+1} + x_{m+2}^* a_{\bullet m+2} \ldots + x_n^* a_{\bullet n} \\ &= (x_1 - cz_1) a_{\bullet 1} + \ldots + (x_m - cz_m) a_{\bullet m} + c a_{\bullet m+1} \\ &= x_1 a_{1\bullet} + \ldots + x_m a_{m\bullet} + c(a_{\bullet m+1} - z_1 a_{\bullet 1} - \ldots, -z_m a_{\bullet m}) \\ &= b \end{aligned}$$

Thus, $x^* = (x_1^*, \ldots, x_n^*)^t$ is also a solution. Since c is an arbitrary number, there are infinitely many solutions.

3) A is now square and invertible. Since $rank(A) = m$, at least one solution exits. Suppose $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)'$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)'$ are two solutions. Then,

$$A\bar{x} = b = A\tilde{x} \rightarrow A^{-1}A\bar{x} = A^{-1}A\tilde{x} \rightarrow \bar{x} = \tilde{x}.$$

The unique solution is:

$$A\bar{x} = b \rightarrow A^{-1}A\bar{x} = A^{-1}b \rightarrow \bar{x} = A^{-1}b.$$

Note:

- If $rank(A) < m$, the system may have no solution (inconsistency) or infinitely many or unique.

Theorem:

Consider a system of linear simultaneous equations, $A_{m\times n}x_{n\times 1} = b_{m\times 1}$.

Define $A^*_{m\times(n+1)} = (A,b) = (a_{\bullet 1}, a_{\bullet 2}, ..., a_{\bullet n}, b)$. Then, the following results hold:

1) If $rank(A) < rank(A^*)$, then, there is no solution for x. (In this case, we say that the system is **inconsistent**). If $rank(A)$ = $rank(A^*)$, there is at least one solution.

2) If $rank(A) = rank(A^*) < n$, then, there are infinitely many solutions for x.

3)If $rank(A) = rank(A^*) = n$, then, there is one unique solution for x.

Proof:

1) Observe that $rank(A^*) = rank(A)$ means that b is a linear combination of the columns of A. That means that the system has at least one solution. $rank(A) < rank(A^*)$ means that b is not a linear combination.

2) Without loss of generality, assume that the first r (r < m) columns of A, $a_{\bullet 1}, a_{\bullet 2}, \ldots, a_{\bullet r}$, are linearly independent. Since b is a linear combination of $a_{\bullet 1}, a_{\bullet 2}, \ldots, a_{\bullet r}$, there exist r real numbers, z_1, z_2, … , $z_r \in \mathbb{R}$ such that $b = z_1 a_{\bullet 1} + z_2 a_{\bullet 2} + \ldots + z_r a_{\bullet r}$. In addition, there exists $c_1, c_2, \ldots, c_r \in \mathbb{R}$ such that $a_{\bullet r+1} = c_1 a_{\bullet 1} + c_2 a_{\bullet 2} + \ldots + c_r a_{\bullet r}$. Set $x_j = z_j - dc_j$ for $j = 1, \ldots, r$, $x_{r+1} = d$ and $x_{r+2} = \ldots = x_n = 0$, where d is an arbitrary real number. Then,

$$\begin{aligned} & x_1 a_{\bullet 1} + x_2 a_{\bullet 2} + \ldots + x_r a_{\bullet r} + x_{r+1} a_{\bullet r+1} + x_{r+2} a_{\bullet r+2} + \ldots + x_n a_{\bullet n} \\ &= z_1 a_{\bullet 1} + \ldots + z_r a_{\bullet r} + d(a_{\bullet r+1} - c_1 a_{\bullet 1} - \ldots - c_r a_{\bullet r}) = b \end{aligned}$$

Since d is an arbitrary number, there are infinitely many n×1 vectors $x = (x_1, \ldots, x_n)^t$ that satisfy $Ax = b$.

3) Since $rank(A) = rank(A^*)$, there is at least one solution. Observe that $A^t A$ is an n×n matrix with $rank(A^t A) = rank(A) = n$. Thus, $A^t A$ is invertible. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)'$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)'$ be two solutions. Then,

$$\begin{aligned} A\bar{x} = b = A\tilde{x} &\rightarrow A^t A\bar{x} = A^t A\tilde{x} \rightarrow (A^t A)^{-1} A^t A\bar{x} = (A^t A)^{-1} A^t A\tilde{x} \\ &\rightarrow \bar{x} = \tilde{x}. \end{aligned}$$

Note:

- The unique solution for 3) is:

$$A\bar{x} = b \rightarrow (A^t A)^{-1} A^t A\bar{x} = (A^t A)^{-1} A^t b \rightarrow \bar{x} = (A^t A)^{-1} A^t b.$$

EX 1:

Check whether the following three vectors are linearly independent:

$v_1 = (1,2,-1)^t; v_2 = (6,4,2)^t; v_3 = (9,2,7)^t$.

- $\begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \underset{\substack{-2\times r1+r2, \\ r1+r3}}{\cong} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{pmatrix} \underset{r2+r3}{\cong} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{pmatrix}$
- $rank \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} = 2$.
- Linearly dependent.

EX 2:

Check whether $u = (4,-1,8)^t$ is a linear combination of v_1 and v_2.

- $\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix} \underset{\substack{-2\times r1+r2, \\ r1+r3}}{\cong} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{pmatrix} \underset{r2+r3}{\cong} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 3 \end{pmatrix}$
- $rank \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix} = 3$.
- The three vectors are linearly independent.
- u cannot be a linear combination of v_1 and v_2.

EX 3:

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Do they span $\mathbb{R}^2$?

- $v_2 = 2v_1$.

EX 4:

How about $v_1 = (1,0)^t$ and $v_2 = (1,1)^t$?

EX 5:

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Is there a solution?

- $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow Ax = b$.

- $rank(A) = 1; rank(A^*) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = rank \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 2$.

  $\rightarrow rank(A) < rank(A^*)$.

  $\rightarrow$ No solution.

EX 6:

$$x_1 + x_2 = b_1$$
$$x_1 + x_2 = b_2$$

For what values of b_1 and b_2 will the system have solution(s)?

- For at least one solution, it should be that $rank\begin{pmatrix} 1 & 1 & b_1 \\ 1 & 1 & b_2 \end{pmatrix} = 1$. It happens only if $b_1 = b_2$. With $b_1 = b_2$, $rank(A) = rank(A^*) = 1 < 2$ = n. Thus, there are infinitely many solutions.

EX 7:

$$x_1 + 3x_2 - 2x_3 = 11$$
$$2x_1 - 5x_2 + 7x_3 = -11$$
$$-x_1 + 2x_2 - 3x_3 = 4$$
$$x_1 + 2x_2 - x_3 = 8$$

Solution exists? How many?

- $$\begin{pmatrix} 1 & 3 & -2 \\ 2 & -5 & 7 \\ -1 & 2 & -3 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \\ 4 \\ 8 \end{pmatrix} \rightarrow Ax = b$$
- $rank(A) \le 3 < 4 = m$.

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -5 & 7 \\ -1 & 2 & -3 \\ 1 & 2 & -1 \end{pmatrix} \underset{\substack{-2\times r1+r2 \\ r1+r3 \\ -r1+r4}}{\cong} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -11 & 11 \\ 0 & 5 & -5 \\ 0 & -1 & 1 \end{pmatrix} \underset{(r2/11)}{\cong} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\underset{\substack{5\times r2+r3 \\ -r2+r4}}{\cong} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow rank(A) = 2 < m = 4$.

- Can show $A^* = \begin{pmatrix} 1 & 3 & -2 & 11 \\ 2 & -5 & 7 & -11 \\ -1 & 2 & -3 & 4 \\ 1 & 2 & -1 & 8 \end{pmatrix} \cong \begin{pmatrix} 1 & 3 & -2 & 11 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
- Since $rank(A) = rank(A^*) = 2 < 3 = n$, there are infinitely many solutions.
- Finding solutions:
  - $A^* \cong \begin{pmatrix} 1 & 3 & -2 & 11 \\ 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ implies that the solutions satisfy:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= 11 \\ -x_2 + x_3 &= -3 \end{aligned}.$$

- Set $\overline{x}_3 = \lambda$ ($\lambda \in \mathbb{R}$).

  $x_1 + 3x_2 = 2\lambda + 11$

  $-x_2 = -\lambda - 3$

- $\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} 2\lambda + 11 \\ -\lambda - 3 \end{pmatrix} \to \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2\lambda + 11 \\ -\lambda - 3 \end{pmatrix}$

- $\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} -1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\lambda + 11 \\ -\lambda - 3 \end{pmatrix} = \frac{-1}{1} \begin{pmatrix} \lambda - 2 \\ -\lambda - 3 \end{pmatrix} = \begin{pmatrix} 2 - \lambda \\ 3 + \lambda \end{pmatrix}$

- $\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix} = \begin{pmatrix} 2 - \lambda \\ 3 + \lambda \\ \lambda \end{pmatrix}$ for any $\lambda \in \mathbb{R}$.

- Can show that these solutions satisfy all of the four original equations.

Theorem:

Consider a system of linear equations, $A_{n\times n} x_{n\times 1} = b_{n\times 1}$ (n equations and n unknowns). Define $A^* = (A, b) = (a_{\bullet 1}, a_{\bullet 2}, ..., a_{\bullet n}, b)$. Then, the following results hold:

1) If $rank(A) = n$, there is one unique solution for a given b:

   $\overline{x} = A^{-1} b$.

2) If $rank(A) < n$ and $rank(A) = rank(A^*)$, there are infinitely many solutions for a given b.

3) If $rank(A) < rank(A^*)$, there is no solution.

[6] Linear Economic Models

(1) A Simple Keynesian Model

- Assumptions:
  - No foreign trade: X (expert) = 0 and M (import) = 0.
  - No tax.
  - Firms' investments (I) and government spending (G) are fixed. (From now on, subscript "o" means fixed variables in a given system).
  - The aggregate private consumption expenditure (C) is a linear function of aggregated income (Y)

- Model
  - GDP identity: $Y = C + I_o + G_o$.
  - Consumption: $C = a + bY$, $a > 0$ and $0 < b < 1$.

- An economic model consists of two types of equations:
  - **definitional equations** which are true by definition (e.g, GDP identity);
  - **behavioral equations** which describe economic agents' economic decisions (e.g., consumption function).

- An economic model consists of two types of variables and constants:
  - **exogenous variables** whose values are not determined by the equations (I_o and G_o);
  - **endogenous variables** whose values are determined by solving the equations (C and Y);
  - **constants** which are fixed intercepts or coefficients.

- The model can be written:

$$\begin{aligned} Y - C &= I_o + G_o \\ -bY + C &= a \end{aligned} \rightarrow \underbrace{\begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} Y \\ C \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} I_o + G_o \\ a \end{pmatrix}}_{b}.$$

  The matrix A and the vector b contain constants and exogenous variables. The vector x contains only endogenous variables.

- An economic model is called **complete** if the number of equations in it equals the number of endogenous variables and has a unique solution.

- The original form of an economic model is called **structural form.** The solution of the model is called **reduced form**:

  Structural form: $Ax = b$.

  Reduced form: $\bar{x} = A^{-1}b$.

- The solution values of endogenous variables are called **equilibrium values**.

  $$\det(A) = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = 1 - b;$$

  - $$\det(A_1) = \begin{vmatrix} I_o + G_o & -1 \\ a & 1 \end{vmatrix} = a + I_o + G_o;$$

  $$\det(A_2) = \begin{vmatrix} 1 & I_o + G_o \\ -b & a \end{vmatrix} = a + b(I_o + G_o).$$

  - $$\begin{pmatrix} \overline{Y} \\ \overline{C} \end{pmatrix} = \frac{1}{1-b} \begin{pmatrix} a + I_o + G_o \\ a + b(I_o + G_o) \end{pmatrix}.$$

- **Comparative static analysis** concerns how equilibrium endogenous variables react to changes in exogenous variables.
  - When G changes by ΔG, how much would $\overline{Y}$ change?

  $$\begin{pmatrix} \Delta\overline{Y} \\ \Delta\overline{C} \end{pmatrix} = \frac{1}{1-b} \begin{pmatrix} \Delta G \\ b\Delta G \end{pmatrix}.$$

  $$\begin{pmatrix} \partial\overline{Y} / \partial G \\ \partial\overline{C} / \partial G \end{pmatrix} = \frac{1}{1-b} \begin{pmatrix} 1 \\ b \end{pmatrix}.$$

(2) Another Simple Keynesian model

- Assumptions:
  - No tax.
  - Export (X) is exogenous.
  - Firms' investments (I) and government spending (G) are exogenous.
  - The aggregate private consumption expenditure (C) is a linear function of aggregated income (Y).
  - Import is also a linear function of Y.

- Model
  - $Y = C + I_o + G_o + X_o - M$.
  - $C = a + bY$, $a > 0$ and $0 < b < 1$.
  - $M = c + dY$, $c > 0$ and $d > 0$.

- The model can be written:

$$\begin{aligned} Y - C + M &= I_o + G_o + X_o \\ -bY + C &= a \\ -dY + M &= c \end{aligned} \quad \rightarrow \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ -b & 1 & 0 \\ -d & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} Y \\ C \\ M \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} I_o + G_o + X_o \\ a \\ c \end{pmatrix}}_{b}.$$

- The solution values of endogenous variables:

$$\det(A) = \begin{vmatrix} 1 & -1 & 1 \\ -b & 1 & 0 \\ -d & 0 & 1 \end{vmatrix} = 1 - b + d;$$

$$\det(A_1) = \begin{vmatrix} I_o + G_o + X_o & -1 & 1 \\ a & 1 & 0 \\ c & 0 & 1 \end{vmatrix} = I_o + G_o + X_o + a - c;$$

$$\det(A_2) = \begin{vmatrix} 1 & I_o + G_o + X_o & 1 \\ -b & a & 0 \\ -d & c & 1 \end{vmatrix} = a - bc + ad + b(I_o + G_o + X_o);$$

$$\det(A_3) = \begin{vmatrix} 1 & -1 & I_o + G_o + X_o \\ -b & 1 & a \\ -d & 0 & c \end{vmatrix} = c + ad - bc + d(I_o + G_o + X_o)$$

- $$\begin{pmatrix} \bar{Y} \\ \bar{C} \\ \bar{M} \end{pmatrix} = \frac{1}{1-b+d} \begin{pmatrix} I_o + G_o + X_o + a - c \\ a - bc + ad + b(I_o + G_o + X_o) \\ c + ad - bc + d(I_o + G_o + X_o) \end{pmatrix}.$$

- When G changes by ΔG, how much would $\bar{Y}$ change?

$$\begin{pmatrix} \Delta\bar{Y} \\ \Delta\bar{C} \\ \Delta\bar{M} \end{pmatrix} = \frac{1}{1-b+d} \begin{pmatrix} \Delta G \\ b\Delta G \\ d\Delta G \end{pmatrix}.$$

(3) Effect of subsidy on equilibrium price and quantity

- Assumption:
  - Perfectly competitive cell phone market.
  - Exogenous subsidy (S) per cell phone.

- $Q_d = a + bP$, $a > 0$ and $b < 0$,

  $Q_s = c + d(P + S_o)$, $c \le 0$ and $d > 0$.

  $Q_d = Q_s$, $bc - ad < 0$.

- $$\underbrace{\begin{pmatrix} 1 & -b \\ 1 & -d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \bar{Q} \\ \bar{P} \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} a \\ c + dS_o \end{pmatrix}}_{b}.$$

  $$\rightarrow \det(A) = \begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix} = b - d ;$$

  $$\det(A_1) = \begin{vmatrix} a & -b \\ c + dS_o & -d \end{vmatrix} = b(c + dS_o) - ad ;$$

  $$\det(A_2) = \begin{vmatrix} 1 & a \\ 1 & c + dS_o \end{vmatrix} = c + dS_o - a .$$

- $$\begin{pmatrix} \bar{Q} \\ \bar{P} \end{pmatrix} = \frac{1}{b-d} \begin{pmatrix} bc - ad + bdS_o \\ c - a + dS_o \end{pmatrix}. \rightarrow \begin{pmatrix} \Delta\bar{Q} \\ \Delta\bar{P} \end{pmatrix} = \frac{1}{b-d} \begin{pmatrix} bd\Delta S \\ d\Delta S \end{pmatrix}.$$

  $$\Delta\bar{P} = \frac{d\Delta S}{b-d};\ \Delta(\bar{P} + S) = \frac{d\Delta S}{b-d} + \Delta S = \frac{b\Delta S}{b-d}.$$

(4) Wage Gaps and International Trade (From Klein, p. 109)

- Trade with developing countries will widen the wage gap between skilled and unskilled workers?

- Assumptions:
  - Two sectors: Textile (T) and Computer (C).
  - Two inputs: Skilled (S) and unskilled workers (U)
  - Technologies:
    - $a_{ST} = S_T / T$ and $a_{UT} = U_T / T$ are fixed.
    - $a_{SC} = S_C / C$ and $a_{UC} = U_C / C$ are fixed.
    - $a_{ST} < a_{SC}$ and $a_{UT} > a_{UC}$.
  - The two output markets are perfectly competitive in the long run so that long-run profits = 0. The two labor markets are also perfectly competitive so that one wage (w_S) for skilled workers and one wage (w_U) for unskilled workers:
    - $w_S S_T + w_U U_T = p_T T \rightarrow (S_T / T) w_s + (U_T / T) w_U = p_T$

      $w_S S_C + w_u U_C = p_C C \rightarrow (S_C / C) w_s + (U_C / C) w_U = p_C$

      where p_T and p_C are prices.
  - The output prices are exogenous to US economy (because they are determined by perfectly competitive world markets).

- Zero Profits imply

$$w_S S_T + w_U U_T = p_T T \text{ and } w_S S_C + w_U U_C = p_C C$$

$$\to \underbrace{\frac{S_T}{T}}_{a_{ST}} w_S + \underbrace{\frac{U_T}{T}}_{a_{UT}} w_U = p_T \text{ and } \underbrace{\frac{S_C}{C}}_{a_{SC}} w_S + \underbrace{\frac{U_C}{C}}_{a_{UC}} w_U = p_C .$$

- $\underbrace{\begin{pmatrix} a_{ST} & a_{UT} \\ a_{SC} & a_{UC} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} w_S \\ w_U \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} p_T \\ p_C \end{pmatrix}}_{b}.$

$$\to \quad \det(A) = a_{ST} a_{UC} - a_{SC} a_{UT} ;$$

$$\det(A_1) = \begin{vmatrix} p_T & a_{UT} \\ p_C & a_{UC} \end{vmatrix} = a_{UC} p_T - a_{UT} p_C ;$$

$$\det(A_2) = \begin{vmatrix} a_{ST} & p_T \\ a_{SC} & p_C \end{vmatrix} = a_{ST} p_C - a_{SC} p_T .$$

- $\begin{pmatrix} \overline{w}_S \\ \overline{w}_U \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} a_{UC} p_T - a_{UT} p_C \\ a_{ST} p_C - a_{SC} p_T \end{pmatrix}.$

- If US trade with developing countries, $\Delta p_T < 0$ (and $\Delta p_C \approx 0$):

$$\begin{pmatrix} \Delta \overline{w}_S \\ \Delta \overline{w}_U \end{pmatrix} \approx \frac{1}{\det(A)} \begin{pmatrix} a_{UC} \Delta p_T \\ -a_{SC} \Delta p_T \end{pmatrix}.$$

(5) Ordinary Least Squares (OLS)

- Wish to explain $y_i = hwage_i$, using $x_{i1} = yrschool_i$, $x_{i2} = yr\exp_i$, $x_{i3} = gender_i$, and etc (i = 1, 2, … , m.)

- Regression model:
  - $y_i = \beta_o + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_{n-1} x_{i,n-1} + u_i$,

    where u_i is an error term that is not related to $x_{i1},..,x_{i,n-1}$
  - For all m people (m > n),

$$y_{m\times 1} \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1,n-1} \\ 1 & x_{21} & \cdots & x_{2,n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{m1} & \cdots & x_{m,n-1} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

$$\equiv X_{m\times n} \beta_{n\times 1} + u_{m\times 1}$$

  - Assume that all columns of X are linearly independent. [This is called the assumption of no perfect multicollinearity.]

    $\rightarrow$ $rank(X) = rank(X^t X) = n$.
  - Ideally, wish to find $\bar{\beta}$ such that $y = X\beta$ and $u = 0_{m\times 1}$.

- But such β generally does not exist unless accidentally the columns of X span y: Since $rank(X) = n < m$, not all $y \in \mathbb{R}^m$ (in fact, very few) are linear combinations of the column of X.

- If $\overline{\beta}$ exists, it should be that $\overline{\beta} = (X^t X)^{-1} X^t y \equiv \hat{\beta}$. Since $X^t X$ is an n×n matrix and $rank(X^t X) = n$, it is invertible.

- Ordinary Least Squares (OLS) estimator: $\hat{\beta} = (X^t X)^{-1} X^t y$.

- Vector of residuals: $\hat{u} = y - X\hat{\beta}$.

  $$\hat{u} = I_m y - X(X^t X)^{-1} X^t y = (I_m - X(X^t X)^{-1} X^t)y \equiv Ny,$$

  which is not zero matrix.

- The matrix $N_{m\times m}$ is a symmetric and idempotent matrix.
  - $NX = [I_m - X(X^t X)^{-1} X^t]X = 0_{m\times m}$.

    $\rightarrow$ $X^t\hat{u} = X^t Ny = 0_{m\times 1}$.
  - All columns of X are orthogonal to $\hat{u}$.