2. Further Topics in Matrix Algebra

(1) Quadratic Form

- Consider:

$$A_{n\times n} = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix};\ x_{n\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where A is symmetric ($a_{ij} = a_{ji}$).

- The following form is called a quadratic form.

$$\begin{aligned} x^t Ax &= \Sigma_{i=1}^n \Sigma_{j=1}^n a_{ij} x_i x_j \\ &= a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \\ &\quad + a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{2n}x_2x_n \\ &\quad + \dots \\ &\quad + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \dots + a_{nn}x_n^2 \\ &= a_{11}x_1^2 + a_{22}x_2^2 + \dots + a_{nn}x_n^2 + 2\begin{pmatrix} a_{12}x_1x_2 + \dots + a_{1n}x_1x_n \\ + a_{23}x_2x_3 + \dots + a_{2n}x_2x_n \\ + \dots + a_{n-1,n}x_{n-1}x_n \end{pmatrix} \end{aligned}$$

EX 1: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^t \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 7x_2^2 + 2\times 3x_1x_2 = 2x_1^2 + 7x_2^2 + 6x_1x_2$.

EX 2: $x_1^2 + 8x_2^2 + 4x_1x_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^t \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Definition: **Positive and Negative Definiteness**

Let $A_{n\times n}$ be a symmetric matrix. If $x^t Ax > 0$, for any $x_{n\times 1} \neq 0_{n\times 1}$, A is said to be positive definite. Conversely, if $x^t Ax < 0$, for any $x_{n\times 1} \neq 0_{n\times 1}$, A is said to be negative definite.

Definition: **Positive and Negative Semidefiniteness**

Let $A_{n\times n}$ be a symmetric matrix. If $x^t Ax \geq 0$, for any $x_{n\times 1} \neq 0_{n\times 1}$, A is said to be positive semidefinite. Conversely, if $x^t Ax \leq 0$, for any $x_{n\times 1} \neq 0_{n\times 1}$, A is said to be negative semidefinite.

EX 1: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$x^t Ax = 2x_1^2 + 2x_1x_2 + 2x_2^2 = 2(x_1^2 + x_1x_2 + x_2^2)$$

$$= 2\left(x_1^2 + x_1x_2 + \frac{1}{4}x_2^2 + \frac{3}{4}x_2^2\right) = 2\left(\left(x_1 + \frac{1}{2}x_2\right)^2 + \frac{3}{4}x_2^2\right) > 0$$

for any x_1 and x_2, unless $x_1 = x_2 = 0$.

EX 2: $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$$x^t Ax = -2x_1^2 + 2x_1x_2 + -2x_2^2 = -2(x_1^2 - x_1x_2 + x_2^2)$$

$$= -2\left(x_1^2 - x_1x_2 + \frac{1}{4}x_2^2 + \frac{3}{4}x_2^2\right) = -2\left(\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{4}x_2^2\right) < 0,$$

for any x_1 and x_2, unless $x_1 = x_2 = 0$.

EX 3: $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

$$x^t A x = x_1^2 + 2x_1x_2 + x_2^2 = \left(x_1 + x_2\right)^2 \geq 0,$$

for any x_1 and x_2 ($x^t A x = 0$ if $x_1 + x_2 = 0$).

EX 4: $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

$$x^t A x = -(x_1^2 - 2x_1x_2 + x_2^2) = -\left(x_1 - x_2\right)^2 < 0,$$

for any x_1 and x_2 ($x^t A x = 0$ if $x_1 - x_2 = 0$).

EX 5: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$$x^t A x = x_1^2 + 4x_1x_2 + x_2^2 = \left(x_1 + 2x_2\right)^2 - 3x_2^2 \geq 0.$$

If $x_1 = -2$ and $x_2 = 1$, $x^t A x = -3 < 0$.

If $x_1 = 1$ and $x_2 = 0$, $x^t A x = 1 > 0$

EX 6: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$.

$$\begin{aligned} x^t A x &= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \\ &= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2 + \frac{a_{11}a_{22} - (a_{12})^2}{a_{11}}x_2^2 \geq 0 \end{aligned}.$$

If $a_{11} > 0$ and $a_{11}a_{22} - (a_{12})^2 > 0$, $x^t A x > 0$ for any $x_{2\times1} \neq 0_{2\times1}$.

If $a_{11} < 0$ and $a_{11}a_{22} - (a_{12})^2 > 0$, $x^t A x < 0$ for any $x_{2\times1} \neq 0_{2\times1}$.

Definition: **Leading Principal Minors**

The leading principal minors of $A_{n\times n} = [a_{ij}]$ are:

$$a_{11}\,, \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Theorem:

A symmetric matrix $A_{n\times n}$ is positive definite iff all of its leading principal minors are positive. A is negative definite iff the leading principal minors alternate in sign: the first being negative, the next positive, etc.

EX: $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -6 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$$a_{11} = -1 < 0\,; \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -6 \end{vmatrix} = 2 > 0\,;\ |A| = -1 < 0\,.$$

EX: $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 3 \end{pmatrix}$

$$a_{11} = 1 > 0\,; \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} = 5 > 0\,;\ |A| = 11 > 0\,.$$

Question:

What if some leading principal minors are zeros?

A **tempting** theorem: (Not real theorem!!!)

A symmetric matrix $A_{n\times n}$ is positive semidefinite iff all of its leading principal minors are non-negative. A is negative semidefinite iff every leading principal minor of odd order is non-positive and every leading principal minor of even order is non-negative.

EX 1: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$a_{11} = 1 > 0$; $\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$; $|A| = 0$.

EX 2: $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$a_{11} = -1 < 0$; $\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$; $|A| = 0$.

Answer:

To check whether a matrix is positive or negative semidefinite, we'd better compute eigenvalues of the matrix.

Definition: **Eigenvalue and Eigenvector**

For $A_{n\times n}$, suppose $\exists$ a scalar λ and an n×n non-zero vector $q_{n\times 1}$ ∍ $Aq = \lambda q$. Then, λ is called an eigenvalue and q is called an eigenvector corresponding to λ.

Theorem:

Let λ be an eigenvalue of $A_{n\times n}$. Then, $\det(A-\lambda I_n) = 0$.

Proof:

$Aq = \lambda q = \lambda I_n q \rightarrow (A-\lambda I_n)x = 0_{n\times 1}$.

If $A-\lambda I_n$ is invertible, $q = (A-\lambda I_n)^{-1} 0_{n\times 1} = 0_{n\times 1}$. But it cannot happen since q is a non-zero vector. Thus, $A-\lambda I_n$ must be singular. That is, $\det(A-\lambda I) = 0$.

Theorem:

If a square matrix $A_{n\times n}$ has an eigenvalue of zero, then $\det(A) = 0$.

Proof:

It is enough to show that A is not invertible. If $\lambda = 0$,

$Aq = 0 \times q = 0_{n\times 1}$ for an non-zero vector x. If A in invertible,

$$q = A^{-1} 0_{n\times 1} = 0_{n\times 1},$$

which is a contradiction.

Finding eigenvalues and eigenvectors

EX 1: $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) = 0$$

$\rightarrow \lambda_1 = 1$ and $\lambda_2 = 2$.

For $\lambda_1 = 1$,

$$\begin{pmatrix} 1-1 & 2 \\ 0 & 2-1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 2q_2 = 0 \\ q_2 = 0 \end{matrix}$$

$\rightarrow q_2 = 0$ and q_1 can be any number.

$\rightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$ where c is any number.

EX 2: $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 2$$

$\rightarrow$ The eigenvalues are not real numbers.

Theorem:

Let $A_{n\times n}$ be an idempotent matrix. Then, the eigenvalues of A are zeros or ones.

Proof:

Let λ be an eigenvalue of A and q be a corresponding eigenvector which is a non-zero vector. Then,

$$\lambda q = Aq = A^2 q = AAq = A(\lambda q) = \lambda Aq = \lambda\lambda q = \lambda^2 q.$$

$$\rightarrow (\lambda - \lambda^2)q = 0_{n\times 1} \rightarrow \lambda(1-\lambda)q = 0_{n\times 1} \rightarrow \lambda = 0 \text{ or } \lambda = 1,$$

because q is a non-zero vector.

Theorem:

Let $A_{n\times n}$ be a symmetric matrix. Then, there exist n real numbers λ_1, λ_2, …, λ_n such that $\det(A - \lambda I_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda)$. These numbers are the eigenvalues of A.

Theorem:

Let $A_{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1, ..., \lambda_n$.

Then, $rank(A)$ equals the number of non-zero eigenvalues.

Finding eigenvalues and eigenvectors of symmetric matrices

EX: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \rightarrow (\lambda - 3)(\lambda - 1) = 0 \rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = 3.$$

Digression:

- The general solutions of quadratic equations:

$$\text{For } ax^2 + bx + x = 0,\ \bar{x} = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- In our example,

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4 \times 1 \times 3}}{2} = \frac{4 \pm 2}{2} = 1, 3.$$

End of Digression

For $\lambda_1 = 1$,

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} q_1 + q_2 = 0 \\ q_1 + q_2 = 0 \end{matrix}$$

$\rightarrow$ Infinitely many solutions.

$\rightarrow$ Choose the normalized eigenvector:

If you choose $q_1 = 1$, then, $q_2 = -1$.

Choose $q_{\bullet 1} = \frac{1}{\sqrt{q_1^2 + q_2^2}}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{1^2 + (-1)^2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

For $\lambda_1 = 3$,

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$

$-q_1 + q_2 = 0$
$q_1 - q_2 = 0$

$\rightarrow$ Infinitely many solutions.

$\rightarrow$ Choose the normalized eigenvector:

If you choose $q_1 = 1$, then, $q_2 = 1$.

Choose $q_{\bullet 1} = \frac{1}{\sqrt{q_1^2 + q_2^2}}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{1^2 + 1^2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

Observe that $q_{\bullet 1}^t q_{\bullet 1} = 1;\ q_{\bullet 2}^t q_{\bullet 2} = 1;\ q_{\bullet 1}^t q_{\bullet 2} = 0$.

We call these two normalized vectors "orthonormal eigenvectors".

Let $Q = [q_{\bullet 1}, q_{\bullet 2}]$. Then, $Q^t Q = I_2$.

Lessons:

- Two eigenvalues for a 2×2 symmetric matrix.
- The eigenvector corresponding to an eigenvalue is not unique. Use normalized eigenvectors.
- Eigenvectors corresponding to different eigenvalues are orthogonal.

EX: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = (1-\lambda)^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 1.$$

For $\lambda_1 = \lambda_2 = 1$,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 0 \times q_1 + 0 \times q_2 = 0.$$

$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ with any real numbers of q_1 and q_2 is an eigenvector.

Choose $q_1 = 0$. Then, for the eigenvector to be normalized, it should be that $q_2 = 1$. So, one normalized vector is:

$$q_{\bullet 1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Choose $q_1 = 1$. Then for the corresponding eigenvector to be normalized, it should be that $q_2 = 0$:

$$q_{\bullet 2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $Q = [q_{\bullet 1}, q_{\bullet 2}]$. Then, $Q^t Q = I_2$.

Lessons:

- Two identical eigenvalues.
- The two eigenvectors corresponding to the common eigenvalue are not unique: For example, you could have chosen:

$$q_{\bullet 1} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}; q_{\bullet 2} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

EX: $A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

$$\det(A-\lambda I_3) = \begin{vmatrix} -3-\lambda & 2 & 0 \\ 2 & -3-\lambda & 0 \\ 0 & 0 & -5-\lambda \end{vmatrix} = -(3+\lambda)^2(5+\lambda)+4(5+\lambda)$$

$$= -(5+\lambda)[(3+\lambda)^2-4] = -(\lambda+5)(\lambda^2+6\lambda+5)$$

$$= -(\lambda+5)(\lambda+1)(\lambda+5) = -(\lambda+1)(\lambda+5)^2$$

$\rightarrow \lambda_1 = -1; \lambda_2 = \lambda_3 = -5$.

Digression:

- Factorizing cubic functions.
- Consider $f(\lambda) = \lambda^3 + 11\lambda^2 + 35\lambda + 25$.
  - Think about the numbers that can divide the intercept (25):
    1, 5, 25 (In fact, they do not have to be positive or integers.)
  - Divide $f(\lambda)$ by (x+1) or (x+5) or (x+25):

$$
\begin{array}{r}
\lambda^2 + 10\lambda + 25 \\
\lambda + 1 \overline{\big) \lambda^3 + 11\lambda^2 + 35\lambda + 25} \\
\lambda^3 + \lambda^2 \qquad\qquad\qquad \\
\hline
10\lambda^2 + 35\lambda + 25 \\
10\lambda^2 + 10\lambda \qquad \\
\hline
25\lambda + 25 \\
25\lambda + 25 \\
\hline
0
\end{array}
$$

- $f(\lambda) = (\lambda + 1)(\lambda^2 + 10\lambda + 25) = (\lambda + 1)(\lambda + 5)^2$.

End of Digression

For $\lambda_1 = -1$,

$$
\begin{pmatrix} -3+1 & 2 & 0 \\ 2 & -3+1 & 0 \\ 0 & 0 & -5+1 \end{pmatrix}
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}
=
\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

A solution of q should be that $q_3 = 0$ and $q_2 = q_1$ where q_1 can be any number. Choose $q_1 = 1$ so that $q_2 = 1$. Then, normalize the eigen-vector:

$$q_{\bullet 1} = \frac{1}{\sqrt{1^2+1^2+0^2}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}.$$

For $\lambda_1 = -1$,

$$\begin{pmatrix}-3+5 & 2 & 0\\ 2 & -3+5 & 0\\ 0 & 0 & -5+5\end{pmatrix}\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}2 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

A solution of q should be that $q_2 = -q_1$ where q_1 and q_3 can be any numbers. Choose $q_3 = 0$ and $q_1 = 1$ so that $q_2 = -1$. Then, normalize the eigenvector:

$$q_{\bullet 2} = \frac{1}{\sqrt{1^2+(-1)^2+0^2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix}.$$

Choose $q_3 = 1$ and $q_1 = 0$ so that $q_2 = 0$:

$$q_{\bullet 3} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Let $Q = [q_{\bullet 1}, q_{\bullet 2}, q_{\bullet 3}]$. Then, $Q^t Q = I_3$.

Theorem:

Let Q be an n×n matrix of the orthonormal eigenvectors corresponding to the eigenvalues of a n×n symmetric matrix, $A_{n\times n}$.

Then, $Q^tQ = I_n$. This means that $Q^t = Q^{-1}$.

Theorem: **Spectral Decomposition**

Let $q_{\bullet 1}, q_{\bullet 2}, ..., q_{\bullet n}$ be the orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a n×n symmetric matrix, $A_{n\times n}$. Define D = $diag(\lambda_1, \lambda_2, ..., \lambda_n)$ and $Q = (q_{\bullet 1}, q_{\bullet 2}, ..., q_{\bullet n})$. Then, $D = Q^t AQ$

Proof:

We first show that $QD = AQ$. Observe that:

$$[Aq_{\bullet 1}, Aq_{\bullet 2}, ..., Aq_{\bullet n}] = [\lambda_1 q_{\bullet 1}, \lambda_2 q_{\bullet 2}, ..., \lambda_n q_{\bullet n}].$$

But

$$[Aq_{\bullet 1}, Aq_{\bullet 2}, .., Aq_{\bullet n}] = A[q_{\bullet 1}, q_{\bullet 2}, ..., q_{\bullet n}] = AQ.$$

$$[\lambda_1 q_{\bullet 1}, \lambda_2 q_{\bullet 2}, ..., \lambda_n q_{\bullet n}] = [q_{\bullet 1}, q_{\bullet 2}, ..., q_{\bullet n}] \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix} = QD$$

Thus, $QD = AQ \to Q^tQD = Q^tAQ \to D = Q^tAQ$.

Corollary:

Let $q_{\bullet 1}, q_{\bullet 2}, ..., q_{\bullet n}$ be the orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a n×n symmetric matrix, $A_{n\times n}$. Define D = $diag(\lambda_1, \lambda_2, ..., \lambda_n)$ and $Q = (q_{\bullet 1}, q_{\bullet 2}, ..., q_{\bullet n})$. Then, $A = QDQ^t$.

Theorem:

Let $A_{n\times n}$ be a symmetric n×n matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Then, the followings are true. (i) A is positive definite, iff all of the eigenvalues are positive. (ii) A is negative definite, iff all of the eigenvalues are negative.

Proof:

(i) Since A is symmetric, for any n×1 vector x,

$$\begin{aligned} x^t Ax &= c^t(QDQ^t)c = (Q^t c)^t D(Q^t c) = y^t Dy \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 \end{aligned},$$

where $y = (y_1, y_2, ..., y_n)^t = Q^t x$ (and $x = Qy$). Now, we will show that if all of the eigenvalues are positive, then $x^t Ax > 0$ for any non-zero vector x. Since Q^t is invertible, y is a non-zero vector as long as x is a non-zero vector. That is, at least one of y_j's is not zero. Thus, we can see that if all of the eigenvalues are positive, then $x^t Ax > 0$.

We now wish to show that if $x^t Ax > 0$ for all non-zero vector x, all of the eigenvalues should be positive. To show this, suppose that one eigenvalue, say λ_1, is non-positive: $\lambda_1 \le 0$. Choose $y_1 = 1; y_2 = ... = y_n = 0$ ($y = (1,0,0,...0)^t$). Since Q^t is invertible, there exists a nonzero vector x such that $y = Q^t x$. With the choice of x, $x^t Ax = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 = \lambda_1 \le 0$. But this is a contradiction if A is positive definite.

Theorem:

Let $A_{n\times n}$ be a symmetric n×n matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Then, the followings are true. (i) Iff A is positive semidefinite, all of the eigenvalues are non-negative. (ii) Iff A is negative semidefinite, all of the eigenvalues are non-positive.

Proof:

(i) Since A is symmetric, for any n×1 vector x,

$$\begin{aligned} x^t Ax &= c^t (QDQ^t)c = (Q^t c)^t D(Q^t c) = y^t Dy \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 \end{aligned},$$

where $y = (y_1, y_2, ..., y_n)^t = Q^t x$ (and $x = Qy$). Now, we will show that if all of the eigenvalues are non-negative (≥ 0), then $x^t Ax \ge 0$ for any non-zero vector x. This result is obvious from the above equation.

We now wish to show that if $x^t Ax \ge 0$ for all non-zero vector x, all of the eigenvalues should be non-negative. To show this, suppose that one eigenvalue, say λ_1, is strictly negative: $\lambda_1 < 0$. Choose

$y_1 = 1; y_2 = ... = y_n = 0$ ($y = (1,0,0,...0)^t$). Since Q^t is invertible, there exists a nonzero vector x such that $y = Q^t x$. With the choice of x, $x^t Ax = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2 = \lambda_1 < 0$. But this is a contradiction if A is positive semidefinite.

EX 1: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\det(A - \lambda I_2) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$

$\lambda_1 = 1$ and $\lambda_2 = 3$.

EX 2: $A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

$\det(A - \lambda I_3) = -(\lambda + 5)(\lambda + 1)(\lambda + 5) = -(\lambda + 1)(\lambda + 5)^2$

$\lambda_1 = -1; \lambda_2 = \lambda_3 = -5$.

Theorem:

Let $M_{m\times m}$ be a symmetric and idempotent matrix. Then, M is positive semidefinite.

Definition: **Trace**

For $A_{n\times n} = [a_{ij}]$, $trace(A) = a_{11} + a_{22} + ... + a_{nn}$.

Theorem:

For $A_{n\times p}$ and $B_{p\times n}$, $trace(AB) = trace(BA)$.

Theorem:

Let $A_{n\times n}$ be a symmetric and square matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Then, $trace(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof:

$$trace(A) = trace(QDQ^t) = trace(DQ^tQ) = trace(D).$$

EX 1: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda_1 = 1$ and $\lambda_2 = 3$.

$trace(A) = 2 + 2 = 4$.

EX 2: $A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

$\lambda_1 = -1; \lambda_2 = \lambda_3 = -5$.

$trace(A) = -3 - 3 - 5 = -11$.

Theorem: **Trace and Eigenvalues**

Let $A_{n\times n}$ be a symmetric and square matrix with eigenvalues $\lambda_1,...,\lambda_n$.

Then, $trace(A) = \lambda_1 + \lambda_2 + ... + \lambda_n$.

Theorem: **Rank of Symmetric Idempotent Matrix**

Let $M_{m\times m}$ be a symmetric and idempotent matrix. Than,

$rank(A) = trace(A)$.

Proof:

Observe that the eigenvalues of an idempotent matrix are zeros or ones. Thus, the number of the eigenvalues of one equals the number of non-zero eigenvalues of M, which in turn equals the rank of M. Observe that the number of eigenvalues of one equals

$$\lambda_1 + \lambda_2 + ... + \lambda_m = trace(M).$$

EX:

Let $A_{m\times n}$ with $rank(A) = n$. Let $M = I_m - A(A^tA)^{-1}A^t$.

M is symmetric and idempotent. Thus,

$$\begin{aligned} rank(M) &= trace(M) = trace(I_m) - trace(X(X^tX)^{-1}X^t) \\ &= m - trace[(X^tX)^{-1}X^tX] = m - trace(I_n) = m - n \end{aligned}.$$

Theorem:

Let $M_{m\times m}$ be a symmetric and idempotent matrix. If M is invertible, $M = I_m$.

Proof:

Since M is invertible, its eigenvalues must be all ones. Then,

$$M = QDQ^t = QQ^t = I_n.$$