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The Delian Problem (Duplication/Doubling of the Cube)

There are lots of (conflicting) stories about the so-called Delian Problem. But the gist of these stories are that the citizens of the Greek island of Delos were afflicted by the plague (alternatively, by internal political unrest) and asked the oracle of the god Apollo what they could do to alleviate the situation. The oracle said that they should construct an altar to Apollo in their temple that was twice the size of the altar, in the shape of a regular cube, that they already had. They tried doubling the ***side*** of the cube, but that didn't work (the plague, or political conflict, continued unabated). So the Delians eventually decided (with the advice of Plato?) that they should construct a cubical altar with twice the ***volume*** of the old one. (Plutarch later [1st- 2nd centuries A.D.] maintains that Plato interpreted the oracle's response in a metaphorical or 'demythologized' way–as a general injunction to the Delians to spend their time studying geometry rather than fighting with one another.)

Interpreted literally, then the Delian Problem is the following: given a cube of side a, to find the linear distance x such that a cube of side x is twice the volume of a cube of side a. In algebraic notation (***which the Greeks did not have***): Given a, to find the length x such that $x^3 = 2a^3$.

According to ancient reports, the mathematician Hippocrates of Chios had, in the 5th century B.C., 'reduced' the Delian Problem of the Duplication of the Cube to the following problem: given two line segments of length a and b, to find the line segments of length x and y in continuing geometric proportion between a and b. In algebraic notation this is the Problem (II) of solving the following equation(s) for x and y given values for a and b:
$a/x = x/y = y/b$ (where a, x, y, and b are all positive lengths).

We do not know how Hippocrates reduced the Delian problem to the latter Problem (II)–which is a matter of showing that, from a general solution of Problem (II), we can obtain a solution for the former (Delian) problem. But we can be morally certain that he did ***not*** demonstrate the reduction in the way that ***we*** would–i.e., algebraically, as follows:

Let $b = 2a$. Then from (two of) the equality of ratios above, it follows that (1) $x^2 = ay$ and (2) $y^2 = 2ax$. Multiply both sides of the equation (1) by y in order to obtain (3) $x^2y = ay^2$. Then use the identity of equation (2) to substitute $2ax$ for y^2 in equation (3) and simplify in order to obtain (4) $x^2y = 2a^2x$. Making use of the identity of equation (1), multiply the left side of equation (4) by x^2 and the right side of (4) by ay and simplify terms in order to obtain (5) $x^4y = 2a^3xy$. Finally, divide both sides of equation (5) by xy in order to obtain (6) $x^3 = 2a^3$. So if one can solve the general Problem (II) of finding the two lengths x and y in continuing geometrical proportion between line segments of length a and b (and let $b = 2a$), we have just shown that the value of x solves the Delian Problem of duplication of the cube of side a: the cube constructed on x is twice the volume of the cube constructed on side a.

By the 4th century B.C., then, the geometrical problem faced by members associated with Plato's academy was to solve Problem (II): to find line segments of length $\boldsymbol{x}$ and $\boldsymbol{y}$, given line segments of arbitrary length $\boldsymbol{a}$ and (longer) length $\boldsymbol{b}$.

Again, we who have the advantage of knowing analytic geometry (which was not developed until the 17th and 18th centuries A.D.) might recognize that quadratic equations (1) and (2) above could be represented on a two-dimensional Cartesian coordinate-system (named after its inventor, Rene Descartes, the French philosopher and mathematician who lived in the first half of the 17th century) as intersecting parabolas. So given a values for $\boldsymbol{a}$ and for $\boldsymbol{b}$ ($\boldsymbol{2a}$, for example), we could graph the algebraic equations and obtain values of $\boldsymbol{x}$ and $\boldsymbol{y}$ as the $\boldsymbol{x}$- and $\boldsymbol{y}$-coordinates of the point of intersection of the two parabolas (assuming that the two parabolas do in fact intersect and at only one point).

From the contemporary perspective, this is the easiest way to understand the solution of Problem (II) attributed to the 4th century B.C. geometer Menaechmus, who was associated with Plato's school, the Academy***. But Menaechmus did not have the aid of algebra, Cartesian-coordinate systems, or analytic geometry***. Yet, amazingly, he did produce a solution of Problem (II) in terms of the intersection of two parabolas. How, without the aid of later mathematical developments, he was able to figure this out is a source of wonder (at least to me–and to quite a few other scholars). But he did. Does that mean that he received unqualified praise and admiration for his solution? Not quite. Why not? Because the construction of an arbitrary parabola, which his solution requires, cannot be done using just a (geometrical!) compass and a (unmarked) straight-edge or ruler. In Greek mathematics, parabolas were conceived as 'conic sections'–more particularly, as plane or two-dimensional curves that are obtained by the intersection of (the surface of) a right circular cone and a plane intersecting the cone which is parallel to some other plane tangential to the surface of the cone. In turns out that there is no way to produce any arbitrarily selected plane curve produced in this way by using just a compass and unmarked straight-edge.

So, the Delian Problem of Duplication of a Cube [but, really, the more general problem (II)] entered history as one of the three famous 'insoluble' problems of Greek geometry. The others are
A. 'squaring the circle' (given a circle of arbitrary radius $\boldsymbol{r}$, to construct a square of side s having the same area as that of the circle of radius $\boldsymbol{r}$–i.e. having the area $\boldsymbol{\pi r^2}$).
B. 'trisecting an angle' (given an arbitrary angle $\boldsymbol{\alpha}$, to construct an angle $\boldsymbol{\beta}$ such that $\boldsymbol{3\beta} = \boldsymbol{\alpha}$).
None of these three problems, duplicating the cube, squaring the circle, and trisecting an angle, can be accomplished using only compass and unmarked straight-edge. But, one might ask, "Why should anyone care about that?"

From the algebraic triple-equation of Problem (II), you will note that another equation, which we haven't mentioned, can be derived: $\boldsymbol{xy} = \boldsymbol{ab}$ or (where we let $\boldsymbol{b} = \boldsymbol{2a}$) (7) $\boldsymbol{xy} = \boldsymbol{2a^2}$. Note that we can regard $\boldsymbol{2a^2}$ as a constant. What kind of curve is the graph of this equation? If we find a unique point of intersection of this curve with the parabola defined by equation (1) $\boldsymbol{x^2} = \boldsymbol{ay}$ above, then we can solve Problem (II) and, consequently, the Delian Problem as follows: multiply the less side of (7) by the left side of the identity (1) and the right side of (7) by the right side of (1) and, then, divide out y to obtain (6) $\boldsymbol{x^3} = \boldsymbol{2a^3}$. The strictly geometrical version of ***this*** solution is also credited to Menaechmus as an 'alternative' solution.