System of Linear DE

$$\begin{bmatrix} dx_1/dt \\ \vdots \\ \vdots \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_{t1} \\ x_{t2} \\ \vdots \\ x_{tn} \end{bmatrix}$$

The solution is $[x_1(t), x_2(t) \cdots x_n(t)]$ such that $\frac{dx}{dt} = Ax(t)$ for any t.

The general solution is the formula for all possible solutions for: $\frac{dx}{dt} = Ax$

How to find the general solution?

1.) Write the canonical form and the change of coordinates
    A_{new} and C
2.) Write the general solution

$$\mathbf{A} = \begin{bmatrix} block_1 & 0 & 0 & \cdots & 0 \\ 0 & block_2 & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & block_k \end{bmatrix}$$ and

$$\mathbf{C} = [cblock_1 \quad cblock_2 \quad \cdots cblock_k]$$

The general solution is:

$$x(t) = [dblock_1 \quad +block_2 \quad +\cdots \quad +block_k]$$

Where each individual dblock can take the form of one of the four situations described below. Just remember, you are not done until you have as many constants (C) as columns in the matrix.

I. Single Real Roots

$$block_j = [\lambda_j] \ , \ cblock_j = [\xi_j] \rightarrow dblock_j = [C_j * e^{(\lambda_{1t})} * \xi_j]$$

II. Single Complex Roots

$$x(t) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} ,$$

$$cblock_j = \begin{bmatrix} a & b \end{bmatrix}$$
$$\rightarrow dblock_j = e^{(\alpha t)} \cdot (C_j(a\cos(\beta t) - b\sin(\beta t)) + C_{j+1}(a\sin(\beta t) + b\cos(\beta t)))$$

III. Repeated Real Roots

$$\lambda = \lambda_{(j+1)} = \cdots = \lambda_{(j+m-1)} = \mu$$

$$block_j = \begin{bmatrix} \mu & 1 & 0 & \cdots & 0 \\ 0 & \mu & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \mu \end{bmatrix}$$, with columns size *m* and rows size *m*.

$$cblock_j = \begin{bmatrix} \xi_j \xi_{j1} \xi_{j2} \cdots \xi_{(jm-1)} \end{bmatrix}$$

$$dblock_j = e^{(\mu t)} \cdot [C_j \xi_j + C_{(j+1)}(\xi_j t + \xi_{(j1)}) + C_{(j+2)}(\xi_j \frac{t^2}{2} + \xi_{(j1)} t + \xi_{(j2)})] + \cdots$$

Keep taking the integral and adding new eigenvectors as needed.

IV. Repeated Complex Roots

$$\alpha_j + i\beta = \alpha_{(j+1)} + i\beta_{(j+1)} = \cdots = \alpha_{(j+m+1)} + i\beta_{(j+m+1)} = \alpha + i\beta$$

$$block_j = \begin{bmatrix} \alpha & \beta & 1 & 0 & \cdots & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta & \alpha \end{bmatrix}$$

$$cblock_j = \begin{bmatrix} a_j b_j & a_{j1} b_{j1} & a_{j2} b_{j2} & \cdots & a_{(jm-1)} b_{(jm-1)} \end{bmatrix}$$

$$dblock_j = \exp(\alpha t)[C_j(a\cos(\beta t) - b\sin(\beta t)) + C_{j+1} \cdot ((a_j \cdot t + a_{j1}) \cdot \cos(\beta t) - (b_j t + b_{j1}) \cdot \sin(\beta t)) + \cdots]$$

For: Sergey Nikitin
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By:Carolina Tostado
Joshua Carroll