Lecture and Exercises (Th. Jan. 26, 2012)

1. 2nd order surfaces. The 2nd order hyper surface is defined by the following equation

$$\sum_{j=1}^{n} a_{jj} x_j^2 + \sum_{k<j} a_{kj} x_k \cdot x_j + \sum_{j=1}^{n} b_j \cdot x_j + d = 0$$

in R^n.

You already know from the previous Lectures that there are three big classes of the 2nd order hyper surfaces: elliptic, hyperbolic and parabolic. The class of the surface is dictated by the quadratic form

$$\sum_{j=1}^{n} a_{jj} x_j^2 + \sum_{k<j} a_{kj} x_k \cdot x_j$$

You need to write the matrix of the quadratic form, subtract λ from its diagonal elements, take the determinant and make it equal to zero. Then you need to find the roots of the resulted polynomial. Looking at the roots you can classify the surface as elliptic, hyperbolic or parabolic.

Exercise

Classify the following 2nd order hyper surface as elliptic, hyperbolic or parabolic.

$$3\cdot(x_1^2+x_2^2+x_3^2+x_4^2) - 6\cdot(x_1\cdot x_2 + x_1\cdot x_3 + x_1\cdot x_4 + x_2\cdot x_3 + x_2\cdot x_4 + x_3\cdot x_4) + x_1 + x_4 = 0.$$

2. Partial Derivatives. A function of two variables

$$y = F(x_1, x_2)$$

can be approximated by a linear function

$$y = F(a_1, a_2) + \frac{\partial}{\partial x_1} F(a)(x_1 - a_1) + \frac{\partial}{\partial x_2} F(a)(x_2 - a_2)$$

near the point (a_1, a_2).

$$\frac{\partial}{\partial x_1} F(a), \quad \frac{\partial}{\partial x_2} F(a)$$

are numbers. First, one calculates partial derivatives

$$\frac{\partial}{\partial x_1}F(x_1,x_2), \quad \frac{\partial}{\partial x_2}F(x_1.x_2)$$

and then evaluates them at $a=(a_1,a_2)$. From the geometrical point of view the linear approximation of $y=F(x_1,x_2)$ at the point $a=(a_1,a_2)$ is a plane tangent to the graph of the function $y=F(x_1,x_2)$ at the point $(a_1,a_2,F(a))$

Exercise
Write the linearization (linear approximation) for the function

$$y=\ln(x_1+x_2^n)$$

at $a=(1,0)$.

Exercise
The function

$$y=F(x_1,x_2,\ldots,x_n)$$

is called uniform of order m if

$$F(t\cdot x_1,t\cdot x_2,\ldots,t\cdot x_n)=t^m\cdot F(x_1,x_2,\ldots,x_n).$$

Prove the Euler theorem of uniform functions of order m,

$$F(x_1,x_2,\ldots,x_n)=\frac{1}{m}\sum_{j=1}^{n}x_j\cdot\frac{\partial}{\partial x_j}F(x).$$

Verify the Euler theorem using the following functions.

$$y=(x_1-2x_2+3x_3)^2, \quad y=\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \quad y=(\frac{x_1}{x_2})^{(\frac{x_2}{x_1})}$$

If we consider linearization in terms of tiny changes (deviations of variables) at any point then the linearization is also called the differential of the function (or the differential of the first degree, or the first differential) and denoted as

$$dy=\sum_{j=1}^{n}\frac{\partial}{\partial x_j}F(x)\cdot dx_j$$

The list of derivatives

$$(\frac{\partial}{\partial x_1}F(x), \frac{\partial}{\partial x_2}F(x), \ldots, \frac{\partial}{\partial x_n}F(x))$$

is called **gradient** of the function F and denoted as

$$grad(F)$$

or

$$\nabla F.$$

The symbol ∇ denotes also the formal list of operators

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$$

"Multiply" formally left and right by F and you will get the gradient:

$$\nabla F(x) = (\frac{\partial}{\partial x_1}F(x), \frac{\partial}{\partial x_2}F(x), \ldots, \frac{\partial}{\partial x_n}F(x)).$$

Exercise
Calculate the gradient $\nabla F(x)$ for the following functions.

$$x_1^n \cdot x_2^m, \quad \frac{x_1}{x_2}, \quad x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3, \quad \frac{x_3}{x_1^2 + x_2^2}$$

The gradient points in the direction of the fastest growth of the function. In other words, the gradient points to the fastest way to the top of the function. Follow the gradient and you will get to the top.