MAT 267 PRACTICE PROBLEMS FOR TEST 3

1. Use triple integration to find the volume of the solid enclosed by the cylinder $x^2+y^2=9$ and the planes $y+z=5$ and $z=1$.

2. Write 5 other iterated integrals that are equal to
$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)dzdydx$$

3. Find the volume of the solid region bounded by the cylinder $x^2+y^2=9, z=0, x=0, y=0$ and $y+x/2=10$.

4. Find the volume of the solid bounded by the cylinder $x^2+y^2=2$, the paraboloid $z=4-x^2-y^2$ and the xy-plane.

5. Evaluate the integral
$$\int_V x\mathrm{d}V$$
inside domain V, where V is bounded by the planes $x=0, y=x, z=0$ and the surface $x^2+y^2+z^2=1$.

6. Evaluate the integral
$$\int_V x^2z\mathrm{d}V$$
inside domain V, where V is bounded by the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=1$. Can you get the answer without evaluating the triple integral?

7. Find the gradient field for $f(x,y)=\ln(\sqrt{x^2+y^2})$. Plot the gradient field together with the level curves.

8. Evaluate the line integral $\int_C F\cdot dr$, where C is given by the vector field $r(t)$.
$$F(x,y)=x\cdot\mathbf{i}+y\cdot\mathbf{j}+z\cdot\mathbf{k},$$
$$r(t)=\cos(t)\cdot\mathbf{i}+\sin(t)\cdot\mathbf{j}+t\cdot\mathbf{k},\ \ 0\le t\le 2\pi$$

9. Evaluate the line integral $\int_C F\cdot dr$, where C is given by the vector field $r(t)$.
$$F(x,y)=-y\cdot\mathbf{i}+x\cdot\mathbf{j},$$
$$r(t)=(2+\cos(t))\cdot\mathbf{i}+(2+\sin(t))\cdot\mathbf{j},\quad 0\le t\le 2\pi$$

10. Find $\int_C xy^2 ds$ where C is $x = \sqrt{4-y^2}$, traversed clockwise.

11. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) =< 2xy + y^3 + 2, x^2 + 3xy^2 + 2y >$ and C is $x = -\sqrt{1-y^2}$ in clockwise direction.

12. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) =< x, e^{y^2} >$ and C is a curve between the points (-1,-1) and (2,-1).

13. Let C be the curve that consists of the line segment from $(0,0)$ to $(2,0)$, the semicircle from $(2,0)$ to $(1, \sqrt{3}\ )$, and the line segment from $(1, \sqrt{3}\ )$ to $(0,0)$. Use Green's theorem to find $\oint_C 6y^2x\, dx + 5x^2y\, dy$. (Note: the semicircle is centered at the origin.)

14. Let C be the curve that consists of the line segment from $(0,0)$ to $(1,0)$, the line segment from $(1,0)$ to $(1,2)$, and the line segment from $(1,2)$ to $(0,0)$. Use Green's theorem to find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x,y) = \langle \cos(x^2) + y^3,\ e^y \sin(y) + 3x^2 \rangle$.

15. For $\mathbf{F}(x,y,z) = \langle 2x^2y,\ 2xy^2 + e^z,\ x + ye^z - \sin(z) \rangle$. Compute
    i) div$\mathbf{F}(x,y,z)$ and div$\mathbf{F}(3,2,0)$,
    ii) curl$\mathbf{F}(x,y,z)$ and curl$\mathbf{F}(3,2,0)$,
    iii) div curl $\mathbf{F}(x,y,z)$ and div curl $\mathbf{F}(3,2,0)$.

16. Determine which vector fields are conservative. Find a potential for each vector field that is conservative or state the reason why $\mathbf{F}$ is not conservative.

    a) $\mathbf{F}(x,y,z) = \langle \sin(xy), \sin(xy), 0 \rangle$

    b) $\mathbf{F}(x,y,z) = \langle x, z, y \rangle$

    c) $\mathbf{F}(x,y,z) = \langle x, z^2, y^2 \rangle$

    d) $\mathbf{F}(x,y,z) = \langle e^{-y^2}, -2xye^{-y^2}, -\sin(z) \rangle$

    e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$ if C is parameterized by $\mathbf{r}(t) = \langle \cos(t/2), \sin(t), t \rangle$, $0 \le t \le 2\pi$ and $\mathbf{F}$ is the vector field from part (d)?

17. Refer to the graphs of the 2D vector fields on page 730 in your book. Find if the gradient and the divergence is positive, negative or zero:

    a) On graph I at the point (2,2)

    b) On graph II at the point (-4,-4)

    c) On graph III at the point (2,2)

18. Find the surface area of the paraboloid $z = x^2 + y^2$ that lies above the annulus $1 \leq r \leq 3$.

19. Find the surface area of the the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices (0,0),(1,0) and (1,2).

20. Evaluate the surface integral $\int\int_S \sqrt{1 + x^2 + y^2}\, dS$ where S is given by $\mathbf{r}(u, v) =< u \cos v,\ u \sin v,\ v >$, $0 \leq u \leq 1,\quad 0 \leq v \leq \pi$.

21. Evaluate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^2\mathbf{i} + e^y\mathbf{j} + \mathbf{k}$ and S is the plane $x + y + z = 1$ in the first octant. S is oriented upward.

SOLUTIONS

1. 36π

2.

$$\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z) dxdydz$$

$$\int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z) dydxdz$$

$$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z) dxdzdy$$

$$\int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z) dydzdx$$

$$\int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z) dzdxdy$$

3. Substitute $z = r\cos\theta, x = r\sin\theta, y = y$. The volume is $\int_0^3 \int_0^{\pi/2} \int_0^{10-r\sin\theta} r\ dyd\theta dr = \int_0^3 \int_0^{\pi/2} 10r - \frac{r^2}{2}\sin\theta drd\theta = \int_0^{\pi/2} 45 - \frac{9}{2}\sin\theta d\theta = 1/2(45\pi - 9)$.

4. Substitute $x = r\cos\theta, y = r\sin\theta, z = z$. The volume is $\int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^{4-r^2} r\ dzd\theta dr = \int_0^{\sqrt{2}} \int_0^{2\pi} 4r - r^3 d\theta dr = 2\pi \int_0^{\sqrt{2}} 4r - r^2 dr = 2\pi(4 - 2^{3/2}/3) = 4\pi(2 - \frac{\sqrt{2}}{3})$

5. $1/8 - \sqrt{2}/16$.

6. 0. There is reflection symmetry of x^2z about the $z = 0$ plane. The domain is also symmetric so the integral should sum to 0.

7. $\nabla f = < \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} >$.

8. The line integral is reduced to

$$\int_0^{2\pi} tdt = \frac{(2\pi)^2}{2}$$

9. The line integral is reduced to

$$\int_0^{2\pi} 2(\sin(t) + \cos(t)) + 1dt = 2\pi$$

10. $\dfrac{32}{3}$ (hint: parametrize the circle x= 2 sin(t), y= 2 cos(t)).

11. 0 (did you find the potential function and use the FT of Line integrals?)

12. 1.5 remark: note that although this is a conservative vector field we cannot find the potential function so we have to calculate the line integral directly. The good news is that it is still independent of path so we can choose any convenient and easy path we want to.

13. -3

14. 2

15. i) $8xy + ye^z - \cos(z)$, 49
ii) $\langle 0,\ -1,\ 2y^2 - 2x^2\rangle$, $\langle 0,\ -1,\ -10\rangle$
iii) 0, 0

16. (Note the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$)

a) $\text{curl}\mathbf{F}(x, y, z) = \langle 0,\ 0,\ (y - x)\cos(xy)\rangle \neq \mathbf{0}$, so $\mathbf{F}$ is not conservative.

b) $\text{curl}\mathbf{F}(x, y, z) = \langle 0,\ 0,\ 0\rangle = \mathbf{0}$, so $\mathbf{F}$ is conservative. A potential is $f(x, y, z) = \frac{x^2}{2} + yz$

c) $\text{curl}\mathbf{F}(x, y, z) = \langle 2y - 2z,\ 0,\ 0\rangle \neq \mathbf{0}$, so $\mathbf{F}$ is not conservative.

d) $\text{curl}\mathbf{F}(x, y, z) = \langle 0,\ 0,\ 0\rangle = \mathbf{0}$, so $\mathbf{F}$ is conservative. A potential is $f(x, y, z) = xe^{-y^2} + \cos(z)$

e) $f(-1, 0, 2\pi) - f(1, 0, 0) = -2$

17. a) On graph I at the point (2,2) $\text{curl}\mathbf{F}(2, 2)$ is negative and $\text{div}\mathbf{F}(2, 2)$ is zero.

b) On graph II at the point (-4,-4) $\text{curl}\mathbf{F}(-4, -4)$ is zero and $\text{div}\mathbf{F}(-4, -4)$ is negative.

c) On graph III at the point (2,2) $\text{curl}\mathbf{F}(2, 2)$ is positive and $\text{div}\mathbf{F}(2, 2)$ is zero.

18. $\frac{\pi}{6}(3\sqrt{3} - 1)$

19. $\frac{1}{6}(27 - 5\sqrt{5})$

20. $\frac{4}{3}\pi$

21. $\frac{7}{2}$

22. 4π