0.1 Canonical basis in $\mathbb{R}^n$

Any string $a \in \mathbb{R}^2$ can be written as follows.

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For any string $a \in \mathbb{R}^3$ we also have

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Following this

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + a_n \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Let us introduce notations.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots \ e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

The set of strings

$$\{e_1, \ e_2, \ \ldots \ e_n\}$$

is called the **canonical** basis in $\mathbb{R}^n$.

In Euclidean space ($\mathbb{R}^n$ equipped with the dot product)

$$e_i \cdot e_j = 0 \quad \text{for} \quad i \neq j \quad \text{and } e_i \cdot e_i = 1 \quad \text{for} \quad i, \ j = 1, \ 2, \ \ldots \ n$$

A basis with such properties is called orthonoraml.

0.2 Wedge product (exterior product) definition

Consider two strings a and b from $\mathbb{R}^n$ the wedge (wedge product) of a and b is

$$a \wedge b$$

It is the product with the antisymmentric (and/or anticommutative) property

$$a \wedge b = -b \wedge a$$

You can open parentheses and factor out numbers

$$(\alpha \cdot a + \beta \cdot c) \wedge b = \alpha \cdot a \wedge b + \beta \cdot c \wedge b$$

where $\alpha, \ \beta$ are numbers and $a, \ b, \ c$ are from $\mathbb{R}^n$.

0.3 Wedge in a linear space

Let $\{e_1,\ e_2,\ \dots\ e_n\}$ be the canonical basis in $\mathbb{R}^n$. Then the set of all possible wedge products between strings from $\mathbb{R}^n$ is

$$\bigwedge\nolimits_2(\mathbb{R}^n) = \{a \wedge b;\ a \text{ and } b \in \mathbb{R}^n\}$$

Let us show that $\bigwedge_2(\mathbb{R}^n)$ is a subset of a real linear space of dimension $\binom{n}{2}$.

$\binom{n}{2}$ denotes the number of ways one can choose an unordered pair of elements from the set of n elements.

Since $a,\ b \in \mathbb{R}^n$

$$\begin{aligned} a &= a_1 \cdot e_1 + a_2 \cdot e_2 + \cdots + a_n \cdot e_n \\ b &= b_1 \cdot e_1 + b_2 \cdot e_2 + \cdots + b_n \cdot e_n \end{aligned}$$

and

$$a \wedge b = \sum_{i<j} (a_i \cdot b_j - a_j \cdot b_i) e_i \wedge e_j$$

where $i,\ j$ are taking values $1,\ 2,\ \dots\ n$.

In other words,

$$a \wedge b = \sum_{i<j} \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \cdot e_i \wedge e_j$$

and $a \wedge b$ is the string of values

$$\det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \qquad j < j \quad \text{and } i, j = 1,\ 2,\ \dots\ n$$

in $\binom{n}{2}$-dimensional real space with the canonical basis

$$e_i \wedge e_j \qquad j < j \quad \text{and } i, j = 1,\ 2,\ \dots\ n$$

For convenience we order pairs $(i,\ j)$ according to lexicographic rule (like words in a dictionary). For example, if $n = 4$ then lexicographical order $(i,\ j)$ $(i, j = 1,\ 2,\ 3,\ 4)$ looks as follows.

$$(1,\ 2),\ (1,\ 3),\ (1,\ 4),\ (2,\ 3),\ (2,\ 4),\ (3,\ 4)$$

We address

$$\mathbb{R}^4 = \{(x_1,\ x_2,\ x_3,\ x_4);\ x_j \in \mathbb{R}\ (j = 1,\ 2,\ 3,\ 4)\}$$

as time-space continuum. For $a,\ b \in \mathbb{R}^4$ the wedge $a \wedge b$ has the following coordinates.

$$\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}, \det \begin{pmatrix} a_1 & a_4 \\ b_1 & b_4 \end{pmatrix},$$

$$\det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}, \det \begin{pmatrix} a_2 & a_4 \\ b_2 & b_4 \end{pmatrix},$$

$$\det\begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \end{pmatrix}$$

0.4 Wedge in Euclidean space. Wedge product magnitude.

A linear real space $\mathbb{R}^n$ with the dot product

$$x \cdot y = x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n$$

becomes the Euclidean space. One can calculate all sorts of stuff with the help of the dot product. In particular one can calculate the magnitude $|x|$ of x as

$$|x| = \sqrt{x \cdot x}$$

$\bigwedge_2(\mathbb{R}^n)$ is a subset of Euclidean space as well and we can use the following dot product.

$$(a \wedge b) \cdot (c \wedge d) = \sum_{i<j} \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \cdot \det\begin{pmatrix} c_i & c_j \\ d_i & d_j \end{pmatrix}$$

The magnitude $|a \wedge b|$ is

$$|a \wedge b| = \sqrt{\sum_{i<j} \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}^2}$$

There is another way to calculate $|a \wedge b|$, in fact, it is

$$|a \wedge b| = \sqrt{\det\begin{pmatrix} a \cdot a & a \cdot b \\ b \cdot a & b \cdot b \end{pmatrix}}$$

As you know from our classes

$$\sqrt{\det\begin{pmatrix} a \cdot a & a \cdot b \\ b \cdot a & b \cdot b \end{pmatrix}}$$

is the area of the parallelogram spanned by a and b.

Actually, it is not difficult to see that

$$\det\begin{pmatrix} a \cdot a & a \cdot b \\ b \cdot a & b \cdot b \end{pmatrix} = \sum_{i<j} \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}^2$$

As you can see

$$\begin{pmatrix} a \cdot a \\ b \cdot a \end{pmatrix} = \sum_{i=1}^{n} a_i \cdot \begin{pmatrix} a_i \\ b_i \end{pmatrix}$$

and, respectively,

$$\begin{pmatrix} a \cdot b \\ b \cdot b \end{pmatrix} = \sum_{i=1}^{n} b_i \cdot \begin{pmatrix} a_i \\ b_i \end{pmatrix}$$

Hence,

$$\det\begin{pmatrix} a \cdot a & a \cdot b \\ b \cdot a & b \cdot b \end{pmatrix} = \det\left(\begin{pmatrix} a \cdot a \\ b \cdot a \end{pmatrix} \quad \begin{pmatrix} a \cdot b \\ b \cdot b \end{pmatrix}\right) =$$

$$\det\left(\sum_{i=1}^n a_i \cdot \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \begin{pmatrix} a\cdot b \\ b\cdot b \end{pmatrix} \right) = \sum_{i=1}^n a_i \cdot \det\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \begin{pmatrix} a\cdot b \\ b\cdot b \end{pmatrix} \right)$$

and performing the same steps for

$$\begin{pmatrix} a\cdot b \\ b\cdot b \end{pmatrix}$$

we obtain

$$\det\left(\begin{pmatrix} a\cdot a \\ b\cdot a \end{pmatrix} \quad \begin{pmatrix} a\cdot b \\ b\cdot b \end{pmatrix} \right) = \sum_{i,j=1}^n a_i b_j \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$$

Taking into account properties of the determinant

$$\det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} = -\det\begin{pmatrix} a_j & a_i \\ b_j & b_i \end{pmatrix}$$

yields

$$\det\begin{pmatrix} a\cdot a & a\cdot b \\ b\cdot a & b\cdot b \end{pmatrix} = \sum_{i<j} (a_i b_j - a_j b_i) \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$$

and finally

$$\det\begin{pmatrix} a\cdot a & a\cdot b \\ b\cdot a & b\cdot b \end{pmatrix} = \sum_{i<j} \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}^2$$

0.5 Star operator $*$

Consider the wedge product

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p$$

where $v_1,\ v_2,\ \ldots\ v_p$ are from $\mathbb{R}^n$ and $p \le n$. The set of all such wedge products is denoted by

$$\bigwedge\nolimits_p(\mathbb{R}^n)$$

It is a subset of a linear space of dimension $\binom{n}{p}$. Let

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p = \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1 i_2 \ldots p} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$$

Then the star operator

$$* : \bigwedge\nolimits_p(\mathbb{R}^n) \to \bigwedge\nolimits_{n-p}(\mathbb{R}^n)$$

is a linear mapping defined as follows.

$$*(v_1 \wedge v_2 \wedge \cdots \wedge v_p) = \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1 i_2 \ldots p} (*(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}))$$

where

$$*(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = (-1)^{P(j_1,\, \ldots\, j_{n-p},\, i_1\, \ldots\, i_p)} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{n-p}}$$

and $j_1 < j_2 < \dots < j_{n-p}$ is such that one can get

$$(1,\ 2,\ \dots\ n)$$

out of

$$j_1,\ \dots\ j_{n-p},\ i_1\ \dots\ i_p$$

performing the number $P(j_1,\ \dots\ j_{n-p},\ i_1\ \dots\ i_p)$ of transpositions.

For example $\bigwedge_2(\mathbb{R}^3)$ has the following canonical basis.

$$e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3$$

One makes $(1,\ 2,\ 3)$ out of $(3,\ 1,\ 2)$ with 2 transpositions and

$$*(e_1 \wedge e_2) = (-1)^2 e_3 = e_3$$

After flipping 2 and 1 (one transpositions) $(2,\ 1,\ 3)$ becomes $(1,\ 2,\ 3)$ and

$$*(e_1 \wedge e_3) = (-1)^1 e_2 = -e2$$

Finally

$$*(e_2 \wedge e_3) = (-1)^0 e_1 = e_1$$

For any $a,\ b \in \mathbb{R}^3$

$$a \wedge b = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} e_1 \wedge e_2 + \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} e_1 \wedge e_3 + \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} e_2 \wedge e_3$$

and

$$*(a \wedge b) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} (*(e_1 \wedge e_2)) + \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} (*(e_1 \wedge e_3)) + \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} (*(e_2 \wedge e_3))$$

Taking into account $*(e_1 \wedge e_2) = e_3,\ \ *((e_1 \wedge e_3) = -e2,\ \ *(e_2 \wedge e_3) = e1$ yields the following.

$$*(a \wedge b) = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} e_3 + \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} (-e_2) + \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} e_1$$

You can see that $*(a \wedge b)$ is identified by the following string from $\mathbb{R}^3$.

$$(\det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}, -\det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}, \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix})$$

This string from $(R)^3$ has a special name it is called the cross product between a and b and denoted by $a \times b$. The cross product is defined ONLY IN $\mathbb{R}^3$. However, the wedge product is defined for any dimension n.

by Sergey Nikitin, September, 2022