Indag. Mathem., N.S., 11 (3), 453-462 September 25, 2000

A logical look at characterizations of geometric transformations under mild hypotheses

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Communicated by Prof. H.S.M. Coxeter at the meeting of April 24, 2000

Without going out-of-doors,
one may know all under heaven;
Without peering through windows,
one may know the Way of heaven.
Lao Tzu, Tao Te Ching

ABSTRACT

For several characterizations of geometric transformations – which state that a map, which satisfies certain conditions like injectivity, surjectivity, bijectivity and preserves certain geometric notions γ_i, must preserve another notion ν as well – we provide the definitional counterpart, i.e. a definition that satisfies certain syntactic constraints of the notion ν in terms of the notions γ_i.
2000 MSC: 51M05 51B10 03C40

1. The basis for our logical reformulations of characterizations of geometric transformations under mild hypotheses is the following[1]

Theorem 1 (Lyndon [11]; Keisler [9, Corollary 1.4a]). *Let* L *be a first order language containing a sign for an identically false formula,* $\mathcal{T}$ *be a theory in* L, *and* $\varphi(\mathbf{X})$ *be an* L*-formula in the free variables* $\mathbf{X} = (X_1, \ldots, X_n)$. *Then the following assertions are equivalent:*

[1] I thank Lou van den Dries for referring me to [9].

(i) *there is a positive existential (positive existential, but negated equality is allowed; positive)* L*-formula* $\psi(\mathbf{X})$ *such that* $\mathcal{T} \vdash \varphi(\mathbf{X}) \leftrightarrow \psi(\mathbf{X})$*;*

(ii) *for any* $\mathfrak{A}, \mathfrak{B} \in Mod(\mathcal{T})$*, and each homomorphism (monomorphism; epimorphism)* $f : \mathfrak{A} \to \mathfrak{B}$*, the following condition is satisfied:*

if $\mathbf{c} \in \mathfrak{A}^n$ *and* $\mathfrak{A} \models \varphi(\mathbf{c})$*, then* $\mathfrak{B} \models \varphi(f(\mathbf{c}))$.

The validity of characterizations of geometric transformations under mild hypotheses can be thus seen *inside* the geometric theory itself, it is an intrinsic property, for which one need not make any reference to maps of models of that theory.

We start with the following

Theorem 2 (Kestelman [10]). *Let* V *be a real pre-Hilbert space of dimension* ≥ 2 *and* $f : V \to V$ *be a map that preserves orthogonality. Then* f *is a similarity.*

This theorem says that mappings that preserve orthogonality must preserve equidistance, collinearity, and ratios as well. But ratios are not a first-order notion (given three points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in V, such that $\mathbf{r} = (1-\lambda)\mathbf{p} + \lambda\mathbf{q}$, λ is said to be a *ratio* of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$), so we should not expect that they will be preserved. In fact, the preservation of ratios is a consequence of the Archimedeanity of the ordered field of real numbers, and Archimedeanity is again not a first-order notion. The most general spaces that we may replace V with and still reach the conclusion that f preserves equidistance and collinearity are *Euclidean spaces* as defined and axiomatized in [13]. V is a vector space over a commutative field K of characteristic $\neq 2$, with dim $V \geq 2$, $q : V \to K$ a quadratic form such that $q(\mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$, and β the associated symmetric bilinear form, i.e. $q(\mathbf{v}) = \beta(\mathbf{v}, \mathbf{v})$. The points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ from V are *collinear*, a relation to be denoted by $L(\mathbf{pqr})$, if $\mathbf{p} = \mathbf{q}$ or $\mathbf{r} = (1-\lambda)\mathbf{p} + \lambda\mathbf{q}$ for some $\lambda \in K$; the pairs of points $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{c}, \mathbf{d})$ are *equidistant* (or the segment $\mathbf{ab}$ is *congruent* to segment $\mathbf{cd}$), a relation to be denoted by $\mathbf{ab} \equiv \mathbf{cd}$, if and only if $q(\mathbf{a} - \mathbf{b}) = q(\mathbf{c} - \mathbf{d})$; and $\mathbf{ab}$ is *perpendicular* to $\mathbf{ac}$, to be denoted by $\mathbf{ab} \perp \mathbf{ac}$ if and only if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are different and $\beta(\mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}) = 0$. The proof given in [10] remains unchanged in this more general situation, and allows us to conclude that f preserves L and $\equiv$. Let $\mathcal{T}_n$ be the theory axiomatized in [13], formulated inside the language $\mathsf{L}_{L\equiv\perp}$ (the first order language with one sort of variables, for 'points' and with L, $\equiv$, and $\perp$ as primitive notions), all of whose models are the n-dimensional Euclidean spaces defined above. The definitional counterpart of the generalized version of Theorem 2 was given by D. Scott [15] (cf. also [14]), who proved that the midpoint operation $M(abc)$ (to be interpreted as 'b is the midpoint of ac') can be defined by an existential fromula in $\mathsf{L}_{\perp}$:

$$M(xyz) :\leftrightarrow [(y = x \wedge y = z) \vee (\exists uv)\,(ux \perp uz \wedge vx \perp vz \wedge xu \perp xv \wedge zu \perp zv \wedge yx \perp yu \wedge yx \perp yv \wedge yz \perp yu \wedge yz \perp yv)], \tag{1}$$

and Pieri's I ($I(abc)$ stands for $ab \equiv ac$) by:

(2) $\qquad I(xyz) :\leftrightarrow [y = z \vee M(yxz) \vee (\exists w)\,(M(ywz) \wedge wx \perp wy \wedge wx \perp wz)]$,

thus equidistance is positively existentially definable in $\mathsf{L}_\perp$ by the following definition

(3) $\qquad ab \equiv cd :\leftrightarrow (\exists uv)\, M(aud) \wedge M(cuv) \wedge I(abv)$.

It is now easy to see that, for every $n \geq 2$, collinearity is positively definable in $L_\perp$. For $n = 2$ we have

(4) $\qquad L(abc) :\leftrightarrow (\exists uv)\, uv \perp ua \wedge uv \perp ub \wedge uv \perp uc$,

whereas for higher dimensions we can express positively existentially that $a_1, \ldots a_m$ lie in a hyperplane, by the formula

$$H(a_1 \ldots a_m) :\leftrightarrow (\exists uv) \bigwedge_{i=1}^{m} uv \perp ua_i,$$

and thus can succesively lower the dimension until we get to the 2-dimensional case, and use inside it (4) to express the collinearity of a, b, c. It is an open problem whether L can be be positively existentially defined in terms of $\perp$ in dimension-free Euclidean planes, i.e. if there is a positive existential definition of L in terms of $\perp$ valid in $\cap_{n\geq 2}\mathcal{T}_n$.

2. We now look at theorems characterizing maps that preserve circles, with the aim of finding the most general framework in which they remain valid, and to find an intrinsic, intra-theoretical expression of these theorems. We begin with

Theorem 3 (Gardner, Mauldin [7, Theorem 18]). *Let H be a real Hilbert space of dimension ≥ 2 and $f : H \to H$ be a bijection that maps circles onto circles. Then f is a similarity.*

Again, the fact that f preserves ratios is a consequence of Archimedeanity, thus, if we formulate this theorem in its entire generality, as one about maps of Euclidean spaces, then all we can expect of f is that it preserves orthogonality, for then the generalized version of Theorem 2 allows us to conclude that f preserves $\equiv$ and, in the finite-dimensional case, L as well. In Euclidean spaces, by *circles* we understand sets of points x that lie in a plane π such that $ox \equiv oa$ for some fixed distinct points o and a. Concyclicity, a quaternary predicate C, with $C(abcd)$ to be read as 'a, b, c, d are four different concyclic points', is a covenient way to express the notion of a circle in a first-order language with points as variables. The condition that f maps circles onto circles may be expressed in three different ways. First, it may be expressed as 'f preserves the concyclicity and nonconcyclicity of four distinct points, i.e. C and $\neg C$'. To see that the hypotheses of the above theorem do imply the preservation of nonconcyclicity, let a, b, c, d be four distinct points with $\neg C(abcd)$. If among a, b, c, d there are three noncollinear points, then there is a circle, whose point-set we denote by K, passing through them. Its image under f, $f(K)$ has to be the point-set of a circle, so the image under f of the fourth point, the one not in K, cannot be in $f(K)$, since f is injective. If the points a, b, c, d are different and collinear, then let K be the pointset of a circle which passes through a and b. Then $f(K)$ is a the pointset of a circle which passes through $f(a)$ and $f(b)$. Suppose that $f(a)$, $f(b)$, $f(c)$, and $f(d)$ are concyclic, and let K' denote the pointset of that circle. Since f is injective, $f(K) \neq K'$. Let p be the intersection of the tangent in $f(a)$ to $f(K)$ with K', and let z be any point in $K' \setminus \{p, f(a), f(b)\}$. Then the line joining $f(a)$ and z has a second intersection point with $f(K)$, say u. Since f is surjective, there is an $x \in K$ and a y on the line ab with $f(x) = u$ and $f(y) = z$ (y has to be on ab, since otherwise there would exist a circle G passing through a, b, y, and $f(G) \neq K'$, thus $f(G) \cap K' = \{f(a), f(b)\}$, thus $f(y)$ would not be in $f(G) \cap K'$, contradiction). Since both x and y are different from a and from b, by the injectivity of f, a, x, y are three different noncollinear points, so their images, $f(a)$, u, z should lie on a circle, which is impossible, since these points are collinear.

For $n = 2$, a stronger version of Theorem 3, in which f is required to preserve only C, was proved in [2].

Other two ways to express this theorem may be obtained in a two-sorted language $\mathsf{L}_{\mathrm{I}\perp}$, with lower-case variables for 'points' and upper-case variables for 'circles', and a binary relation of incidence between points and circles, I, with $p\mathrm{I}K$ to be read as 'p is incident with K', and the ternary relation $\perp$ among points that we have already encountered. One of these two versions asks f to preserve I and $\neg$I, the other to preserve I and circle inequality, both of which are easily seen to follow from the condition that f map circles onto circles and be one-to-one.

Theorem 4. *Let $\mathfrak{M}$, $\mathfrak{N}$ be Euclidean spaces of dimension $n \geq 2$, and $f : \mathfrak{M} \to \mathfrak{N}$ a surjective map that satsifies one of the following conditions:*

(i) *f preserves C and $\neg C$,*

(ii) *f preserves* I, $\neg$I, *and $\neq$ between point variables,*

(iii) *f preserves* I *and $\neq$ between both point and circle variables.*

Then f also preserves $\perp$ (orthogonality), thus $\equiv$ and, if n is finite, L as well. For $n = 2$ the same conclusion holds with (i) *or* (ii) *weakened to*

(i)′ *f preserves C,*

(ii)′ *f preserves* I *and $\neq$ between point variables.*

Proof. Suppose f satisfies (i). According to Theorem 1 we have to define $\perp$ in terms of C. Checking that the following is a definition of $\perp$ in terms of C and L (regardless of dimension) is a simple exercise in linear algebra (see Fig. 1).

$$ab \perp ac :\leftrightarrow (\exists mnp)\, L(bmp) \wedge L(cap) \wedge L(cnm) \wedge L(anb) \wedge C(acbm) \wedge C(anmp). \tag{5}$$

That L may be defined in terms of C is readily seen from

$$L(abc) :\leftrightarrow (\forall d)\, \neg C(abcd).$$

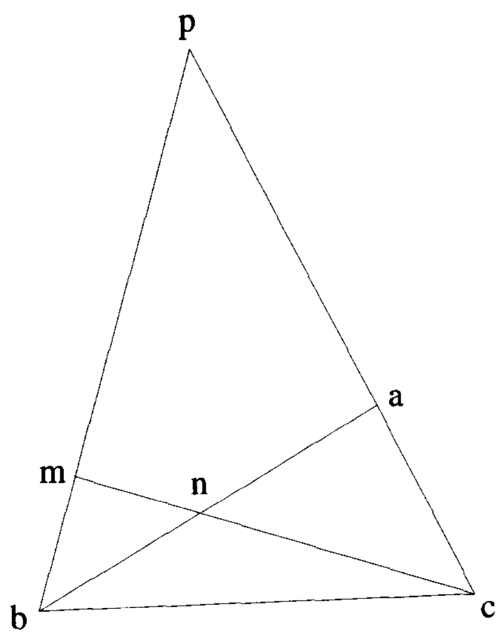

Figure 1: Definition of $ab \perp ac$

Suppose $n = 2$ and (i)$'$ holds. We need to show that L may be defined *positively* in terms of C, and

$$L(a_1a_2a_3) :\leftrightarrow (\exists x_1)(\exists x_2)(\exists y_1)(\exists y_2)(\forall p)(\exists q_1)(\exists q_2) \bigvee_{i \neq j} a_i = a_j$$

$$\vee \left[C(a_1a_2x_1x_2) \bigwedge_{i=1}^{2} C(a_2a_3x_iy_i) \wedge \left(p = a_1 \vee C(a_2a_3pq_1) \vee \left(\bigwedge_{i=1}^{2} C(a_1x_ipq_i) \right) \right) \right]$$

is such a definition, which states that, in case a_1, a_2, a_3 are all different, there are two points x_1 and x_2, not on the line a_2a_3, not collinear with a_1, such that any point p, different from a_1, which is on the line a_2a_3, cannot be on any one of the lines a_1x_1 or a_1x_2, i.e. that a_1 is the intersection point of a_2a_3 and both a_1x_1 and a_1x_2. This clearly holds if a_1, a_2, a_3 are collinear, and does not hold otherwise, since a_2a_3 must intersect one of the lines a_1x_1 and a_1x_2 (as only one of them could be parallel to a_2a_3), and the intersection point is not a_1.

Suppose that f satisfies (ii). One can readily translate the definition (5) of $\perp$ in $\mathsf{L}_{\mathrm{I}\perp}$ by replacing every occurrence of $C(xyuv)$ with $(\exists K) \neq (xyuv) \wedge (x, y, u, v\mathrm{I}K)$ (with different K for different quadruples (x, y, u, v)).[2] Therefore, to show that f preserves $\perp$ we need to show that L is definable in terms of I, without using $\neq$ between circle variables. Such a definition is:

$$L(pqr) :\leftrightarrow p = q \vee q = r \vee r = p \vee (\forall K)(\forall x)(\exists K')\,(p, q, x\mathrm{I}K) \rightarrow x = p \vee (q, r, x\mathrm{I}K'). \tag{6}$$

It states that three different points p, q, r are collinear if and only if for all circles K through p and q, the only points on K which may be collinear with q and r are

[2]Here and in the sequel, for improved readability, $\bigwedge_{1 \leq i \leq n, 1 \leq j \leq m} a_i\mathrm{I}K_j$ was denoted by $(a_i, \ldots, a_n\mathrm{I}K_1, \ldots K_m)$, and $\neq (a_1, \ldots, a_n)$ stands for $\bigwedge_{i \neq j} a_i \neq a_j$.

p and q (the noncollinearity of three points is equivalent to the existence of a circle through them).

In case f satisfies (iii) the desired result follows from the positive definability of $\neg\mathrm{I}$ in terms of I and $\neq$ between point and circle variables, with definition

$$(7) \qquad \neg x\mathrm{I}K : \leftrightarrow (\exists K')(\exists p)(\exists q)\, K \neq K' \wedge \neq (p,q,x) \wedge (p,q,x\mathrm{I}K') \wedge (p,q\mathrm{I}K).$$

It states that x is not on K if and only if there is a circle K', different from K, that passes through x and intersects K in two points p and q, both different from x.

By replacing every occurence of $\neg\mathrm{I}$ in the definiens of (7) (after rephrasing it without the use of $\rightarrow$) by the definiens of (8) corresponding to it, we obtain a definition of L, positive in I, but which contains $\neq$ between both point and circle variables.

To prove that f satisfying (ii)$'$ preserves $\perp$ as well, all we need to show is that L is positively definable in terms of I, with negated equality allowed. Such a definition is

$$\begin{aligned} L(a_1a_2a_3) : \leftrightarrow &(\exists x_1)(\exists x_2)(\exists G_1)(\exists G_2)(\forall p)(\exists L_1)(\exists L_2) \bigvee_{i\neq j} a_i = a_j \\ &\vee [\neq (x_1,x_2,a_1,a_2,a_3) \wedge (a_1,x_1,x_2\mathrm{I}K) \bigwedge_{i=1}^{2} (a_2,a_3,x_i\mathrm{I}G_i) \\ &\wedge (p = a_1 \vee (p \neq a_2 \wedge p \neq a_3 \wedge (a_2,a_3,p\mathrm{I}L_1) \\ &\vee (\bigwedge_{i=1}^{2} (a_1,x_i,p\mathrm{I}L_i))]. \end{aligned} \qquad \square$$

A definition of $\perp$ in terms of I, valid in Euclidean planes, without imposing any syntactical constraints on the definiens, was given in [12, Theorem 2].

Corollary. *The quaternary relation C may serve, in a first-order language with only one sort of variables, to be interpreted as 'points', as the only primitive notion for axiomatizing Euclidean spaces of finite dimension ≥ 2. These may also be axiomatized in a language with two sorts of individuals, for 'points' and 'circles', using the binary relation* I.

A like-minded result on maps preserving circles is the following

Theorem 5 (Carathéodory [5]). *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an injection that maps circles onto circles. Then f is a similarity.*

An intrinsic expression for this theorem may be formulated in $L_{\mathrm{I}\perp}$. If f is an injection between two Euclidean spaces of dimension 2 that maps circles onto circles, then f ought to preserve $\perp$ as well. The condition that f be 'surjective on circles' translates into the syntactic constraint on the formula that should define $\perp$ in terms of I that requires it to be an existential formula with negated equality between points and 'bounded universal quantification' over points allowed (we abbreviate this syntactic constraint as b.u.e.). By bounded universal quantification over points we understand quantifications of the type $(\forall p)\, pIK \rightarrow \varphi$, for which we write (in analogy to set theory or arithmetic) $(\forall pIK)\, \varphi$. To prove that $\perp$ is b.u.e.-definable in terms of I, all we need do is to show that collinearity of points, L, is b.u.e.-definable in terms of I, since we can get an existential definition of $\perp$ in terms of L and I (thus a b.u.e. definition in terms of I) by eliminating C from the definiens in (5) as indicated in the proof of Theorem 4. The desired definition of L is:

$$L(p_1p_2p_3) :\leftrightarrow \bigvee_{i \neq j} p_i = p_j \vee [(\exists K_1)(\exists K_2)\, K_1 \neq K_2 \wedge (p_1, p_2 \mathrm{I} K_1, K_2)$$
$$\bigwedge_{j=1}^{2} ((\forall x \mathrm{I} K_j)(\exists L_j) x \neq p_3 \wedge (x = p_1 \vee (p_2, p_3, x \mathrm{I} L_j))].$$

It states that three different points p_1, p_2, p_3 are collinear if and only if there exist two different circles K_1 and K_2 through p_1 and p_2, not passing through p_3, such that every point x, different from p_1 and p_2, on one of these two circles is not on the line p_2p_3 (where the noncollinearity of three points is again expressed by the existence of a circle through them). If p_1, p_2, p_3 were not collinear, then the line p_2p_3 could be tangent to at most one of the two circles K_1 and K_2, and thus would have to intersect at least one of the circles in a point x that is different from p_1. No circle would pass through p_2, p_3, x.

Since we have used the negation of equality among circles in this definition, we need to show that $K_1 \neq K_2$ is b.u.e-definable. The definition is:

$$K_1 \neq K_2 \leftrightarrow (\exists a_1 a_2)(\forall x_1 \mathrm{I} K_1)(\forall x_2 \mathrm{I} K_2) \bigvee_{i=1}^{2} (x_1 = a_i \wedge x_2 = a_i) \vee x_1 \neq x_2.$$

The model theoretic counterpart of the theorem we proved syntactically is:

Theorem 6. *Let $\mathfrak{M}$, $\mathfrak{N}$ be Euclidean spaces of dimension 2, and $f : \mathfrak{M} \rightarrow \mathfrak{N}$ be an injection (i.e. it preserves $\neq$ between points) that preserves incidence,* I, *and such that $f_{|\mathbf{K}}$ is surjective for all circles* **K**. *Then f must preserve L and $\perp$ as well.*

The last theorem, which we state in its most general form, and whose logical counterpart we express is:

Theorem 7 (Carathéodory [5]; Aczél, McKiernan [1]). *Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a one-to-one map that maps circles (real circles or lines) onto circles (real circles or lines). Then f is either a Möbius transformation or a conjugate Möbius transformation.*

The elementary content, i.e. the purely geometric content, that does not depend on topological properties or the Archimedeanity of $\mathbb{R}$, of this theorem amounts to:

Theorem 8. *Let $\mathfrak{M}, \mathfrak{N}$ be models of Miquelian Möbius geometry with fields of characteristic $\neq 2$, $f : \mathfrak{M} \to \mathfrak{N}$ be a mapping which is one-to-one on both points and circles, and which preserves incidence. Then f preserves circle-orthogonality as well.*

This theorem is a generalization of Theorem 7 in a different sense as well: we no longer require that the map f be onto on circles, but just that it map different circles into different circles.

An axiom system for Miquelian Möbius planes in terms of point-circle incidence can be found in [3, p. 205f.], and the van der Waerden and Smid representation theorem is proved for them in [3, Satz III.2.1]. It is shown in [3, Satz III.6.3] that for Miquelian Möbius planes of characteristic $\neq 2$ there is a unique orthogonality relation satisfying three natural orthogonality axioms, (OI), (OII), (OIII). In logical terms, this amounts to the implicit definability of circle-orthogonality (which we denote by $\perp_c$) in models of the theory of Miquelian Möbius planes of characteristic $\neq 2$. Let $\mathcal{M}$ denote the theory of Miquelian Möbius planes of characteristic $\neq 2$ with the orthogonality axioms (OI), (OII), (OIII). By the Beth definability theorem ([4], [8, Theorem 6.6.4]) implicit definabilty of $\perp_c$ is equivalent with the explicit definability of $\perp_c$ in terms of I, the definition being valid in $\mathcal{M}$. The syntactic counterpart of Theorem 8 is a stronger result than plain explicit definability of $\perp_c$, for it states the existence of a definition of $\perp_c$ by a positive existential sentence in terms of point-circle incidence, in which negated equality is allowed for both points and circles. The definition that proves Theorem 8 is (see Fig. 2):

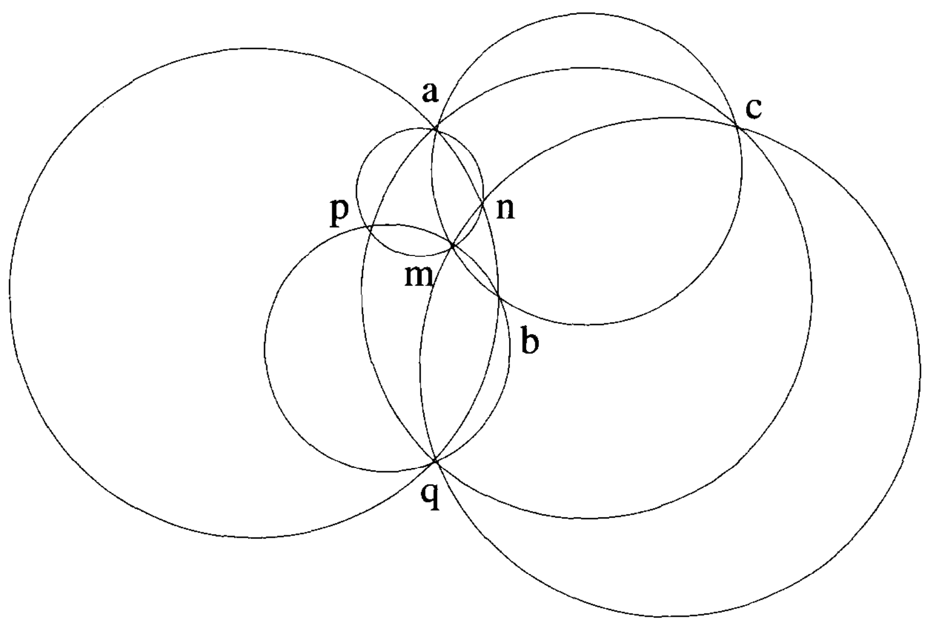

Figure 2: Definition of $K_1 \perp_c K_2$

$$
(8) \quad
\begin{aligned}
K_1 \perp_c K_2 :\leftrightarrow\ & (\exists a)(\exists b)(\exists c)(\exists m)(\exists n)(\exists p)(\exists q)(\exists U)(\exists V)(\exists X)(\exists Y)\, K_1 \neq K_2 \\
& \wedge \neq (abcmnpq) \wedge \neq (UVXY) \wedge (q\mathrm{I}K_1, K_2, U, V) \\
& \wedge (a, b, n\mathrm{I}K_1) \wedge (a, c, p\mathrm{I}K_2) \wedge (c, m, n\mathrm{I}U) \wedge (b, m, p\mathrm{I}V) \\
& \wedge (a, b, c, m\mathrm{I}X) \wedge (a, m, n, p\mathrm{I}Y).
\end{aligned}
$$

To see that this sentence holds in $\mathcal{M}$, notice that it is a rephrasing of (5) in the context of Möbius geometry. To prove $\rightarrow$, notice that if $K_1 \perp_c K_2$, then, by (OII), K_1 and K_2 have exactly 2 points in common, say a and q. There is a Möbius transformation that maps q into ∞. With $q = \infty$, K_1 and K_2 become lines, if we ignore the point ∞ and consider them as lying in the Euclidean plane from which the Möbius plane was obtained by adjoining ∞, and (8) becomes (5), which is valid, and since Möbius transformations preserve all the notions involved, so is (8). To prove $\leftarrow$ we apply again a Möbius transformation mapping q to ∞, and argue analogously.

So far, the only explicit definition of circle perpendicularity that I am aware of in the literature goes back to [6, p. 465], where it is shown to be valid in a Möbius geometry provided with a richer structure:

$$
\begin{aligned}
K_1 \perp_c K_2 :\leftrightarrow\ & K_1 \neq K_2 \wedge (\exists a)(\exists b)(\exists c)(\exists U)(\exists V)\, a \neq b \wedge (a, b\mathrm{I}K_1, K_2) \\
& \wedge c\mathrm{I}K_1 \wedge \tau(a, U, K_2) \wedge \tau(b, V, K_2) \wedge \tau(c, U, V)
\end{aligned}
$$

where $\tau(a, K, L)$ stands for 'a is the point of tangency of the circles K and L', which requires bounded universal quantification when expressed in terms of I, such as in

$$
\tau(a, K, L) :\leftrightarrow (a\mathrm{I}K, L) \wedge (\forall x\mathrm{I}K)(\forall y\mathrm{I}L) x \neq y \vee x = a.
$$

Notice that we have used Theorem 1 to provide the syntactic equivalent for the theorems above, although we have not assumed that the corresponding languages contain a sign for an identically false formula. That one does not need such an assumption follows from the fact that $(\exists a)\, aa \perp aa$, $(\exists a)\, C(aaaa)$, and $(\exists x)\, x \neq x$, may serve as such identically false formulas for the theories dealt with in Theorems 2, 4 (i) or (i)$'$, and 4 (ii) or (ii)$'$ or (iii) or 6 or 8 respectively.

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(Received December 1999)