ON THE EUCLIDEAN GEOMETRY OF THE DOUBLE-EDGED RULER

VICTOR PAMBuccIAN

The aim of this note is to show that plane Euclidean geometry can be axiomatised by quantifier-free axioms in languages containing operations that correspond to the geometric constructions that can be performed by a double-edged ruler.

The double-edged ruler is an instrument which can be used in the following ways: (i) as a ruler (to draw the line joining two points, as well as to construct the point of intersection of two intersecting lines); (ii) to draw the two parallels to a given line that lie at unit distance from that line; (iii) to draw two parallel lines at unit distance apart, one of which passes through A, the other of which passes through B, whenever the distance between the points A and B, \( d(A, B) \), is \( \geq 1 \) (there are two such pairs of parallels whenever \( d(A, B) > 1 \)).

The subject of quantifier-free (universal) axiomatisations of elementary geometry goes back to [4], where plane Euclidean geometry of ruler and segment-transporter constructions was axiomatised by universal axioms in a language containing two quaternary operation symbols and three individual constants, with 'points' as individual variables. It was further expanded in [2], [11], and [6][10].

The possibility of such axiomatisations is suggested by the well known fact (going back to Adler [1], see also [3]) that the set of points that can be constructed in the standard Euclidean plane by using a double-edged ruler in all three possible ways coincides with the set of points that can be constructed with ruler and compass. However, this result does not indicate if and how a quantifier-free axiom system may be set up, for results on the constructibility of points always assume the underlying geometry as 'given', whereas in a quantifier-free axiomatisation the geometry itself needs to be determined by axioms that refer to geometric constructions.

1 The axiomatisations

We first consider a language for constructions with a restricted double-edged ruler, which can be used only in ways (i) and (ii). Let \( \mathcal{S} \) be a bi-sorted first order language \( \mathcal{S} \), whose two sorts of variables will be referred to as 'points' (upper case) and 'lines' (lower case), that contains: three individual constants \( A_0, A_1, A_2 \), standing for three non-collinear points; \( \wp \), a binary operation symbol with points (say \( A \) and \( B \)) as arguments and the line joining them, \( \wp(A, B) \), as value (whenever \( A \neq B \), an arbitrary line, otherwise); \( t \), a binary operation symbol with lines (say \( g \) and \( h \)) as arguments and the intersection point of the lines \( g \) and \( h \), \( t(g, h) \), as value (whenever \( g \) and \( h \) are not parallel and not equal, an arbitrary point, otherwise); \( \alpha_i \) (for \( i = 1,2 \)), a unary operation symbol with a line (say \( l \)) as variable and the line \( \alpha_i(l) \), such that \( \{\alpha_1(l), \alpha_2(l)\} \) are the two lines that are parallel to \( l \) at unit distance from \( l \), as value.\(^2\)

\(^1\)Partially supported by a TRIS grant from ASU West.
\(^2\) For logical issues related to many-sorted languages see [5].
In [9] we have axiomatised Euclidean planes of characteristic \( \neq 2 \) in a bi-sorted first-order language \( L_1 \) that contains \( A_0, A_1, A_2, \varphi, \gamma \) as non-logical constants, with \( \gamma(P, l) \) standing for the line perpendicular to \( l \) in \( P \) whenever \( l = \varphi(P, Q) \) for some \( Q \neq P \), arbitrary, otherwise. We shall prove that an operation equivalent to \( \gamma \) can be defined without using quantifiers by means of the primitive notions in \( S \).

We first introduce some abbreviations that will improve the readability of the axioms:

\[
\begin{align*}
\lambda(A, B, C) & : \leftrightarrow A = B \lor A = C \lor \varphi(A, B) = \varphi(A, C), \\
\bar{\lambda}(A, B, C) & : \leftrightarrow A \neq B \land B \neq C \land C \neq A \land \varphi(A, B) = \varphi(A, C),
\end{align*}
\]

and, for \( A, B, P \) with \( \overline{\lambda}(A, B, P) \) and \( i \in \{1, 2\} \) (see Figures 1 and 2):

\[
\begin{align*}
\mu_i(A, B, P) & := \psi(\varphi(A, B), \varphi(P, \psi(\varphi(A, B), \varphi(A, P)))) \\
\delta_i(A, B, P) & := \psi(\varphi(A, B), \varphi(\alpha_i(\varphi(A, B), \varphi(A, P)))) \\
\varepsilon(A, B, P) & := \psi(\varphi(A, B), \delta_1(A, B, P), \alpha_2(\varphi(A, B))) \\
\omega(A, B, P) & := \psi(\varphi(A, B), \varepsilon(A, B, P), \alpha_1(\varphi(A, B))) \\
\sigma(A, B, P) & := \psi(\varphi(A, B), \omega(A, B, P), \delta_2(A, B, P)).
\end{align*}
\]

The abbreviation \( \lambda(A, B, C) \) may be read as ‘\( A, B, C \) are collinear points’, \( \bar{\lambda}(A, B, C) \) as ‘\( A, B, C \) are three different collinear points’, \( \mu_i(A, B, P) \) stands for the midpoint of \( AB \), and \( \sigma(A, B, P) \) stands for the reflection of \( B \) in \( A \) (i.e., \( A \) is the midpoint of \( BS\sigma(A, B, P) \)). We further define (see Fig. 3):

\[
\begin{align*}
\rho(A, B, P) & := \varphi(P, \sigma(\mu_1(P, B, A), A, B)) \\
\zeta(A, B, P) & := \sigma(\varphi(A, B), A_1(\varphi(A, P))), A, P) \\
\tau(A, B, P) & := \psi(\varphi(A, B), A_1(\varphi(A, P)), A, P)) \\
\nu(A, B, P) & := \psi(\nu_1(\varphi(A, B), A_1(\varphi(A, B), A))), A) \\
\psi(A, B, P) & := \psi(\varphi(A, B), A_1(\varphi(\psi(A, B), A))) \\
\kappa(A, B, P) & := \psi(\zeta(A, B, P), \tau(A, B, P), \varphi(A, \nu(A, B, P)))
\end{align*}
\]

and have that \( \rho(A, B, P) \) is the parallel through \( P \) to \( AB \) in case \( \overline{\lambda}(A, B, P) \), and that \( \psi \) is perpendicular to \( AB \).

With these abbreviations we are now ready to define our main abbreviation, which is

\[
\Gamma(A, \varphi(A, B), P) := \rho(\kappa(A, B, P), \psi(A, B, P), A)
\]

and which may be read as ‘the line perpendicular in \( A \) to \( \overline{\varphi}(\varphi(A, B), P) \).

We have provided in [9] a quantifier-free axiom system for Euclidean planes of characteristic \( \neq 2 \) in the language \( L_1 \) containing \( A_0, A_1, A_2, \varphi, \gamma \) as primitive notions, where \( \gamma(P, l) \) denotes the perpendicular in \( P \) on \( l \) whenever \( l = \varphi(P, Q) \) with \( P \neq Q \), an arbitrary line, otherwise. Having defined \( \Gamma(A, \varphi(A, B), P) \), which is a variant of \( \gamma(A, \varphi(A, B)) \) for although the former contains an extra variable as argument, the values of the two are the same, we can now closely follow the axiom system presented in [9] to obtain the underlying Euclidean plane structure of our geometry. Adding to the axiom system thus obtained an axiom stating that the distance between \( \varphi \) and \( \alpha_1(l) \) is constant, we obtain a complete description of the operations in \( S \) and an axiom system for Euclidean planes with a unit length, in which all angles are biseable (or, in other words, which have free mobility).

We first introduce some additional abbreviations, for line intersection (\( \chi \)), line parallelity (\( || \)), line parallelity or equality (\( \|| \)), line perpendicularity (\( \perp \)), segment congruence (Pieri’s relation \( \eta \)), \( \eta \), and \( \theta \):

\[
\begin{align*}
\chi(\varphi(A, B), \varphi(C, D)) & : \leftrightarrow \lambda(A, B, \psi(\varphi(A, B), \varphi(C, D))) \land \lambda(C, D, \psi(\varphi(A, B), \varphi(C, D)))) \\
\varphi(X, Y) \parallel \varphi(U, V) & : \leftrightarrow \varphi(X, Y) \neq \varphi(U, V) \land \neg \chi(\varphi(X, Y), \varphi(U, V)),
\end{align*}
\]
\( \varphi(X, Y) \parallel (\varphi(U, V) :\iff \varphi(X, Y) = \varphi(U, V) \lor (X, Y) \parallel \varphi(U, V). \)

For \( A, B, P \) with \( -\lambda (A, B, P) \) and \( C \neq D \) we define \( \vdash \) by \(^3\) \( (\varphi(A, B), \varphi(C, D), P) :\iff \varphi(A, B) \neq \varphi(C, D) \land \chi(\varphi(A, B), \varphi(C, D)) \land \Gamma(\varphi(A, B), \varphi(C, D), \varphi(A, B), P) = \varphi(C, D). \)

For points \( A, B, C, P \) with \( A, B, P \) not collinear, with \( A \neq B \) and \( A \neq C \), we define the relation \( \pi(A, B, C, P) \) to be read as "segment \( AB \) is congruent to segment \( AC \)" by:

\[ \pi(A, B, C, P) :\iff \sigma(A, B, P) = C \]

\[ \forall \lnot \lambda (A, B, C) \land \varphi(\mu_1 (B, C, A), \varphi(\mu_1 (B, C, A), C), A). \]

We next introduce \( \eta \) by

\[ \eta(g, A, B) \iff g = \varphi(A, B) \land A \neq B, \]

to be read as "\( g \) is the line joining the different points \( A \) and \( B \)’, and \( \Theta_i(A, B, P) \) for \( i \in \{1, 2\} \) by

\[ \Theta_i(A, B, P) :\iff \Gamma(\varphi(A, B), P), \alpha_i(\varphi(A, B)). \]

We are now ready to state the axioms.\(^4\)

A 1 \( \varphi(A, B) = \varphi(B, A), \)

A 2 \( A \neq B \land B \neq C \land D \neq B \land \varphi(A, B) = \varphi(C, D) \rightarrow \varphi(D, B) = \varphi(B, C), \)

A 3 \( \forall_{0 \leq i, j \leq 2, i \neq j, i \neq j + 1} (\eta(g, \tau(a_{i, j}, g), \tau(a_{i, j+1}, g)) \lor \eta(g, A_i, \tau(\alpha_i(A_i, A_{i+1}), g)), \)

A 4 \( \lnot \lambda (A_0, A_1, A_2), \)

A 5 \( \lnot \lambda (A, B, P) \rightarrow \varphi(A, B), \parallel \rho(A, B, P) \land P \neq \sigma(\mu_1 (P, B, A), A, B), \)

A 6 \( \lnot \lambda (A, B, X) \rightarrow \tau(\varphi(A, X), \varphi(B, X)) = X, \)

A 7 \( C \neq D \land \varphi(A, B) \parallel \varphi(C, D) \land \lnot \lambda (A, B, X) \rightarrow \chi(\varphi(A, X), \varphi(C, D)), \)

A 8 \( C \neq D \land C \neq D' \land \lnot \lambda (A, B, P) \land \lnot \lambda (A', B', P') \land \vdash (\varphi(A, B), \varphi(C, D), P) \land \varphi(A, B) \parallel \varphi(A', B', P') \land \varphi(C, D') \parallel \varphi(C', D'), (\varphi(A', B'), \varphi(C', D'), P'), \)

A 9 \( \lnot \lambda (P_1, P_2, P_3) \land \lnot \lambda (P_1, P_2, B) \land B \neq P_3 \vdash (\varphi(P_1, P_2, B), \varphi(P_1, B), P_1) \vdash (\varphi(P_1, P_3), \varphi(P_2, B), P_2) \rightarrow \vdash (\varphi(P_1, P_2), P_1, \varphi(P_2, B), P_3), \)

A 10 \( \lnot \lambda (A, B, P) \rightarrow \Gamma(A, \varphi(A, B), P) \neq \varphi(A, B), \)

as well as the minor Desargues axiom (des) and Fano’s axiom in the following form:

A 11 \( \lnot \lambda (A, B, C) \land \varphi(A, B) \parallel \varphi(C, D) \land \varphi(B, C) \parallel \varphi(A, D) \land \chi(\varphi(A), \varphi(B), \varphi(C), \varphi(D)) \land \mu_1 (A, C, B) = \tau(\varphi(A, C), \varphi(B, D)). \)

It follows from [9, p. 12-14] that all models of \( \Sigma = \{A1 \rightarrow A11, des\} \) are Euclidean planes associated to a quadratic field extension where the field characteristic is \( \neq 2 \), and that \( \varphi, \tau, \Gamma, \mu_1 \) have the desired interpretation therein. To obtain the desired interpretation for \( \sigma, \alpha_1 \) and \( \alpha_2 \) we add the following axioms \( i \in \{1, 2\}: \)

\(^3\)In this definition, by \( \Gamma(\tau(\varphi(A, B), \varphi(C, D)), \varphi(A, B), P) \) we mean \( \Gamma(\tau(\varphi(A, B), \varphi(C, D)), \varphi(\tau(\varphi(A, B), \varphi(C, D), X), P), \)

\( \)where \( X \) is \( A \) or \( B \), whichever is \( \neq \tau(\varphi(A, B), \varphi(C, D)). \)

\(^4\)Addition in the indices of the \( A_i \) is mod 3.
A12 \( \lambda(A, B, P) \rightarrow \lambda(A, B, \sigma(A, B, P)) \land \mu_1(B, \sigma(A, B, P), P) = A \),

A13 \( A \neq B \rightarrow \phi(A, B) \parallel \alpha_\epsilon(\phi(A, B)) \),

A14 \( \lambda(A, B, P) \rightarrow \mu_1(\theta_1(A, B, P), \theta_2(A, B, P), B) = A \).

A15 \( \lambda(A, B, A', B') \land \phi(A, B) \neq \phi(A', B') \rightarrow \theta_1(A, B, A') = \mu_1(A, A', B) \land \mu_1(A', B', A) = \mu_1(A, A', B) \land \mu_1(A, A', B) \land (\pi(A, \theta_1(A', B', A), \sigma(\mu_1(A, A', B), \theta_1(A, B, A', B), B) \land \lambda(A, A', \theta_1(A, B, A')) \lor (\pi(A, \theta_1(A', B', A), \sigma(\mu_1(A, A', B), \theta_1(A, B, A', B), B) \land \lambda(A, A', \theta_1(A, B, A')))).\)

From axiom A12 it follows that \( \sigma \) has the desired interpretation, A13 and A14 imply that for every line \( l \) \( \alpha_\epsilon(l) \) and \( \alpha_\zeta(l) \) are two parallel lines equidistant from \( l \), and A15 ensures that for different lines \( l \) and \( l' \), the distances between \( \alpha_\epsilon(l) \) and \( l \), and \( \alpha_\zeta(l') \) and \( l' \) are the same. It follows that

**Representative Theorem 1.** \( \mathcal{M} \) is a model of \( \Sigma \cup \{A12 - A15\} \) if and only if \( \mathcal{M} \) is isomorphic to a Euclidean plane associated to a quadratic field extension with characteristic \( \neq 2 \) in which all angles are bisectable, and in which the primitive notions of \( S \) have the desired interpretation.

We can obtain ordered Euclidean geometry with free mobility and with a unit distance by adding a ternary relation symbol \( \zeta \) for the betweenness relation to \( S \) and the following axioms:

A16 \( \zeta(A, B, C) \rightarrow \lambda(A, B, C) \)

A17 \( \lambda(A, B, C) \rightarrow \zeta(A, B, C) \lor \zeta(B, C, A) \lor \zeta(C, A, B) \),

A18 \( \zeta(A, B, C) \rightarrow \zeta(C, B, A) \),

A19 \( \lambda(A, B, P) \rightarrow \zeta(A, \mu_1(A, B, P), B) \),

A20 \( (\lambda(P_1, P_2, P_3) \lor \lambda(P_n, M, N)) \land \zeta(P_1, M, P_2) \rightarrow \sqrt{\lambda(P, \chi(M, N), P, \chi(P, M, N), \phi(P, P_3)) \land \zeta(P, \chi(M, N), P, \chi(P, P_3), P_3)).\)

To obtain the ordered Euclidean geometry of ruler and compass constructions, i.e. Cartesian planes over ordered Euclidean fields, we extend the language of \( S \cup \{\zeta\} \) by another pair of binary operation symbols, standing for the operation (iii) of the double edged ruler, \( \beta_1 \) and \( \beta_2 \). This language will be denoted by \( S' \). For points \( A \) and \( B \) with \( d(A, B) \geq 1 \), \( (\beta_1(A, B), \beta_2(A, B)) \) may be read as 'the pair of lines (different if \( d(A, B) > 1 \) that pass through \( A \) such that \( \alpha_\epsilon(\beta_1(A, B)) \) and \( \alpha_\zeta(\beta_2(A, B)) \) pass through \( B \) for some \( i, j \in \{1, 2\} \)'. For points \( A \) and \( B \) that are less than 1 apart, \( \beta_1(A, B) \) is an arbitrary line. The axioms that ensure that the \( \beta \) have precisely the above meaning are (here \( i, k \) take values in \( \{1, 2\} \), so there are four axioms A211;\( i, k \) and two axioms A22);

A211;\( i, k \) \( \lambda(A, B, P) \land \lambda_\zeta(A, \phi(A, B), \alpha_\zeta(\Gamma(A, \phi(A, B), P)), B) \rightarrow \eta(\beta_k(A, B), \alpha_1(\alpha_1(A, B), \beta_k(A, B)) \land (\sqrt{\eta_A^2 \eta_1(\alpha_1(A, B), \beta_k(A, B)) \land \eta_1(\alpha_1(A, B), \alpha_1(A, B), \alpha_1(\beta_k(A, B)))), \)

\( \rightarrow \eta(\beta_k(A, B), \alpha_1(\alpha_1(A, B), \beta_k(A, B)) \land (\sqrt{\eta_A^2 \eta_1(\alpha_1(A, B), \beta_k(A, B)) \land \eta_1(\alpha_1(A, B), \alpha_1(A, B), \alpha_1(\beta_k(A, B)))), \)
A 22 \rightarrow \lambda(A, B, P) \wedge \exists \alpha(T(A, \varphi(A, B)), B) \\
\rightarrow \beta_1(A, B) \neq \beta_2(A, B), \text{ where } \xi(X, Y, Z) : \leftrightarrow \xi(X, Y, Z) \lor Y = Z. \text{ We now have the following}

**Representation Theorem 2**. \( \mathcal{M} \) is a model of \( \Sigma \cup \{A12 - A22\} \) if and only if \( \mathcal{M} \) is isomorphic to a Cartesian plane over a Euclidean ordered field, in which the primitive notions of \( S' \) have the desired interpretation.

---

**Figure 1**

---

**Figure 2**

---

**Figure 3**
References


Received February 5, 1999
Department of Integrative Studies,
Arizona State University West,
Phoenix, AZ 85069-7100, U.S.A.
e-mail: pamb@math.west.asu.edu