VICTOR PAMBUCCIAN

The Simplest Axiom System for Plane Hyperbolic Geometry*

Abstract. We provide a quantifier-free axiom system for plane hyperbolic geometry in a language containing only absolute geometrically meaningful ternary operations (in the sense that they have the same interpretation in Euclidean geometry as well). Each axiom contains at most 4 variables. It is known that there is no axiom system for plane hyperbolic consisting of only prenex 3-variable axioms. Changing one of the axioms, one obtains an axiom system for plane Euclidean geometry, expressed in the same language, all of whose axioms are also at most 4-variable universal sentences. We also provide an axiom system for plane hyperbolic geometry in Tarski's language $\mathsf{L}_{B\equiv}$ which might be the simplest possible one in that language.

Keywords: Hyperbolic geometry, constructive axiomatization, Euclidean geometry, simplicity.

1. Introduction

Elementary hyperbolic geometry was born in 1903 when Hilbert [11] provided, using the end-calculus to introduce coordinates, a first-order axiomatization for it by adding to the axioms for plane absolute geometry (groups I,II,III) a hyperbolic parallel axiom stating that through any point P not lying on a line l there are two rays r_1 and r_2, not belonging to the same line, which do not intersect l, and such that every ray through P, which is contained in the angle formed by r_1 and r_2, does intersect l. The details of the coordinatization were worked out later by J. C. H. Gerretsen [6]. [7], P. Szász [33], [34] (cf. also [10, Ch. 7, §41-43]), and different coordinatizations were proposed by W. Szmielew [36] and Doraczyńska [3] (cf. also [30, II.2], and for a constructive axiomatization using operations for hyperbolic parallels [24]). In this paper we ask and answer the question regarding the simplest axiomatization of elementary hyperbolic geometry. From the many possible ways to look at simplicity (cf. [16]) we choose the syntactic criterion which declares that axiom system to be simplest for which the maximum number of variables which occur in any of its axioms, written in

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prenex form, is minimal. This maximal number of variables will of course depend on the language in which one chooses to express the axiom system. We shall first express the axiom system in Tarski's one-sorted first-order language $\mathsf{L} = \mathsf{L}_{B\equiv}$, with individual variables to be interpreted as *points*, and with two relation symbols, a ternary one, B, and a quaternary one $\equiv$, with $B(abc)$ to be read as 'point b lies between a and c ', and $ab \equiv cd$ to be read as 'segment ab is congruent to segment cd'. This axiom system consists of at most 5-variable sentences, with a single exception, an axiom which is a 6-variable statement. One can show that hyperbolic geometry expressed in L cannot be axiomatized by means of 4-variable prenex statements alone. It remains open whether hyperbolic geometry can be axiomatized in L by means of 5-variable prenex sentences alone.

Based on this L-axiom system we provide an axiom system in a one-sorted first-order language, with individuals standing for points, without any relation symbol, but containing several ternary operation symbols, all of whose axioms, with one exception, a purely existential 2-variable axiom, are universal at most 4-variable sentences. There is, by a theorem of D. Scott [31] for axiom systems for Euclidean geometry which is valid in the hyperbolic case as well, no axiom system with individuals to be interpreted as points, in a language without individual constants in which the set of primitive notions is invariant under isometries, for plane hyperbolic geometry, consisting of at most 3-variable sentences, since all the at most 3-variable sentences which hold in plane hyperbolic geometry hold in all higher-dimensional hyperbolic spaces as well.

Changing one axiom in the axiom system for plane hyperbolic geometry, we obtain an axiom system for plane Euclidean geometry over Euclidean ordered fields, which is also minimal as far as the number of variables occuring in its axioms is concerned.

The axiom system consists of several axioms which were first proposed as axioms for absolute geometry by Rigby [28], [29], who improved upon axioms proposed by Forder [5] (see also [35]).

It is also worth noting that hyperbolic geometry *can* be axiomatized by using the single binary operation of point reflection σ, but cannot be axiomatized in a quantifier-free manner by means of only binary operations. In this sense our axiomatization, which uses only at most ternary operations, is also the simplest possible. For axiomatizations of various geometries by means of universal axioms in languages without relation symbols, a programme started in [15], see [25] and the bibliography therein. We have shown earlier ([19], [20], [22]) that plane Euclidean geometry over Euclidean ordered fields can be axiomatized by an L-axiom system all of whose axioms are at most

5-variable prenex sentences, and that this is best possible, as well as that it can be axiomatized by universal at most 4-variable sentences and a purely existential 3-variable sentence in a language containing only ternary operation symbols. One may conclude that from this point of view hyperbolic geometry is no more complex than Euclidean geometry.

A somewhat related result was obtained in [23], where it was shown that plane hyperbolic geometry and Euclidean geometry over Pythagorean ordered fields can be axiomatized by quantifier-free axioms in a common language containing only operation symbols. However, the aim of that paper was different, as the axiom systems display no variable or arity of the operations simplicity, and the goal was to provide an axiom system for plane absolute ruler and segment transporter geometry, from which to move on to either Euclidean or hyperbolic by adding a new operation and corresponding axioms.

In the present paper, our axiom systems do not contain subsystems for the plane absolute geometry of ruler and segment transporter.

2. The $\mathsf{L}_{B\equiv}$-Axiom System

To improve the readability of the axioms, we shall use the defined notion (which is to be read as an abbreviation) of collinearity, L, defined by

$$L(abc) :\leftrightarrow B(abc) \vee B(bca) \vee B(cab). \tag{1}$$

We shall omit quantifiers in the case of universal sentences.

A 1. $ab \equiv cd \rightarrow cd \equiv ab$,

A 2. $ab \equiv cd \rightarrow ab \equiv dc$,

A 3. $ab \equiv aa \rightarrow a = b$,

A 4. (i) $ab \equiv cd \wedge cd \equiv ce \rightarrow ab \equiv ce$,
(ii) $ab \equiv cd \wedge cd \equiv ae \rightarrow ab \equiv ae$,

A 5. $B(abc) \rightarrow B(cba)$,

A 6. $B(abd) \wedge B(bcd) \rightarrow B(abc)$,

A 7. $a \neq b \wedge ((B(abc) \wedge B(abd)) \vee (B(abc) \wedge B(dab)) \vee (B(bca) \wedge B(bda))) \rightarrow L(acd)$,

A 8. $p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \rightarrow L(abc)$, A 9. $a \neq b \wedge ac \equiv ad \wedge bc \equiv bd \wedge B(abe) \rightarrow ec \equiv ed$,

A 10. $B(abc) \wedge (B(ade) \vee B(aed)) \wedge ab \equiv ad \wedge ac \equiv ae \rightarrow B(ade) \wedge bc \equiv de$,

A 11. $B(abc) \wedge B(dbe) \wedge ba \equiv bd \wedge bc \equiv be \rightarrow ac \equiv de$,

A 12. $ab \equiv ad \wedge ((B(abc) \wedge B(ade)) \vee (B(cab) \wedge B(ead)) \wedge ac \equiv ae \rightarrow dc \equiv be$,

A 13. $c \neq a \wedge B(cad) \wedge ab \equiv ad \wedge (B(ced) \vee B(cde)) \wedge cb \equiv ce \rightarrow B(ced)$,

A 14. $(\forall ab)(\exists c)\,[B(acb) \wedge ca \equiv cb]$,

A 15. $(\forall abc)(\exists d)[B(cad) \wedge ab \equiv ad]$,

A 16. $(\forall abc)(\exists d)\,[\neg L(abc) \rightarrow (B(adc) \vee B(bdc)) \wedge da \equiv db]$,

A 17. (i) $(\forall abcd)(\exists e)\,[B(bad) \wedge ab \equiv ad \wedge B(bcd) \rightarrow ae \equiv ad \wedge be \equiv bc]$,
(ii) $(\forall abc)(\exists d)\,[B(abc) \rightarrow ac \equiv ad \wedge ad \equiv bd]$,

A 18. $(\forall abc)(\exists d)(\forall x)\,[B(abc) \wedge b \neq a \wedge b \neq c \rightarrow ((B(abx) \wedge ba \equiv bx) \rightarrow da \equiv dx) \wedge ((B(adx) \wedge da \equiv dx) \rightarrow ca \equiv cx)]$,

A 19. $(\forall abc)(\exists de)\,[\neg L(abc) \rightarrow (B(acd) \vee B(adc)) \wedge de \equiv ab \wedge ad \equiv ae \wedge bd \equiv be]$,

A 20. $\neg L(xyz) \wedge B(xay) \wedge ax \equiv ay \wedge B(ybz) \wedge by \equiv bz \wedge B(zcx) \wedge cz \equiv cx \rightarrow \neg L(abc)$,

A 21. $(\exists ab)\,[a \neq b]$,

A 22. $(\forall abcd)(\exists e)\,[a = b \vee b = c \vee c = d \vee d = a \vee (B(abe) \wedge ba \equiv be \wedge \neg ca \equiv ce) \vee (B(bce) \wedge cb \equiv ce \wedge \neg db \equiv de) \vee (B(cde) \wedge dc \equiv de \wedge \neg ac \equiv ae) \vee (B(dae) \wedge ad \equiv ae \wedge \neg bd \equiv be)]$.

A8 is an upper-dimension axiom, stating that, if three points are equidistant from two fixed points, then they lie on the same line; A9 states that if two congruent trilaterals share a common side, then the segments joining the corresponding third vertices with a point on the extension of the common side are congruent as well; A10 is a special variant of Euclid's Common Notion 3, "If equals are subtracted from equals, then the remainders are equal", and A11 of Euclid's Common Notion 2, "If equals are added to equals, then the wholes are equal"; A12 is a form of the side-angle-side congruence axiom for two triangles which share an angle, or whose congruent angles are vertical angles; A13 is an axiom stating the triangle inequality; A14 states the existence of the midpoint of a segment; A15 is a segment transport axiom; A16 states that the perpendicular bisector of a side of a triangle intersects

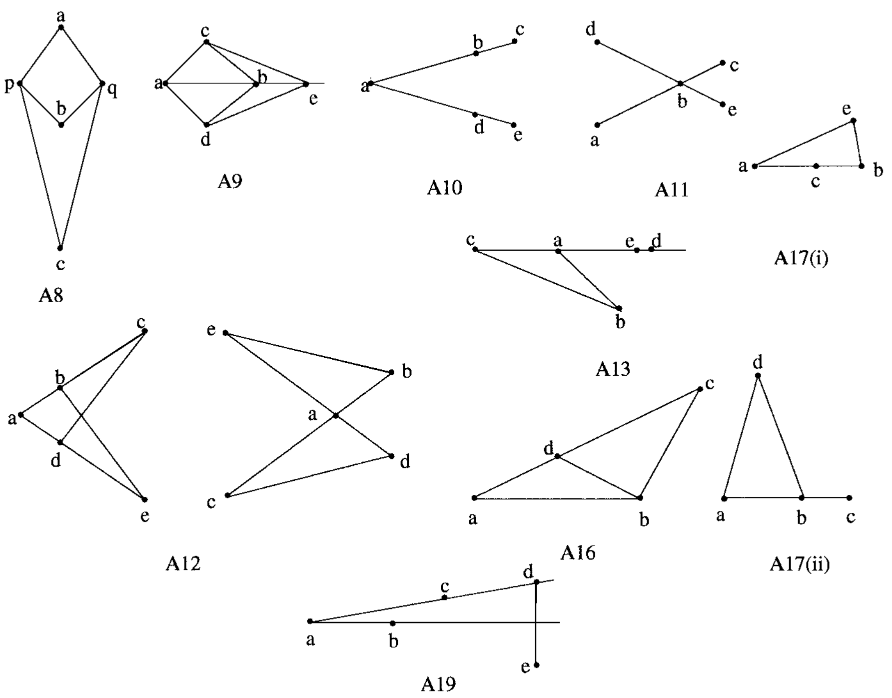

Figure 1. Axioms A8, A9, A10, A11, A12, A13, A16, A17, and A19

one of the remaining sides as well; A17(i) states that two circles, the radius of one being precisely the distance between the two centres, and the radius of the other being less than or equal to twice the distance between the two centres, intersect; A17(ii) that two circles with equal radii, greater than or equal to the distance between the centres of the two circles, intersect; A18 states the existence of a right triangle with a given hypotenuse and with given footpoint of the altitude to the hypotenuse; A19 is a sharpened version of Aristotle's axiom, which states that, given an angle, one can find on one of its sides a point for which the segment formed by itself and its reflection in the other side of the angle is congruent to a given segment; A20, an axiom first considered by Hjelmslev [12, p. 474], states that the midpoints of the sides of a triangle cannot be collinear, and A22 states that the metric is non-Euclidean by stating that there exists no rectangle.

The importance of Aristotle's axiom in the foundations of hyperbolic geometry has been pointed out in [8].

Let $\Sigma := \{A1 - A22\}$. We shall designate the midpoint c of ab given by A14 by $M(ab)$. Axiom A14 only states the existence of a midpoint, not its uniqueness. Its uniqueness can be proved (see (9) below).

3. Σ is an axiom system for plane hyperbolic geometry

The proof that follows consists of several results, which, given their large number, we have decided not to call lemmas, but to simply number them. The bulk of the proof consists in proving the transitivity of the congruence relation, (23), as well as proving all the other axioms of Sörensen's [32] axiom system for non-elliptic metric planes.

By A15, $(\exists b')B(abb') \wedge bb \equiv bb'$, thus $b = b'$ (by A3 and A1). We have thus shown that

$$B(abb). \tag{2}$$

By A14 $(\exists m)B(amb) \wedge ma \equiv mb$. Since, by (2), we also have $B(maa)$ and $B(mbb)$, we conclude from A10 that

$$aa \equiv bb. \tag{3}$$

Suppose $ab \equiv cc$. Given that we also have $cc \equiv aa$ (by (3)), we conclude, using A4(ii), that $ab \equiv aa$, and by using A3 that $a = b$. Thus

$$ab \equiv cc \to a = b. \tag{4}$$

By A15 with $c = b$, we get $(\exists d)\, ab \equiv ad$, and by A1 and A4(i), we deduce

$$ab \equiv ab. \tag{5}$$

From (5) and A2 we get

$$ab \equiv ba. \tag{6}$$

$$B(abc) \wedge ab \equiv ac \to b = c. \tag{7}$$

PROOF . Suppose the hypothesis (antecedent) of (7) holds. By (2) and (5) we notice that the antecedent of A10 in which $d = c$ and $e = c$ holds, so the consequent, $bc \equiv cc$ must hold as well, i. e. $b = c$ (by (4)).

We now turn to the proof of

$$B(aba) \to a = b. \tag{8}$$

By A14 $(\exists m)\, B(bma) \wedge mb \equiv ma$; from $B(aba)$ and $B(bma)$ we get $B(abm)$ by A6, thus $B(mba)$ by A5, so $a = b$ (by (7)).

For any two points a, b with $a \neq b$, all points x for which $L(abx)$ holds are said to be on the *line* ab. We want to show that any line ab contains at least 5 points. By A14 $(\exists x)\, B(axb) \wedge xa \equiv xb$, and $x \neq a$, $x \neq b$ by A3 and A1. By A15 $(\exists y)\, B(aby) \wedge ba \equiv by$, $(\exists z)\, B(baz) \wedge ab \equiv az$ and $y \neq a$, $z \neq b$

(by (8)), $y \neq b$, $z \neq a$ (by A3). By A5 we have $B(yba), B(bxa), B(zab)$, and we deduce from the first two that $B(ybx)$ (by A6), so $x \neq y$ (by (8)). From $B(zab)$ and $B(axb)$ we deduce that $B(zax)$ (by A6), thus $x \neq z$ (by (8)). If $y = z$, then from $B(aby)$ and $B(baz)$ (i. e. $B(bay)$) we deduce that $B(aba)$ (by A6), thus $a = b$ (by (8)), a contradiction. So $y \neq z$ as well, and a, b, x, y, z are five different points on the line ab. W. Szmielew has proved in [38, Th. 7.2.7] that, if there are at least 5 points on every 'line', then a relation B that satisfies A5, A6, A7, and (8) satisfies all universal properties of the order relation on a line (i. e. the universal properties of a linear order.) We shall freely use the fact that our B has these properties in subsequent proofs and refer to it by (*).

As in [28, 2.4] and [29, 3.3] we can now prove that the midpoint of a segment ab, defined as a point on line ab which is equidistant from a and b, which by (7) has to lie between a and b, is unique, i. e. that

$$a \neq b \wedge L(abm) \wedge ma \equiv mb \wedge L(abn) \wedge na \equiv nb \rightarrow m = n. \tag{9}$$

Using (*), A15, (7), A4(i), we can now prove that

$$(\forall abc)(\exists^{=1} b')\ a \neq c \rightarrow (B(ab'c) \vee B(acb')) \wedge ab \equiv ab', \tag{10}$$

as well as that the d in A15 is unique whenever $c \neq a$. For $a \neq c$ we will designate the b' in (10) by $T(abc)$ and the d in A15 by $T'(abc)$.

We also have

$$B(abc) \rightarrow (\exists b')\, B(ab'c) \wedge cb \equiv ab'. \tag{11}$$

PROOF . By A14, $(\exists m)\, B(amc) \wedge ma \equiv mc$, and by A15 $(\exists b')\, B(bmb') \wedge mb \equiv mb'$. By (*) we have $B(abm)$ or $B(amb)$, and in both cases, using (*), A1, A2, and A12 or A10, we deduce $cb \equiv ab'$, and, using A10 and (*), we get $B(ab'c)$.

A consequence of (11), (10) and A5 is

$$B(abc) \wedge (B(cb'a) \vee B(cab')) \wedge ab \equiv cb' \rightarrow B(cb'a). \tag{12}$$

We now define the notion of segment inequality for segments sharing a common endpoint by

$$ab \geq ac :\leftrightarrow (\exists c')\, B(ac'b) \wedge ac \equiv ac'. \tag{13}$$

Any two segments sharing a common endpoint are comparable, i. e.

$$ab \geq ac \vee ac \geq ab. \tag{14}$$

If $a = c$, then (14) can be seen by setting $c' = a$ in (13). Suppose $a \neq c$. Let $b' := T(abc)$. If $B(ab'c)$, then $ac \geq ab$, as seen by choosing $c' = b'$ in (13). If $B(acb')$, then we must have $a \neq b$, since $ab \equiv ab'$ and $b' \neq a$ (by (8), A3), so, by A10, we have $B(aT(acb)b)$, so $ab \geq ac$, by (13) with $c' = T(acb)$, establishing (14).

A useful expected property of the segment-inequality relation is

$$ab \geq ac \rightarrow (\exists b')\, B(acb') \wedge ab \equiv ab'. \tag{15}$$

To establish it, notice that, if $a = c$ we may choose $b' = b$ and we are done. If $a \neq c$, then, since, with $b' = T(abc)$, we have $(B(acb') \vee B(ab'c)) \wedge ab \equiv ab'$ and, by (13) $(\exists c')\, B(ac'b) \wedge ac \equiv ac'$, we get, by A1 and A10, that $B(acb')$.

From (10), A10, (8), A3, A1 we deduce

$$(\forall abcc')(\exists b')\, ac \equiv ac' \wedge B(abc) \rightarrow B(ab'c') \wedge ab \equiv ab' \wedge bc \equiv b'c'. \tag{16}$$

We now prove that the $\geq$ relation is transitive, i. e. that

$$bc \geq ba \wedge ab \geq ac \rightarrow cb \geq ca. \tag{17}$$

Proof . By (13) $(\exists p)\, B(bpc) \wedge ba \equiv bp$ and $(\exists q)\, B(aqb) \wedge ac \equiv aq$. By (16) and A5 $(\exists q')\, B(bq'p) \wedge bq \equiv bq' \wedge qa \equiv q'p$, so, by A1, A2, A4(i), $ac \equiv pq'$. By (*) $B(cpq')$. By (11) $(\exists r)\, B(crq') \wedge q'p \equiv cr$. From $ac \equiv pq'$ and $q'p \equiv cr$ we get, using A1, A2, A4(ii), $ca \equiv cr$. By (*) we have $B(crb)$ as well, so $cb \geq ca$, by (13) with $c' = r$.

Using (13), (10), A10, A4(i), and (*), we get

$$ab \geq ac \wedge ac \geq ad \rightarrow ab \geq ad. \tag{18}$$

We shall now prove that in every trilateral there is a side which is greater or equal than the other two, i. e. that

$$(ab \geq ac \wedge ba \geq bc) \vee (bc \geq ba \wedge cb \geq ca) \vee (ca \geq cb \wedge ac \geq ab). \tag{19}$$

Proof . By (14) $ab \geq ac$ or $ac \geq ab$. Suppose $ab \geq ac$. By (14) $ba \geq bc$ or $bc \geq ba$. If $ba \geq bc$ then we are done, as the first disjunct of (19) holds. If $bc \geq ba$ then, by (17), we have $cb \geq ca$, and we are done, as the second disjunct of (19) holds. Analogously for $ac \geq ab$.

We can also prove that "if equals are added to equals, the wholes are equal" (Euclid's Common Notion 2), in the following sense:

$$B(abc) \wedge B(ab'c') \wedge ab \equiv ab' \wedge bc \equiv b'c' \rightarrow ac \equiv ac'. \tag{20}$$

PROOF . By (10) $(\exists c'')\,(B(ab'c'') \vee B(ac''b')) \wedge ac \equiv ac''$. By A10 we have $B(ab'c'')$ and $bc \equiv b'c''$. By A2, A4(ii), (*) and (7) we get $c'' = c'$.

Another variant of the same Common Notion 2 we can prove is

$$B(a'ab) \wedge B(abb') \wedge aa' \equiv bb' \rightarrow ba' \equiv ab'. \tag{21}$$

PROOF . With $m = M(ba)$ we have $ma' \equiv mb'$ (by (20), bearing in mind that $B(maa')$ and $B(mbb')$ by (*)), so A11 gives $ba' \equiv ab'$.

We can also prove that the subtraction of "equals" is valid also in the following case

$$B(abc) \wedge B(ab'c') \wedge ac \equiv ac' \wedge bc \equiv b'c' \rightarrow ab \equiv ab'. \tag{22}$$

PROOF . By (10) $(\exists b'')\,(B(ab''c') \vee B(ac'b'')) \wedge ab \equiv ab''$. By A10 we have $B(ab''c')$ and $bc \equiv b''c'$. By A4(ii), A1, and A2 we have $c'b' \equiv c'b''$. By (*) and (7) we get $b'' = b'$.

From now on, we shall refer to a use of A1, A2, and A4(i) as a use of (†), and when using the first two axioms we shall write "by (**)". We now turn to the proof of a key result, the transitivity of the congruence relation, which is a 6-variable statement, namely

$$ab \equiv cd \wedge cd \equiv ef \rightarrow ab \equiv ef. \tag{23}$$

PROOF . By (19) one of
(i) $ae \geq ac$ and $ea \geq ec$, (ii) $ac \geq ae$ and $ca \geq ce$, (iii) $ce \geq ca$ and $ec \geq ea$ must hold. We need to deal only with cases (i) and (ii), since (iii) can be reduced to (ii) by noticing that (23) can be rewritten using (**) as

$$ef \equiv cd \wedge cd \equiv ab \rightarrow ef \equiv ab, \tag{24}$$

and (iii) for (23) corresponds to (ii) for (24).

We now distinguish two cases: (α) $ab \geq ae$ and (β) $ae \geq ab$ (by (14) one of the two has to hold).

Suppose (α) holds. Then, by (15), $(\exists x)\, B(aex) \wedge ab \equiv ax$. By A17(ii) $(\exists p)\, ax \equiv ap \wedge ap \equiv ep$. By A4(i) $ab \equiv ap$, and since $ab \equiv cd$ as well, we get, by (†), $cd \equiv ap$ and $cd \equiv ep$. From this and $cd \equiv ef$ we get, by A1 and A4(ii), $ep \equiv ef$. From this and $ab \equiv ep$ we get, by A4(i), $ab \equiv ef$, proving (23).

We can now prove yet another variant of Common Notion 2, which reads

$$B(fae) \wedge B(acb) \wedge af \equiv ac \wedge ea \equiv bc \rightarrow ab \equiv ef. \tag{25}$$

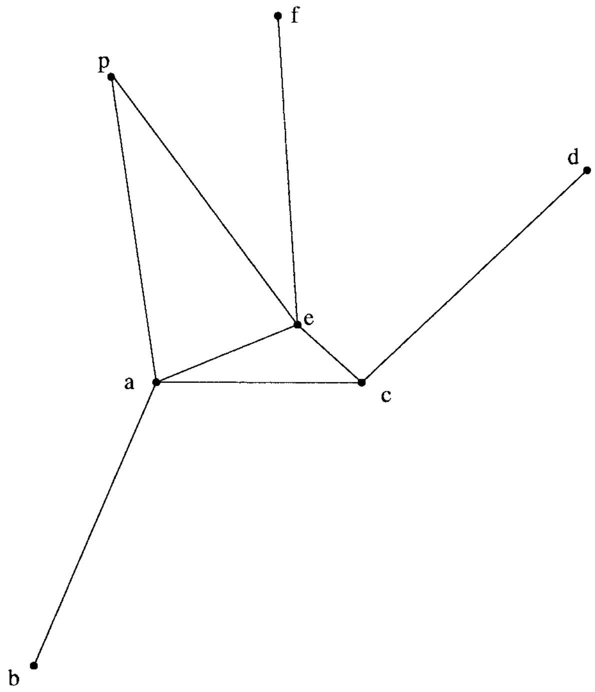

Figure 2. Proof of (23) in case $ab \geq ae$.

PROOF . Let $d = T'(aec)$. Then $B(cad)$ and $ad \equiv cb$ (by (†)). By (21) we get $ab \equiv cd$, and, by A11 and (**), $cd \equiv ef$. Since, by (13), $ab \geq ac$ and, by (11) and (**), $ab \geq ae$, we are in case (α), so we may conclude $ab \equiv ef$.

So we need to prove (23) only in case (β) holds.

Suppose (i) holds, and a, c, e are three different points (else there is nothing to prove, as (23) follows from A4(i) and A1). By (15) $(\exists h)\, B(ach) \wedge ae \equiv ah$. By A17(ii) $(\exists p)\, ah \equiv ap \wedge ap \equiv cp$. By A4(i) we get $ae \equiv ap$. By A15 $(\exists d')\, B(pcd') \wedge cd \equiv cd'$ and $(\exists b')\, B(pab') \wedge ab \equiv ab'$.

We now turn to the proof of

$$d'p \geq d'e. \tag{26}$$

By A15 $(\exists x)\, B(d'cx) \wedge ce \equiv cx$. Since $ea \geq ec$, $(\exists y)\, B(eya) \wedge ec \equiv ey$ (by (13)). By (10) $(\exists z)\, (B(azp) \vee B(apz)) \wedge ay \equiv az$. Since we also have $ae \equiv ap$ and $B(aye)$ (by A5), we have $B(azp)$ and $ye \equiv zp$ (by A10). By (†) we get $ec \equiv pz$. By (10) $(\exists u)\, (B(puc) \vee B(pcu)) \wedge pz \equiv pu$, thus $B(puc)$ (by A10), and $ec \equiv pu$ (by A4(i)). Since we also have $ce \equiv cx$, we get, by (†), $cx \equiv pu$. Since $B(puc)$, we must have $B(cxp)$ by (12) and (*). By A13 $d'x \geq d'e$.

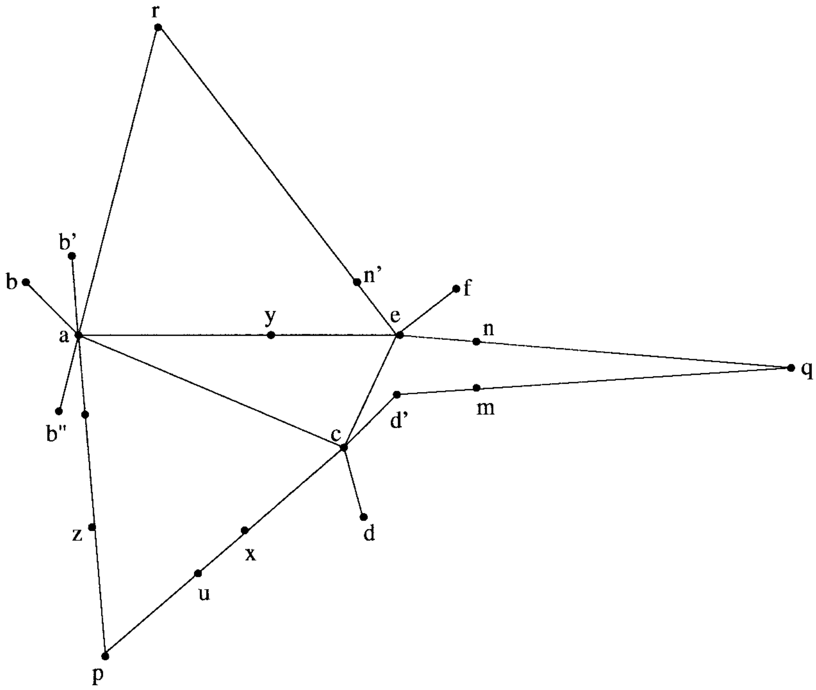

Figure 3. Proof of (23) in case ae is the longest side of trilateral ace, and $ae \geq ab$.

Since $B(cxp)$, $B(d'cx)$, $B(pcd')$, by (*) we get $B(d'xp)$, so $d'p \geq d'x$. By (18) we have (26).

By (26), (15), A4(i), and A17(ii) $(\exists q)\, d'p \equiv d'q \wedge d'q \equiv eq$. Since $pa \equiv pc$ (by (**)), $ab' \equiv cd'$ (by (†)), $B(pab')$ and $B(pcd')$, we get $pb' \equiv pd'$ (by (20)). Let $q' = T(eqa)$. Repeated use of (†) gives $pb' \equiv eq$ and $pb' \equiv eq'$. Let $q_0 = T'(ab'e)$. Given that $ap \equiv ae$ as well (by A1), we have $pb' \equiv eq_0$ (by A11), so by (*), A15, and (7), $q_0 = q'$, so $ab' \equiv aq'$ and $B(eaq')$, thus $ab \equiv aq'$ (by A4(i)). Let e' be the reflection of e in a (i. e. $B(eae') \wedge ae \equiv ae'$ (which exists by A15)). Then $B(aq'e')$, given that $ab \equiv aq'$, $B(eaq')$, $ae \geq ab$ (by (β)), by (13) and A10. Given that, as we have just established, the conditions of (i. e. the antecendent of) A17(i) are satisfied, $(\exists r)\, ae \equiv ar \wedge eq' \equiv er$, thus $eq \equiv er$ as well (by A4(i)). By (16) $(\exists m)\, B(d'mq) \wedge d'c \equiv dm \wedge cp \equiv mq$ and $(\exists n)\, B(qne) \wedge qm \equiv qn \wedge md' \equiv ne$. By (10) $(\exists n')(B(rn'e) \vee B(ren')) \wedge rn' \equiv ra$. By A15 $(\exists b'')\, B(rab'') \wedge ab' \equiv ab''$. Since $ae \equiv ap$, $ae \equiv ar$, we get $ap \equiv ar$ (by (†)). By A11, and (**) we get $b'p \equiv b''r$. Given that $ab \equiv ab'$, $cd \equiv cd'$ and $ab \equiv cd$, $pb' \equiv pd'$, we get, by (†), $pd' \equiv rb''$. Since $pd' \equiv d'q$, we get $d'q \equiv rb''$ (by (†)). But $d'q \equiv eq$, thus, by (†) $rb'' \equiv eq$. Given that $eq \equiv er$,

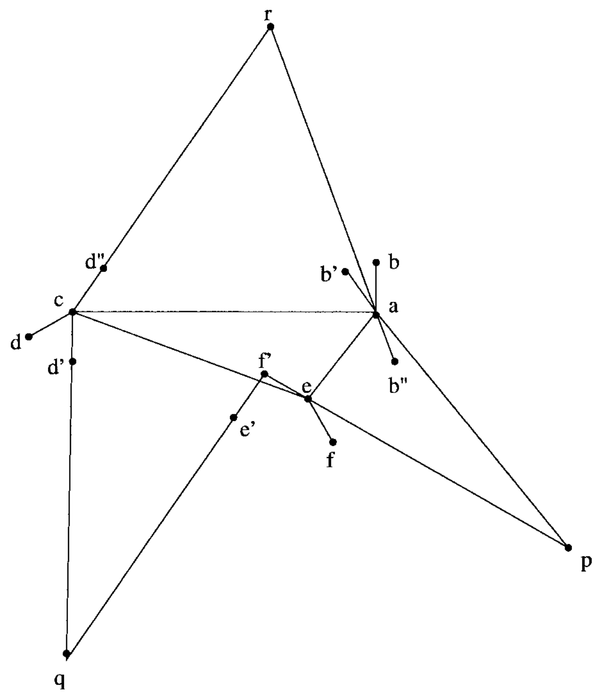

Figure 4. Proof of (23) in case ac is the longest side of trilateral ace, and $ae \geq ab$.

we get $re \equiv rb''$ (by (†)). Since $B(rn'e)$ (by A10) and $rn' \equiv qn$ (by (†), using $qn \equiv qm$ (by A1), $qm \equiv cp$ (by (**)), $cp \equiv pa$ (by (**)), $pa \equiv ar$ (by (**)), $ar \equiv rn'$ (by (**))). Since $B(qne)$ $eq \equiv er$, $nq \equiv n'r$ and $B(rn'e)$, we must have $en \equiv en'$ (by (22)). Since, by A1, $rb'' \equiv re$, $ra \equiv rn'$, and also $B(rab'')$, $B(rn'e)$, we have, by A10, $ab'' \equiv n'e$. Given that $ab \equiv ab''$ and $ef \equiv en'$, and also $ab'' \equiv en'$ (by A2), we get $ef \equiv ab''$ and then $ab \equiv ef$ by applying (†).

Suppose (ii). Since $ac \geq ae$, by A17(ii), (15), and A4(i), $(\exists p)\, ac \equiv ap \wedge ap \equiv ep$. By A15 $(\exists f')\, B(pef') \wedge ef \equiv ef'$ and $(\exists b')\, B(pab') \wedge ab \equiv ab'$. A proof identical to that of (26) shows that $pf' \geq cf'$, and thus, by (15), A4(i), and A17(ii), $(\exists q)\, f'p \equiv f'q \wedge f'q \equiv cq$. By (10) and A10, $(\exists e')\, B(f'e'q) \wedge f'e \equiv f'e' \wedge ep \equiv e'q$, $(\exists d')\, B(qd'c) \wedge qe' \equiv qd' \wedge e'f' \equiv d'c$. By (†) we get $ef \equiv d'c$. Since $cd \equiv ef$ as well, we get $cd \equiv cd'$ (by A4(ii), (**)). Repeated applications of (†) give $ac \equiv qd'$. Let $t = T'(cd'a)$ and $u = T'(abc)$. Given that $ct \equiv au$ (by (†)), $B(act)$, $B(cau)$, we also have $at \equiv cu$ (by (21)) and $cq \equiv at$ (by (25)), thus $cq \equiv cu$ (by A4(ii)). Since $ac \geq ae$ and $ae \geq ab$ (by (β)), we have $ac \geq ab$ (by (18)), so $B(auc')$ (by (13) and A10, bearing in mind that $au \equiv ab$ (by (†) and A4(ii)) and $ac' \equiv ac$), where c' denotes the reflection of c in a. By A17(i) $(\exists r)\, ac \equiv ar \wedge cu \equiv cr$, thus, by A4(i), $cq \equiv cr$, as well. By (10) and A10 $(\exists d'')\, B(cd''r) \wedge cd' \equiv cd'' \wedge d'q \equiv d''r$. By A15 $(\exists b'')\, B(rab'') \wedge ab \equiv ab''$. Using (†) we get $ar \equiv ap$, $ab'' \equiv ab'$, so, by A11, $rb'' \equiv pb'$. By (†) we get $d'q \equiv ac$, thus $d'q \equiv ar$, and, by A4(ii) and (**), $ar \equiv d''r$. Since we also have $cd \equiv cd''$ (by A4(i), A1) and $cd \equiv ab$ (by A1), $ab \equiv ab''$, we get, by A4(i) and A1, $cd'' \equiv ab''$. From $d''c \equiv ab''$ (by (**)), $rd'' \equiv ra$ (by (**)), and $B(rd''c)$, $B(rab'')$, we get, by (20), $rc \equiv rb''$. Since $cq \equiv cr$, $cq \equiv qf'$ (by (**)), $qf' \equiv f'p$ (by (**)) as well, we get $b''r \equiv f'p$ (by (†)) and, bearing in mind that $rb'' \equiv pb'$, we obtain $f'p \equiv pb'$ (by A4(ii) and (**)). Since $pe \equiv pa$ (by (**)) as well, and bearing in mind that $B(pef')$ and $B(pab')$ we get $ef' \equiv ab'$ (by (**) and A10). Since $ab \equiv ab'$ and $ef \equiv ef'$ as well, we get $ab \equiv ef$ (by (†)).

From now on, we shall use A1, A2, A4, (4), (8), (6), (23) without mentioning them, or by mentioning use of (‡). Given two points o and a we denote the unique point a' for which $B(aoa')$ and $oa \equiv oa'$ by $\sigma_o(a)$.

We can show that somewhat more than A21 holds, namely that

$$(\exists abc)\, \neg L(abc). \tag{27}$$

Proof . Let $a \neq b$, as given by A21. By A17(ii) $(\exists c)\, ab \equiv ac \wedge ac \equiv bc$, so we cannot have $L(abc)$, for else c would have to be the midpoint of ab, thus $B(acb)$, which together with $ac \equiv ab$ implies $c = b$ (by (7)), which in turn implies $a = b$ (by (4)).

We now prove that reflections in points are isometries, i. e. that

$$ab \equiv \sigma_o(a)\sigma_o(b). \tag{28}$$

Proof . If $o = a$ or $o = b$ then $\sigma_o(a) = o$ or $\sigma_o(b) = o$, and (28) is part of the definition of σ. Suppose $a \neq o$ and $b \neq o$. Let $a' = \sigma_o(a)$, $b' := \sigma_o(b)$, $a'' := T'(oab)$ and $b'' := T'(oba)$. By (*), we have $(B(oa''b') \vee B(ob'a''))$ and $(B(oa'b'') \vee B(ob''a'))$. Since $ob' \equiv ob''$ and $oa' \equiv oa''$, we have, by A10, $B(oa''b') \wedge B(oa'b'')$ or $B(ob'a'') \wedge B(ob''a')$. By A12 $ab \equiv a''b''$ and $a'b' \equiv a''b''$, thus $ab \equiv a'b'$.

We now prove that segment transport is possible on any given ray, i.e. that

$$(\forall abcd)(\exists e)\, B(cde) \wedge ab \equiv de. \tag{29}$$

Proof . Let $o := M(ad)$, $b' = \sigma_o(b)$. By (28) $ab \equiv db'$. By A15 $(\exists e)\, B(cde) \wedge db' \equiv de$, thus $ab \equiv de$.

As in (10), we can prove, using (*), (29), and A15, that

$$(\forall abcd)(\exists^{=1}e)\, c \neq d \rightarrow (B(ced) \vee B(cde)) \wedge ab \equiv ce. \tag{30}$$

We also have

$$B(abc) \wedge ab \equiv ab' \wedge ac \equiv ac' \wedge bc \equiv bc' \rightarrow B(ab'c'). \tag{31}$$

PROOF . If $a = b'$, then $B(ab'c')$ holds for all c'. Suppose $a \neq b'$. Let $c'' := T(acb')$. Since $ac'' \equiv ac'$, and $b'c'' \equiv b'c'$, $B(ab'c'')$ (by A10), we get $c''c'' \equiv c''c'$ (by A9), implying $c'' = c'$, so $B(ab'c')$.

We can now prove a result that could be stated as "isometries preserve the betweenness relation", i. e.

$$B(abc) \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c' \rightarrow B(a'b'c'). \tag{32}$$

PROOF . Let $o := M(aa')$. By A15 $(\exists b'')\, B(oab'') \wedge ab \equiv ab''$, $(\exists c'')\, B(oac'') \wedge ac \equiv ac''$. By (*) and A10 $B(ab''c'')$ and $bc \equiv b''c''$. By (28) $ab'' \equiv a'\sigma_o(b'')$, $ac'' \equiv a'\sigma_o(c'')$, $b''c'' \equiv \sigma_o(b'')\sigma_o(c'')$. By (*) and (31) we get $B(a'\sigma_o(b'')\sigma_o(c''))$. By (‡) we get $a'\sigma_o(b'') \equiv a'b'$, $a'\sigma_o(c'') \equiv a'c'$, and $\sigma_o(b'')\sigma_o(c'') \equiv b'c'$, and the desired conclusion follows from (31).

We now show that in A9, $B(abe)$ can be replaced by $L(abe)$. Note that A9 remains true if $B(abe)$ is replaced by $B(bae)$ given the commutativity of $\wedge$ (we write $bc \equiv bd \wedge ac \equiv ad$ instead of $ac \equiv ad \wedge bc \equiv bd$). To prove that it remains valid if $B(abe)$ is replaced by $B(aeb)$ as well, notice that, by A9, $\sigma_b(a)c \equiv \sigma_b(a)d$. By (*), $B(\sigma_b(a)be)$, so we can apply A9 to get $ec \equiv ed$. We have thus shown that

$$a \neq b \wedge ac \equiv ad \wedge bc \equiv bd \wedge L(abe) \rightarrow ec \equiv ed. \tag{33}$$

Using (33) and the definiteness of $\equiv$ we get

$$a \neq b \wedge L(abc) \wedge ac \equiv ac' \wedge bc \equiv bc' \rightarrow c = c'. \tag{34}$$

We can extend A10 to

$$\begin{aligned} &B(abc) \wedge (B(a'b'c') \vee B(a'c'b')) \wedge ab \equiv a'b' \wedge ac \equiv a'c' \\ &\rightarrow B(a'b'c') \wedge bc \equiv b'c'. \end{aligned} \tag{35}$$

PROOF . Let $o := M(aa')$, $b'' := \sigma_o(b)$, $c'' = \sigma_o(c)$. By (28) and (32), $B(a'b''c'')$, $ab \equiv a'b''$, $ac \equiv a'c''$, $bc \equiv b''c''$. The desired conclusion now follows from A10 and (‡).

We also have

$$(\forall abca'b')(\exists^{=1}c')\,(a \neq b \wedge L(abc) \wedge ab \equiv a'b') \\ \rightarrow (L(a'b'c') \wedge ac \equiv a'c' \wedge bc \equiv b'c'). \tag{36}$$

PROOF . Suppose $B(abc) \vee B(bca)$. By (30) $(\exists c')\,(B(a'b'c') \vee B(a'c'b')) \wedge ac \equiv a'c'$. The desired conclusion now follows from (35) and (34). Suppose $B(cab)$. By (29) $(\exists c')\,B(b'a'c') \wedge ac \equiv a'c'$, so (*) and (34) imply the desired conclusion.

We first define what we mean by perpendicularity, i. e. what we mean when we write 'line ab is perpendicular to line bc'.

$$ab \perp bc :\leftrightarrow a \neq b \wedge b \neq c \wedge ac \equiv a\sigma_b(c). \tag{37}$$

Notice that, by (33) and A7, we have

$$ab \perp bc \wedge L(abx) \wedge x \neq b \rightarrow xb \perp bc. \tag{38}$$

Perpendicularity, as defined by (37), is, according to (38), a relation between the *line* ba, with b as a distinguished point on it, and the segment bc. We show next that it is the *line* bc, with b as a distinguished point on it, rather than the segment bc, which is the relevant notion involved in perpendicularity.

$$ab \perp bc \wedge L(bcd) \wedge d \neq b \rightarrow ab \perp bd. \tag{39}$$

PROOF . Let $a' := T(bca)$, $d' := T(bda)$. By (38) we have $a'b \perp bc$. By A12 $a'd \equiv d'c \wedge a'\sigma_b(d) \equiv d'\sigma_b(c)$ or $a'\sigma_b(d) \equiv d'c \wedge a'd \equiv d'\sigma_b(c)$. By (38) we have $d'c \equiv d'\sigma_b(c)$. Thus $a'd \equiv a'\sigma_b(d)$, and since $a' \neq b$ and $b \neq d$, we have $a'b \perp bd$. Since $L(a'ba)$ and $a \neq b$, we have $ab \perp bd$ (by (38).

It follows from (28) that $\perp$ is symmetric, i. e. that

$$ab \perp bc \rightarrow cb \perp ba. \tag{40}$$

We can show, using A17(ii), that to every line ab there is a perpendicular to that line. Suppose $a \neq b$. According to A17(ii) $(\exists c)\,ab \equiv ac \wedge ac \equiv bc$. If we denote by o the midpoint of ab, then $co \perp oa$ (by (37)). We can now prove exactly as in [28, §4] that there is a unique perpendicular through any point to any line, and that the reflection in every line exists (and the reflected point is unique) and is an isometry. Notice that this is possible since the only plane separation axiom used in all the proofs we need from

[28, §4] is A11. For $a \neq b$, we shall denote by $F(abc)$ the footpoint of the perpendicular from c to ab.

We now turn to the proof of

$$\neg L(abx) \wedge L(abc) \wedge L(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c' \quad (41)$$
$$\wedge ax \equiv a'x' \wedge bx \equiv b'x' \to xc \equiv x'c'.$$

Proof . Let o be the midpoint of aa' and reflect a', b', c', x' in o. Given that reflections in points are isometries, it turns out that it is enough to prove (41) with $a' = a$. So, assume the hypothesis in (41) with $a' = a$ holds, and that $b \neq b'$ (else we must have $c = c'$, and so $x = x'$ or else, if $x \neq x'$, then (41) is (33)) and let m be the midpoint of bb'. The reflection of b', c', x' in am (a line if $a \neq m$, the point a if $a = m$) maps b' and c' into b and c (c' is mapped into c since reflections in lines or points are isometries, and also (34)) and x' into x'', and $ax'' \equiv ax'$, thus $ax'' \equiv ax$, $b'x' \equiv bx''$, thus $bx \equiv bx''$, $c'x' \equiv cx''$. By (33) we have $cx \equiv cx''$, thus also $cx \equiv c'x'$.

We have now reached an important step in the proof, as we have proved that all of Sörensen's [32] (cf. also [25, p. 385–386]) axioms for non-elliptic metric planes — i. e. $L(aba)$ (by (1) and (2)), $L(abc) \to L(cba) \wedge L(bac)$ (by (1) and A5), $a \neq b \wedge L(abc) \wedge L(abd) \to L(acd)$ (by (1), A7 and A6), (3), (4), (6), A1, (23), (36), (41), existence and uniqueness of the reflection in a line, existence and uniqueness of the reflection in a point, existence of the footpoint of a perpendicular to a line from a point outside it, reflections are isometries, A20, "If x, y, z are three different collinear points, and the segments xy and yz have midpoints, then there is a point which is equidistant from x and z" (a trivial consequence of A14), and (27) — hold in $(\Sigma \setminus \{$A18, A19, A22$\}) \cup \{(1)\}$, so all models of Σ are nonelliptic metric planes (cf. [1]) with free mobility (by A14, A15) and non-Euclidean metric (by A22).

By Bachmann's Haupt-Theorem (cf. [2]) the point- and line-sets of every model $\mathfrak{M}$ of Σ are subsets of the point- and line-sets of a projective-metric plane $\mathfrak{P}(K,k)$ over a Pythagorean field K, such that the point-set $P_{\mathfrak{M}}$ of $\mathfrak{M}$ consists entirely of points $(a,b,1)K$, and contains the point $(0,0,1)K$, and the line-set $L_{\mathfrak{M}}$ of $\mathfrak{M}$ contains every line of $\mathfrak{P}(K,k)$ which passes through a point in $P_{\mathfrak{M}}$.

A *projective-metric plane* $\mathfrak{P}(K,k)$ consists of points, $(a,b,c)K$, with at least one of a,b,c nonzero, lines $[u,v,w]K$, with at least one of u,v,w nonzero, an incidence relation between points and lines, such that point $(a,b,c)K$ is incident with line $[u,v,w]K$ if and only if $au + bv + cw = 0$, an orthogonality and a congruence relation defined by

$$[u,v,w]K \perp [u',v',w']K \Leftrightarrow uu' + vv' + kww' = 0,$$

and, for **a**, **b**, **c**, **d** with $Q(\mathbf{a}) \neq 0$, $Q(\mathbf{b}) \neq 0$, $Q(\mathbf{c}) \neq 0$, $Q(\mathbf{d}) \neq 0$,

$$\mathbf{ab} \equiv \mathbf{cd} \Leftrightarrow \frac{F(\mathbf{a},\mathbf{b})^2}{Q(\mathbf{a})Q(\mathbf{b})} = \frac{F(\mathbf{c},\mathbf{d})^2}{Q(\mathbf{c})Q(\mathbf{d})}, \tag{42}$$

where by $\mathbf{u}$ we have denoted $(x_u, y_u, 1)K$, $F(\mathbf{u},\mathbf{v}) = k(x_u x_v + y_u y_v) + 1$ and $Q(\mathbf{x}) = F(\mathbf{x},\mathbf{x})$.

We shall write from now on (a, b) for all points $(a, b, 1)K$ in $P_{\mathfrak{M}}$.

From A19 we conclude, just like in [23, p. 134-135], that $-k \in K^2$, and thus that it may be normalized to $k = -1$.

We now want to prove that the inner form of Pasch's axiom holds in Σ. It was shown by H. N. Gupta [9] that the full Pasch axiom follows from the inner form of the Pasch axiom together with the other axioms of plane absolute geometry, all of which have been shown to be consequences of Σ.

We first show that

$$(a, 0) \in P_{\mathfrak{M}} \Rightarrow 1 - a^2 \in K^2. \tag{43}$$

PROOF . Since $(0,0)$ and $(a,0)$ have a midpoint $(x,0)$, we get, using (42), $1 - a^2 = (1 - ax)^2$.

Also,

$$\text{There exists } a \in K \text{ such that } (a^2, 0) \in P_{\mathfrak{M}}. \tag{44}$$

PROOF . Let $\mathbf{o} := (0,0)$, $\mathbf{p} := (b,0)$, for some $b \neq 0$ (such a point must exist by A21) and let $\mathbf{m}$ be the footpoint of the perpendicular from $\mathbf{p}$ to the line $y = \frac{1}{2b}x$ (i. e. the line $[-1, 2b, 0]K$, which belongs to $L_{\mathfrak{M}}$ since it passes through $\mathbf{o}$, a point in $P_{\mathfrak{M}}$). Let $\mathbf{o}' := \sigma_{\mathbf{m}}(\mathbf{o})$. Since $\mathbf{po} \equiv \mathbf{po}'$, we find, using (42), that the coordinates of $\mathbf{o}'$ are $(2b(b + (2b)^{-1})^{-2}, (b + (2b)^{-1})^{-2})$. The footpoint $\mathbf{f}$ of the perpendicular from $\mathbf{o}'$ to the y-axis (the line $[1,0,0]K$) is $(0, (b + (2b)^{-1})^{-2})$ and $T(\mathbf{ofp})$ is $((b + (2b)^{-1})^{-2}, 0)$.

We now show that

$$\mathbf{x} = (x, 0) \in P_{\mathfrak{M}} \Rightarrow x \in K^2 \vee -x \in K^2. \tag{45}$$

PROOF . We show that, with $\mathbf{o} = (0,0)$, $\mathbf{a} = (a^2, 0)$, $\mathbf{a}' = (-a^2, 0)$, as in (44), an arbitrary point $\mathbf{x} = (x, 0)$ is such that $B(\mathbf{aox})$ if and only if $-x \in K^2$, and is such that $B(\mathbf{a}'\mathbf{ox})$ if and only if $x \in K^2$. By A18, if $B(\mathbf{aox})$, then there exists a point $\mathbf{y} = (0, y)$ such that the lines $\mathbf{xy}$ ($= [y, x, -xy]K$) and $\mathbf{ay}$ ($= [y, a^2, -a^2 y]K$) are perpendicular, i. e. such that $y^2 + a^2 x - a^2 x y^2 = 0$, so $x = -y^2(a^2(1 - y^2))^{-1}$, i. e. by (43), $-x \in K^2$. In case $B(\mathbf{a}'\mathbf{ox})$, we get $x = y^2(a^2(1 - y^2))^{-1}$, i. e. $x \in K^2$ (by (43)). The desired conclusion now follows from the fact that for every $\mathbf{x} = (x, 0)$ we must have one of $B(\mathbf{aox})$ or $B(\mathbf{a}'\mathbf{ox})$.

Given (45), we can now define an order relation on the coordinates of points in $\mathfrak{M}$ by letting $a < b$ if and only if $b - a \in K^2 \setminus \{0\}$.

$$\text{If } \mathbf{o} = (0,0),\ \mathbf{a} = (a,0),\ \mathbf{b} = (0,b), \qquad (46)$$
$$\text{then } B(\mathbf{oab}) \Leftrightarrow (0 < a < b \vee b < a < 0)$$

PROOF . By A18 there is $\mathbf{y} = (a, y)$ such that $\mathbf{oy}$ is orthogonal to $\mathbf{yb}$, i. e. such that $y^2 = (b - a)a$.

With $F(abx)$ denoting, for a, b, x with $\neg L(abx)$, the footpoint of the perpendicular from x to the line ab, we also have

$$\neg L(abd) \wedge B(abc) \to B(aF(adb)F(adc)). \qquad (47)$$

PROOF . Since we have free mobility and isometries preserve the betweenness relation, we may take $\mathbf{a} = (0,0)$, $\mathbf{ad}$ to be the x-axis, $\mathbf{b} = (x, \lambda x)$, $\mathbf{c} = (y, \lambda y)$. There is nothing to prove if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are not different. Let $\mathbf{b}' = (a, 0)$ and $\mathbf{c}' = (b, 0)$ be two points on $\mathbf{ad}$ such that $\mathbf{ab} \equiv \mathbf{ab}'$, $\mathbf{ac} \equiv \mathbf{ac}'$, $\mathbf{bc} \equiv \mathbf{b}'\mathbf{c}'$. The first two give $a^2 = x^2(\lambda^2 + 1)$ and $b^2 = y^2(\lambda^2 + 1)$, and from the third we get, using the equations just obtained, $ab = xy(1 + \lambda^2)$. Since $B(\mathbf{oa}'\mathbf{b}')$, we must have, by (46), $ab \in K^2$ and $0 < a < b$ or $b < a < 0$, thus we must have either $0 < x < y$ or $y < x < 0$, thus, with $\mathbf{x} = (x, 0)$ and $\mathbf{y} = (y, 0)$, $B(\mathbf{axy})$ (by (46)).

We can now prove the inner form of the Pasch axiom, i. e.

$$(\forall abcde)(\exists f)\, \neg L(acd) \wedge B(abc) \wedge B(ade) \to (B(bfe) \wedge B(cfd)). \qquad (48)$$

PROOF . Notice that, by A18, $\mathfrak{M}$ is metrically convex, i. e. if $\mathbf{a}, \mathbf{b} \in P_{\mathfrak{M}}$, then $P_{\mathfrak{M}}$ also contains all points $\mathbf{x}$ on the line $\mathbf{ab}$ for which there exists a point $\mathbf{d} = (m, n, 1)K$ in $\mathfrak{P}(K, -1)$ such that $\mathbf{dx} \perp \mathbf{xa}$ and $\mathbf{ad} \perp \mathbf{db}$. In particular, if $\mathbf{x} = (\alpha, \beta)$ is on $\mathbf{ab}$ and both α and β are between (in the sense of the order relation $<$ we have introduced earlier) the x- and y- coordinates of $\mathbf{a}$ and $\mathbf{b}$, then $B(\mathbf{axb})$. It is now a matter of routine computation to find the intersection point of the lines $\mathbf{cd}$ and $\mathbf{be}$, where we may choose $\mathbf{a} = (0,0)$ and $\mathbf{ad}$ to be the x- axis, and to prove that its coordinates lie between those of the endpoints of the segments. Given (47), this establishes (48).

We now know that $\mathfrak{M}$ has to be an absolute plane, so we may use Pejas's [27] classification of absolute planes and conclude as in [23, p. 134–135] that $\mathfrak{M}$ must be isomorphic to the Klein space over K, a Euclidean ordered field.

We have thus proved that

THEOREM 3.1. *Σ is an axiom system for plane hyperbolic geometry over Euclidean ordered fields, i. e. every model of Σ is isomorphic to the 2-dimensional Klein model over a Euclidean ordered field.*

4. The constructive axiom system

We shall now provide a constructive axiom system for plane hyperbolic geometry, expressed in the language $\mathsf{L}_{con} := \mathsf{L}(T', C_1, C_2, K_1, K_2, P, A', H_1, H_2)$, which contains only ternary operation symbols having the following intended interpretations: $T'(abc)$ is the point d on the ray opposite to ray $\overrightarrow{ac}$ with $ad \equiv ab$, provided that $a \neq c$ or $a = b$, and arbitrary otherwise; $C_i(abc)$, for $i = 1, 2$, stand for the two points d for which $da \equiv db$ and $da \equiv ac$, provided that $a \neq b$ and $B(abc)$, arbitrary points otherwise; $K_i(abc)$, for $i = 1, 2$, stand for the two points d for which $ad \equiv ab$ and $bd \equiv bc$, provided that c is strictly between b and $\sigma_a(b)$, two arbitrary points, otherwise; $P(abc)$ stands for the point d for which $da \equiv db$ and $B(adc) \vee B(bdc)$, provided that a, b, c are three non-collinear points, an arbitrary point, otherwise; $A'(abc)$ stands for the point d on the ray $\overrightarrow{ac}$ for which $dd' \equiv ab$, where d' is the reflection of d in the line ab, provided that a, b, c are three noncollinear points, arbitrary, otherwise; $H_i(abc)$, for $i = 1, 2$, stand for the two points d for which $db \perp ba$ and $ad \perp dc$, provided that a, b, c are three different points with $B(abc)$, an arbitrary point, otherwise.

We could have chosen to assign a conventional value to these operations whenever they are left "arbitrary", but we felt that there was no good reason to do so, as each such convention adds an axiom to our system without carrying any geometric meaning.

To shorten and improve the readability of the axioms, we introduce the following abbreviations:

$$\sigma(ab) := T'(abb), \tag{49}$$

$$T(abc) := T'(ab\sigma(ac)), \tag{50}$$

$$\alpha(ab) := C_1(\sigma(ab)bb), \tag{51}$$

$$M(ab) := P(\alpha(ab)\sigma(a\alpha(ab))\alpha(ba)), \tag{52}$$

$$\beta(abc) := \sigma(T'(acb)\alpha(T'(acb)a)), \tag{53}$$

$$\pi(abc) := P(\alpha(T'(acb)a)\beta(abc)c), \tag{54}$$

$$\tau_1(abc) := T'(\pi(abc)c\alpha(T'(acb)a)), \tag{55}$$

$$\tau_2(abc) := T'(\pi(abc)c\beta(abc)), \tag{56}$$

$$\mu(abc) := M(\tau_1(abc)\tau_2(abc)), \tag{57}$$

$$R(abc) := \sigma(\mu(abc)c), \tag{58}$$

$$ba \perp ac :\Leftrightarrow a \neq b \wedge b \neq c \wedge c \neq a \wedge T(c\sigma(ab)b) = b, \tag{59}$$

$$B(abc) :\Leftrightarrow T'(bac) = a \vee b = c, \tag{60}$$

$$Z(abc) :\Leftrightarrow a \neq b \wedge b \neq c \wedge T'(bac) = a, \tag{61}$$

$$\begin{aligned} ab \equiv cd : \quad \Leftrightarrow \quad & (a = b \wedge c = d) \vee (a \neq b \wedge ((a \neq c \\ & \wedge T(a\sigma(M(ac)d)b) = b) \vee (a = c \wedge T(adb) = b))), \end{aligned} \tag{62}$$

σ being defined for all values of the arguments, α and M being defined whenever $a \neq b$, T whenever $a \neq c$, and the remaining operations whenever a, b, c are such that $\neg L(abc)$.

The intuitive meaning of some of these abbreviations are: $T(abc)$ is the point d on the ray $\vec{ac}$ for which $ad \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise; $M(ab)$ stands for the midpoint of ab, provided that $a \neq b$; $R(abc)$ stands for the reflection of c in ab, provided that a, b, c are not collinear, an arbitrary point, otherwise; $\perp$, B, $\equiv$ have the same meaning as in the previous section, as does L, which is defined as in (1); $Z(abc)$ stands for b lies between a and c, being different from a and different from c.

The axioms are A7, A21, and

C 1. $T'(aab) = a$,

C 2. $b \neq a \wedge c \neq a \rightarrow T'(abT'(acb)) = b$,

C 3. $a \neq c \wedge a \neq b \wedge T(a\sigma(M(ac)d)b) = b \rightarrow T(c\sigma(M(ca)b)d) = d$,

C 4. $a \neq c \rightarrow \sigma(M(ac)c) = a$,

C 5. $a \neq b \rightarrow M(ab) = M(ba)$,

C 6. $a \neq d \wedge T(abd) = d \wedge T(acd) = d \rightarrow T(abc) = c$,

C 7. $T'(baa) = a \rightarrow a = b$,

C 8. $b \neq d \wedge c \neq d \wedge T'(bad) = a \wedge T'(cbd) = b \rightarrow T'(bac) = a \vee b = c$,

C 9. $a \neq b \wedge b \neq c \wedge c \neq a \wedge c \neq d \wedge T(adc) = c \wedge T(bdc) = c \rightarrow d = R(abc)$,

C 10. $\neg L(abd) \wedge \neg L(acd) \wedge R(abd) = R(acd) \rightarrow L(abc)$,

C 11. $L(abc) \wedge a \neq b \wedge a \neq c \wedge b \neq c \wedge T(adc) = c \wedge T(bdc) = c \rightarrow d = c$,

C 12. $\neg L(abc) \wedge B(abd) \rightarrow T(dR(abc)c) = c$,

C 13. $a \neq c \wedge B(abc) \wedge T(adc) = c \rightarrow B(aT(abd)d) \wedge bc \equiv T(abd)d$,

C 14. $a \neq b \wedge B(abc) \wedge T(bda) = a \rightarrow ac \equiv dT'(bcd)$, C 15. $a \neq b \wedge B(abc) \wedge T(adb) = b \rightarrow dc \equiv bT(acd)$,

C 16. $a \neq b \wedge B(bac) \wedge T(adb) = b \rightarrow dc \equiv bT'(acd)$,

C 17. $a \neq b \rightarrow B(aM(ab)b) \wedge M(ab)a \equiv M(ab)b$,

C 18. $c \neq a \rightarrow B(cT(cbT'(abc))T'(abc))$,

C 19. $c \neq a \rightarrow T(aT'(abc)b) = b$,

C 20. $c \neq a \wedge T'(abc) = a \rightarrow a = b$,

C 21. $\neg L(abc) \rightarrow P(abc) \neq a \wedge T(P(abc)ba) = a$
$\wedge(B(aP(abc)c) \vee B(bP(abc)c))$,

C 22. $\neg L(abc) \rightarrow \neg L(M(ab)M(bc)M(ca))$,

C 23. (i) $Z(bc\sigma(ab)) \rightarrow T(aK_j(abc)b) = b \wedge T(bK_j(abc)c) = c$,
(ii) $B(abc) \wedge a \neq b \rightarrow T(C_j(abc)ba) = a \wedge T(aC_j(abc)c) = c$,

C 24. (i) $Z(bc\sigma(ab)) \rightarrow R(abK_1(abc)) = K_2(abc)$,
(ii) $B(abc) \wedge a \neq b \rightarrow R(abC_1(abc)) = C_2(abc)$,

C 25. $Z(abc) \rightarrow ab \perp bH_j(abc) \wedge aH_j(abc) \perp H_j(abc)c$,

C 26. $Z(abc) \rightarrow \sigma(bH_1(abc)) = H_2(abc)$,

C 27. $\neg L(abc) \rightarrow A'(abc) \neq a \wedge (B(aA'(abc)c) \vee B(acA'(abc)))$
$\wedge ab \equiv A'(abc)R(abA'(abc))$,

C 28. $\neg ab \perp bc \vee \neg bc \perp cd \vee \neg cd \perp da \vee \neg da \perp ab$.

To see that $\Pi = \{$A7, A21, C1-C28$\}$ represents an axiom system for plane hyperbolic geometry, notice that all the axioms of Σ can be deduced from $\Pi \cup \{(49)-(62)\}$. The deductions are routine, so we shall only mention the axioms in Π needed in the proofs (the abbreviations involved, which will also be used in the proofs, will not be mentioned).

By C2 and C1 we have $T(add) = d$, so C6 implies

$$a \neq d \wedge T(abd) = d \rightarrow T(adb) = b. \tag{63}$$

A1 follows from (63), C3, C4; A2 from C3 and C5; A3 (in fact (4)) from C1 and (2) (an immediate consequence of (60)); A4 from C6, C5, A1, and (4); A5 from C1 and C2; A6 from C7 and C8; A8 from C9, C10, and C11; A9 from C12; A10 from C13 (if $a = c$ in A10, then b, d, e must be equal to a as well, so the consequent is $B(aaa) \wedge aa \equiv aa$); A11 from C14; A12 from

C15, C16, A13 from C18, A14 from C17 (when $a = b$, $c = a$ will do); A15 from C19 and C20; A16 from C21; A17 from C23; A18 from C25; A19 from C27; A20 from C22; A22 from C28.

It follows that every model of $\Pi \cup \{(49-(62)\}$ is isomorphic to the Klein model over some Euclidean ordered field, the relations $\equiv$ and B having the desired interpretation. From C19 and C2, C21, C23, C24, C25, C26, and C27 we deduce that T', P, K_j, C_j, H_j, and A' have the desired interpretations as well.

The operations C, K and H have been introduced in pairs, as well as axioms C24, C26 added to the axiom system for the sole reason of having a set of primitives which is invariant under (hyperbolic) motions.

We have proved that

THEOREM 4.1. Π *is an axiom system for plane hyperbolic geometry. Its models are isomorphic to 2-dimensional Kleinian models of hyperbolic geometry with Euclidean ordered fields, the operations of* L_{con} *having the desired interpretation.*

5. Axiom systems for Euclidean geometry

Let

C 29. $a \neq b \rightarrow H_1(\sigma(ab)ab)\sigma(ab) \perp \sigma(ab)H_2(\sigma(ab)ab)$

be the constructive counterpart of $\neg A22$, the axiom of the Euclidean metric. It states somewhat more than $\neg A22$, which states the existence of a rectangle, as it implies (with the help of a few universal axioms of absolute geometry) the existence of a square.

THEOREM 5.1. (i). $\Sigma' = \{$A1-A20,$\neg$A22$\}$ *is an axiom system for plane Euclidean geometry over Euclidean ordered fields, i. e. all its models are isomorphic to Cartesian planes coordinatized by Euclidean ordered fields.*

(ii). $(\Pi \setminus \{$C28, A21$\}) \cup \{$C29$\}$ *is an axiom system for plane Euclidean geometry over Euclidean ordered fields, the operations having the desired interpretation.*

PROOF . (i). We have shown that Sörensen's axiom for non-elliptic metric planes follow from $\Sigma \setminus \{A18, A19, A22\}$, so they are consequences of Σ' as well. Thus all models of Σ' are metric-Euclidean planes with free mobility. Using A19 we can show that the Euclidean parallel postulate holds in Σ'. Let p be a point outside of line ab, with $a \neq b$, and let q be a point different from p and such that pq is perpendicular to $pF(abp)$. Let x be a point which does not lie on pq. Then, by A19 (as well as A15 and universal properties of congruence valid in metric-Euclidean planes), there exists a point y on ray $\overrightarrow{px}$, such that $yF(pqy) \equiv pF(abp)$. Since the metric of the plane is Euclidean, y must lie in ab, so line px intersects line ab.

From here on the proof is identical to that presented in [37].

(ii) follows from (i).

6. Simplicity

Both Π and $(\Pi \setminus \{\text{C28, A21}\}) \cup \{\text{C29}\}$ are axiom systems that are simplest possible from several points of view. Their language is the simplest, since it contains only ternary operations, and it is impossible to axiomatize Euclidean geometry by means of binary operations only, as shown in [4]. So, if we want one language in which to axiomatize both Euclidean and hyperbolic geometry, then we need to have at least one ternary operation. They are also simplest in that all their axioms are prenex sentences with at most 4 variables. By a theorem of D. Scott [31], which, although stated for the Euclidean case only is valid (with the same proof) in the hyperbolic case as well, every axiom system in a language that does not contain individual constants and whose set of primitive notions is invariant under isometries (in our case they are for those values for which they are meaningfully defined, which is all we need), for n-dimensional Euclidean or hyperbolic geometry, has to contain an axiom which is a prenex statement containing $n+2$ variables. They are also simplest in the sense of quantifier complexity, all axioms, with one exception, which is purely existential, being universal axioms. Although one can axiomatize hyperbolic geometry by means of a binary operation (namely by means of point-reflection, as done in [26] using only $\forall\exists$ axioms), one cannot axiomatize it constructively by means of any set of binary operations, each of which are invariant under the group of hyperbolic motions or that of direct hyperbolic motions (so we are only interested in binary operations ω, such that $\omega(ab) = c \Leftrightarrow \varphi(a,b,c)$, where φ is an existential formula in $\mathsf{L}_{B\equiv}$ with a, b, c as only free variables, and such that, for fixed a and b there is only one (or at most two) c satisfying $\varphi(a,b,c)$ in hyperbolic geometry.) To see this, we introduce the notion of a *local* operation. Let $(K,<)$ be an non-Archimedean ordered Euclidean field, let P be the ideal of its *infinitely small* elements, i. e. $P = \{x \in K \,:\, (\forall n \in \mathbb{N})\, x < \frac{1}{n}\}$, and let $\mathfrak{F}$ designate the set $\{(x,y) \in K \times K \,|\, 1 - x^2 - y^2 > 0 \text{ and } \notin P\}$. An operation is called *local* if it leaves $\mathfrak{F}$ invariant, i. e. if its value is in $\mathfrak{F}$ whenever its arguments are in $\mathfrak{F}$. Notice that every binary geometric operation ω is local, as can be seen by induction on the number of existential quantifiers in the definiens φ of

ω. Thus we cannot 'reach' faraway points by means of any set of geometric binary operations in the following precise sense: We cannot constructively define A', which is a non-local operation (in fact the only non-local operation in our constructive language) by means of local operations, as $\mathbf{a} = (0,0)$, $\mathbf{b} = (\frac{1}{2},0)$, and $\mathbf{c} = (\frac{1}{2},p)$, with $p \in P \setminus \{0\}$, belong to $\mathfrak{F}$, but $A'(\mathbf{abc})$ doesn't. Since no succesive application of local operations will move the set $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ out of $\mathfrak{F}$, no finite set of local operations can define the same operation as A'.

As far as Σ is concerned, we do not know if it is simplest possible among all $\mathsf{L}_{B\equiv}$ axiom systems for plane hyperbolic geometry, in the sense that we do not know whether there is an axiom system consisting of prenex at most 5-variable statements for it.

We do know that Σ' is not simplest in this sense, since there is an axiom system consisting of prenex at most 5-variable sentences for it (cf. [17], [18], [21]).

We also know that plane hyperbolic geometry cannot be axiomatized in $\mathsf{L}_{B\equiv}$ by means of prenex at most 4-variable statements. This can be seen by means of Ehrenfeucht-Fraïssé games (see [13] or [14]) as done in [21], by using the same strategy as in [21] to prove that the Duplicator has a winning strategy in the Ehrenfeucht-Fraïssé game with 4 moves.

One of the models is $\mathfrak{K}$, the Kleinian model for plane hyperbolic geometry over the reals, and the other model, $\mathfrak{M}$, is one with a non-transitive congruence relation, being the 2-dimensional Klein model over the field of real numbers, with the usual order structure, but with a congruence relation that is strictly included in the usual congruence relation of $\mathfrak{K}$.

Let $\mathfrak{K}= \langle U, \mathbf{B}_{\mathbb{R}}, \equiv_{\mathbb{R}}\rangle$ be the Klein model over the real numbers, i. e. B and D will have the usual interpretation of "Betweenness" and "Equidistance", and U stands for the unit disk in $\mathbb{R}^2$.

Let $\mathbf{L}_{\mathbb{R}}(\mathbf{abc})$ stand for "$\mathbf{a},\mathbf{b},\mathbf{c}$ are three collinear points in $\mathfrak{K}$".

Let $\mathfrak{M}= \langle U, \mathbf{B}_{\mathbb{R}}, \equiv_{\mathfrak{M}}\rangle$, where $\mathbf{a}_1\mathbf{a}_2 \equiv_{\mathfrak{M}} \mathbf{a}_3\mathbf{a}_4$ iff $\mathbf{a}_1\mathbf{a}_2 \equiv_{\mathbb{R}} \mathbf{a}_3\mathbf{a}_4$ and one of the following is true

(i) $\mathbf{a}_i\mathbf{a}_j$ bisects $\mathbf{a}_k\mathbf{a}_l$ in $\mathfrak{K}$ for some $\{i,j,k,l\} = \{1,2,3,4\}$;

(ii) $\mathbf{L}_{\mathbb{R}}(\mathbf{a}_i\mathbf{a}_j\mathbf{a}_k)$ for some i,j,k with $i \neq j \wedge j \neq k \wedge k \neq i$ and $i,j,k \in \{1,2,3,4\}$;

(iii) $\mathbf{a}_i\mathbf{a}_j \equiv_{\mathbb{R}} \mathbf{a}_i\mathbf{a}_k$ for some i,j,k with $i \neq j \wedge j \neq k \wedge k \neq i$ and $i,j,k \in \{1,2,3,4\}$;

(iv) the measure of one of the angles between $\mathbf{a}_1\mathbf{a}_2$ and $\mathbf{a}_3\mathbf{a}_4$ is $\frac{\pi}{n}$ for some $n \in \mathbb{N} \setminus \{0\}$.

Since all the prenex 4-variable sentences (in fact, more than just the prenex 4-variable sentences, but all we need are prenex ones, see [14]) which are true in $\mathfrak{K}$ are true in $\mathfrak{M}$ as well, and $\mathfrak{M}$ does not satisfy (23), we have shown that an $\mathsf{L}_{B\equiv}$ axiom system for hyperbolic geometry must contain a prenex 5-variable sentence.

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VICTOR PAMBUCCCIAN
Department of Integrative Studies
Arizona State University West
P. O. Box 37100
Phoenix, AZ 85069-7100, U.S.A.
pamb@math.west.asu.edu