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Distribution of Fiber Intersections in Two-Dimensional Random Fiber Webs – A Basic Geometrical Probability Model

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Abstract Fundamental theories governing the number of fiber intersections in random non-woven fiber webs were developed based on the planar geometry of fiber midpoints distributed in a two-dimensional Poisson field. First, the statistical expectation and variance for the number of fiber intersections in unit web area were obtained as functions of a fixed number of fibers with equal lengths. The theories were extended to the case of a two-dimensional Poisson field by assuming that the number and locations of the fibers are random. The theories are validated by a newly developed computer simulation method employing the concept of "seeding region" and "counting region." Unlike all previously published papers, it was shown for the first time that the expectations and variances obtained theoretically matched that from computer simulations almost perfectly, validating both the theories and simulation algorithms developed.

Key words fiber intersections, intersection geometry, fiber web, non-wovens, expectation, variance, edge effect, Poisson field

It is of great mathematical significance and practical importance that the number of fiber intersections and uniformity within a unit area of a non-woven fiber web changes with the number and lengths of fibers as well as with aggregate length. The numerical relationships, while most important for imparting the desired physical and mechanical properties of the resulting fabric, are complex and not easily understood due to the geometrical and probabilistic nature of fiber arrangement within a web that provides the intersection distribution. The distribution not only determines the uniformity of the basis weight but also such physical properties as strength, elongation, air and water permeabilities, acoustics, and filtering efficiency of the resulting fabrics as well as the optimal control strategies with respect to the desired properties [1, 2]. As the number of fibers per unit fabric area increases, the physical counting of the intersections becomes an almost impossible task, making the theory-based computer simulation the only viable alternative.

The major thrust of this study is to understand the basic nature of fiber arrangement within a two-dimensional random fiber web in terms of geometrical probability under certain simplified assumptions, and to extend the results to a more general case. First, a given number of fibers of equal lengths will be considered within a unit area as a starting point. Of primary importance then is to compute the probability of intersection between any two fibers within a given area when, say, the midpoints of the two fibers are assumed to be located within the area. This will be the founding block for deriving the mean and variance of the number of fiber intersections in the area when n fibers are placed within what we call the "seeding area." As

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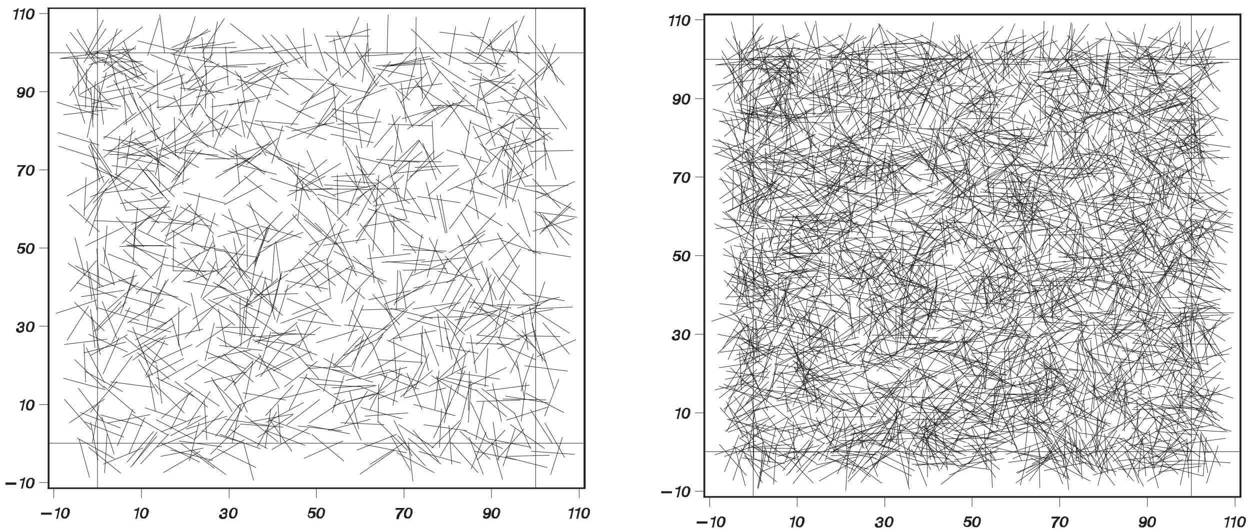

Figure 1 Two simulated random fiber webs with 1000 and 3000 fibers of length $l = 10$.

errors and biases are commonly introduced in counting the fiber intersections, we also have defined and designated the “counting area” inside the seeding area in performing computer simulations that are compatible with the newly developed theories. Thus, the theories and computer simulation methods are to be made to match each other perfectly. In essence, this paper will prove that both the theories and the computer simulation methods used in the past have been erroneous, as exhibited by the gap between the theoretical and actual counts in the published papers.

As illustrations, Figure 1 shows two simulated two-dimensional random fiber webs formed by 1000 and 3000 fibers, respectively, of length 10 thrown at random within a “seeding area” of 120 × 120. Based on computer simulations, the resulting fiber webs are found to contain 2131 and 19,303 fiber intersections respectively, for the two webs within the “counting area” (the 100 × 100 square). Similarly, Figure 2

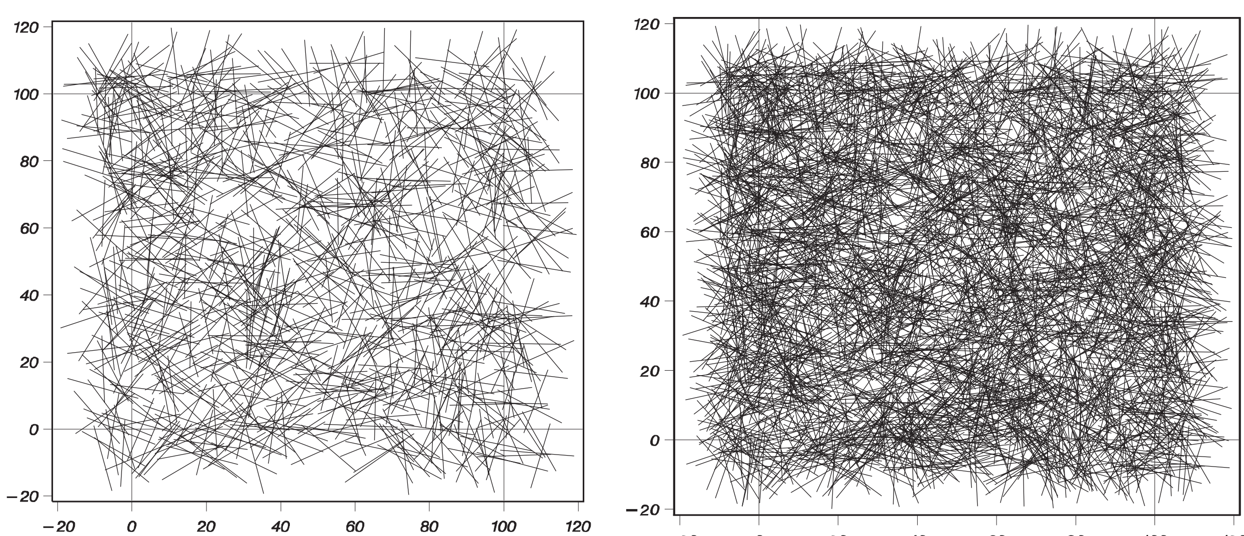

Figure 2 Two simulated random fiber webs with 1000 and 3000 fibers of length $l = 20$.

shows two simulated random fiber webs formed by 1000 and 3000 fibers of length 20 thrown at random within a "seeding area" of 140 × 140 while retaining the same "counting area". Here, the number of fiber intersections is shown to be 6119 and 53,974, respectively, for the two webs. These results are based on the theories and simulation method to be derived and explained in the subsequent sections.

It could be said that the probabilistic model relating to the number of fiber intersections has an origin at the Buffon's needle problem [3,4]. In the classical approaches, there was no distinction between what we call the "seeding area" and the "counting area" in deriving the intersection probability between any two fibers and in computing the mean and variance of the number of fiber intersections. However, this lack of distinction has led both theoretical development and simulation to a dead end.

Two earlier papers, one by Kallmes and Corte [5] and Williams et al. [6] and a follow up work by Yi et al. [7], show that the intersection probability between two straight fibers of length "l" thrown in an area A is

$$p = \frac{2l^2}{\pi A} \tag{1}$$

However, they assumed that all parts of the fibers lie entirely within the area A without considering the border effects reflecting the probability of an intersection outside A. That is, if the two fibers are located near the boundary of A, the intersection probabilities given by equation (1) would underestimate the true probability of intersection since they could form an intersection outside the designated area. In such a case, the intersection probability between two fibers depends on the common area of two regions, A, and the parallelogram formed by two fibers of length l. In terms of the intersection probabilities they have shown in their papers, the area of the parallelogram designated by $l^2|\sin\theta|$ did not completely overlap the area A in some cases, thus underestimating the true intersection probability. Under this setting, the fibers near the border were shown to have a smaller chance to form an intersection compared with those in the middle.

Kallmes and Corte [5], Piekaar and Clarenburg [8], Komori et al. [9], and Williams et al. [6] all produced the same expectation for Y, the number of fiber intersections as, $E(Y) = \binom{n}{2} 2l^2/\pi|A|$ by multiplying the intersection probability p in equation (1) by the total number of ways two fibers can be paired out of n. Aside from its validity, however, $E(Y)$ in their papers was proven to be different from the simulation results due to the bias introduced in computing the intersection probability p if nothing else.

Komori et al. [9] derived the variance of the number of intersections formed by straight fibers as a special case of curled fibers in two-dimensional fiber assemblies. Assuming that n straight fibers of length l are randomly placed within area A of a fiber assembly, they showed that the variance of the number of intersections would be

$$Var(Y) = \frac{2nl^2}{\pi A} - \frac{4nl^4}{\pi^2 A^2} = n\frac{2l^2}{\pi A}\left(1 - \frac{2l^2}{\pi A}\right)$$

This result was obtained with the following assumptions, which we will prove inappropriate in a later section:

1. The event that the ith fiber will intersect the jth fiber is independent of the event that the ith fiber will intersect other fibers.
2. The number of fiber intersections follows a binomial distribution with the probability of intersection as $p = 2l^2/\pi A$ uniformly for all possible pairs.

Williams et al. [6], on the other hand, derived the variance by introducing covariance terms to produce

$$Var(Y) = p\binom{n}{2}\left(1 - p\binom{n}{2}\right) + 2p^2\left(\binom{n}{3} + \binom{n}{2}\right)$$

where $p = 2l^2/\pi A$ as before.

In their derivation,

$$E(H_{m\text{-}n}\,H_{q\text{-}r}) = \begin{cases} 2p^2, & \text{if } m = q, \\ p^2, & \text{otherwise} \end{cases}$$

where $E(H_{m\text{-}n}\,H_{q\text{-}r})$ was defined as the event that fiber m intersects fiber n while fiber q intersects fiber r, and p is the probability for a single fiber to hit another fiber. They claimed that the probability of fiber $m = q$ hitting either n or r is $2p$ whereas the probability of fiber $m = q$ hitting the other fiber (n or r) is simply p. Thus

$$E(H_{m\text{-}n}\,H_{q\text{-}r}) = 2p \cdot p, \quad \text{where } m = q \tag{2}$$

However, the multiplication of the two probabilities requires independence of the two events that cannot be proven, even for the equal length case.

Suh [10] presented a new set of equations for $E(Y)$ and $Var(Y)$ using a conditional probability approach showing that $E(Y)$ is identical to that previously obtained but the variance must be

$$Var(Y) = \frac{l^2 n^2}{\pi A}\frac{1 + 4l^2}{\pi A}$$

His efforts in subsequent years to validate this, however, repeatedly failed first due to the limitations in sample sizes

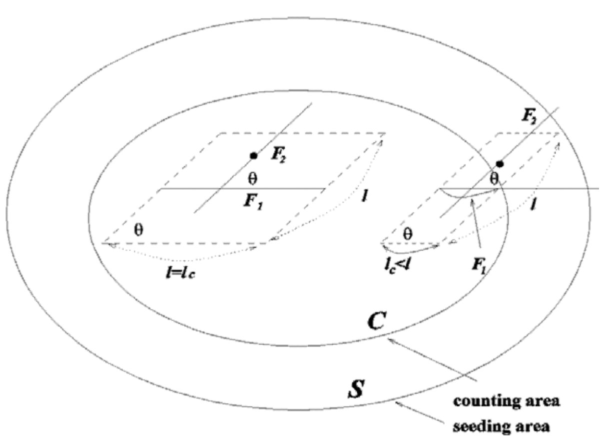

Figure 3 Intersection probability reflecting the edge effect.

owing to the enormous computing time required for the variance simulation, and secondly due to the bias generated by ignoring the edge effects. Only through the recent advance in computing speed can the validity of a theory be checked adequately against the results obtainable from simulations employing a sufficiently large number of replications, as presented in this paper.

An entirely new result will be shown below by employing an appropriate fiber intersection probability, followed by a generalization into a Poisson field, and the verification of the new theories via extensive computer simulations. Overall conclusions based on the theories and simulations will be given.

Geometrical Probability Governing the Fiber Intersections

The Probability Model for a Fiber Intersection

For theoretical modeling, we assume that the midpoints of the fibers in a two-dimensional random fiber web are uniformly distributed within a "seeding area" S with fiber orientation angles that are also uniformly distributed with respect to the horizontal axis. We then designate C to be the "counting area" located $l/2$ distance inside the boundary of the seeding area S in order to take into account the edge effects.

Suppose that two fibers of length l are dropped within area S in such a way that the midpoints of the fibers are randomly distributed within the seeding area as shown in Figure 3. Without loss of generality, we let the first fiber F_1 be parallel to the horizontal axis and the angle θ formed by the second fiber F_2 against F_1 is assumed to be uniformly distributed within $[0, \pi]$. Note that the position of the midpoint and the orientation angle θ can be assumed independent of each other. By letting $L(= l_c)$ be a random variable representing the length of the line segment of F_1 contained within C, we must have $0 \le L \le l$.

Figure 3 shows the feasible region where the F_2 of length l can intersect with the section of F_1 contained within C equal to l_c; that is, within the region, the midpoint of F_2 must lie within the parallelogram formed by the two sides that are parallel to F_1 and F_2 in such a way that the two end points of fiber segment F_1 become the midpoints of the two sides parallel to F_2 for a given angle θ formed by the two fibers.

Further, we write

$$I(F_1, F_2) = \begin{cases} 1 & \text{if } F_2 \text{ intersect } F_1 \text{ in } C, \\ 0 & \text{otherwise} \end{cases}$$

As the line segment has been restricted so that the parallelogram lies entirely within S, the edge effects can be fully corrected in the suggested probabilistic model. Since the common area of the parallelogram and C is $l_c l \sin\theta$, the intersection probability for a given θ is the ratio of this parallelogram and the area of S. Since θ is assumed to be uniformly distributed within $[0, \pi]$ with probability $1/\pi$, the conditional probability that F_2 intersects with F_1 in C given $L = l_c$ is

$$\begin{aligned} & P_r[F_2 \text{ intersects } F_1 \text{in } C \mid L = l_c] \\ &= \int_0^{\pi} P_r[I(F_1, F_2) = 1 \mid \theta, L = l_c] f(\theta) d\theta \\ &= \int_0^{\pi} \frac{l_c l \sin\theta}{|S|} \frac{1}{\pi} d\theta = \frac{2 l_c l}{\pi |S|} \end{aligned}$$

Since $E(L) = l|C|/|S|$ after taking expectation with respect to L [11], the probability of intersection between the two fibers of length l becomes

$$p = \frac{2l^2|C|}{\pi|S|^2}\left(= \frac{2E^2(L)}{\pi|C|}\right) \tag{3}$$

Equation (3) may be regarded as a form of equation (1) adjusted for the edge effects by differentiating the counting area and the seeding area in that A in equation (1) is equivalent to S when $|S|/|C| \approx 1$ for a sufficiently large S and C relative to l, thus making $p \approx 2l^2/\pi|S|$.

Expectation for the Number of Fiber Intersections

Suppose that n fibers of length l are randomly thrown onto the seeding area S with respect to their midpoints. Then, the expectation and variance of the number of fiber intersections Y are of primary importance among others.

Let us define an indicator random variable as

$$Y_{ij} = \begin{cases} 1 & \text{if fibers } i \text{ and } j \text{ meet within } C, \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$Y = \sum_{1 \le i <} \sum_{j \le n} Y_{ij}.$$

By multiplying the p in equation (3) to the total number of ways two fibers may intersect among n,

$$E(Y) = \sum_{1 \le i <} \sum_{j \le n} E(Y_{ij}) = \binom{n}{2} \frac{2l^2|C|}{\pi|S|^2} = \frac{n(n-1)l^2|C|}{\pi|S|^2} \quad (4)$$

If $|S|/|C| \approx 1$, then $E(Y) \approx n(n-1)/l^2/\pi|S|$.

Variance for the Number of Fiber Intersections

We consider now the variance $Var(Y)$ for the total number of fiber intersections Y, when all fibers are of equal lengths l. Then,

$$\begin{aligned} Var(Y) &= \sum_{1 \le i <} \sum_{j \le n} Var(Y_{ij}) \\ &+ \sum_{1 \le i <} \sum_{j \le n,} \sum_{1 \le k <} \sum_{l \le n} Cov(Y_{ij}, Y_{kl}) \end{aligned} \quad (5)$$

which will exclude the cases where $i = k$ and $j = l$ simultaneously.

Since the covariance terms would be zero unless a given fiber intersects two other fibers simultaneously, that is, $Cov(Y_{ij}, Y_{kl}) = 0$ for all $i \neq j \neq k \neq l$, the second part of equation (5) can be partitioned to three covariance terms:

$$\begin{aligned} Var(Y) &= \sum_{1 \le i <} \sum_{j \le n} Var(Y_{ij}) + 2 \sum_{i<k,} \sum_{j<k,} \sum_{i<j} Cov(Y_{ik}, Y_{jk}) \\ &+ 2 \sum_{i<j,} \sum_{j<k} \sum Cov(Y_{ij}, Y_{jk}) \\ &+ 2 \sum_{i<j,} \sum_{i<k,} \sum_{j<k} Cov(Y_{ij}, Y_{ik}) \end{aligned} \quad (6)$$

Note that Y_{ij} has a Bernoulli distribution with the intersection probability p of equation (3) but Y does not follow a binomial distribution with p due to different fiber lengths caused by the edge effect. For the first term of the above expression,

$$\sum_{1 \le i <} \sum_{j \le n} Var(Y_{ij}) = \binom{n}{2} p(1-p)$$

where $p = 2l^2|C|/\pi|S|^2$.

In obtaining the second part of equation (6), $Cov(Y_{ik}, Y_{jk})$, consider the probability that two fibers of length l simultaneously intersect the line segment l_c truncated by the counting area C.

Theorem 1. Given that the truncated length segment of a fiber (say, Fiber 1) within C is $L = l_c$, the events that two other fibers (say, Fibers 2 and 3) will intersect the length segment simultaneously are conditionally independent, or

$$\begin{aligned} &P_r(Y_{12} = 1, Y_{13} = 1 | L = l_c) \\ &= Pr(Y_{12} = 1 | L = l_c) Pr(Y_{13} = 1 | L = l_c) \end{aligned}$$

Proof: Let (X_1, Y_1) be the location of the midpoint of the Fiber 1 on the plane and Θ_1 the angle the first fiber forms with the x-axis. When the position of the first fiber is known, obviously the events that other two fibers meet the first fiber are independent of each other. Thus,

$$\begin{aligned} &Pr(Y_{12} = 1, Y_{13} = 1 | X_1 = x_1, Y_1 = y_1, \Theta_1 \\ &= Pr(Y_{12} = 1 | X_1 = x_1, Y_1 = y_1, \Theta_1 = \theta_1) \\ &\times Pr(Y_{13} = 1 | X_1 = x_1, Y_1 = y_1, \Theta_1 = \theta_1) \\ &= \left(\frac{2ll_c}{\pi|S|}\right)^2 \end{aligned} \quad (7)$$

$$\begin{aligned} &Pr(Y_{12} = 1, Y_{13} = 1 | L = l_c) \\ &= E_{X_1 Y_1 \Theta_1 | L = l_c} Pr(Y_{12} = 1, Y_{13} = 1 | L = l_c, X_1 = x_1, Y_1 = y_1, \Theta_1 = \theta_1) \\ &= E_{X_1 Y_1 \Theta_1 | L = l_c} Pr(Y_{12} = 1, Y_{13} = 1 | X_1 = x_1, Y_1 = y_1, \Theta_1 = \theta_1) \\ &= E_{X_1 Y_1 \Theta_1 | L = l_c} \left(\frac{2ll_c}{\pi|S|}\right)^2 \end{aligned}$$

The second equation holds true since the size l_c of L is determined by the location X_1, Y_1, and Θ_1.

However, because equation (7) does not depend on x_1, y_1, and θ_1, we have

$$\begin{aligned} Pr(Y_{12} = 1, Y_{13} = 1 | L = l_c) &= \left(\frac{2ll_c}{\pi|S|}\right)^2 \\ &= P_r(Y_{12} = 1 | L = l_c) P_r(Y_{13} = 1 | L = l_c) \end{aligned} \quad \square$$

From Theorem 1, therefore, the event that Fiber k meets other two Fibers i, j simultaneously is composed of two conditional independent events as

$$E(Y_{ik}Y_{jk}|L = l_c) = P_r(Y_{ik} = 1, Y_{jk} = 1|L = l_c)$$
$$= Pr(Y_{ik} = 1|L = l_c)Pr(Y_{jk} = 1|L = l_c)$$
$$= \left(\frac{2l_c l}{\pi|S|}\right)^2$$

Taking the expectation with respect to L,

$$E(Y_{ik}, Y_{jk}) = Pr(Y_{ik} = 1, Y_{jk} = 1) = \frac{4l^2E(L^2)}{\pi^2|S|^2}$$

Therefore, it follows that

$$\begin{aligned} Cov(Y_{ik}Y_{jk}) &= E(Y_{ik}Y_{jk}) - E(Y_{ik})E(Y_{jk}) \\ &= \frac{4l^2E(L^2)}{\pi^2|S|^2} - \left(\frac{2l^2|C|}{\pi|S|^2}\right)^2 \\ &= \left(\frac{2l}{\pi|S|}\right)^2\left(E(L^2) - \left(\frac{l|C|}{|S|}\right)^2\right) = \left(\frac{2l}{\pi|S|}\right)^2 Var(L) \end{aligned}$$

Since the number of ways three different fibers i, j and k can be chosen from n fibers is $\binom{n}{3}$, and by noting that

$$Cov(Y_{ij}Y_{jk}) = Cov(Y_{ij}Y_{ik}) = Cov(Y_{ik}Y_{jk})$$

the total number of covariance terms by combining the second, third and fourth terms of equation (6) is $6\binom{n}{3}$. Hence, the variance of the total number of fiber intersection is given by

$$\begin{aligned} Var(Y) &= \binom{n}{2}\frac{2l^2|C|}{\pi|S|^2}\left(1 - \frac{2l^2|C|}{\pi|S|^2}\right) + 6\binom{n}{2}\left(\frac{2l}{\pi|S|}\right)^2 Var(L) \\ &= \frac{n(n-1)l^2|C|}{\pi|S|^2}\left(1 - \frac{2l^2|C|}{\pi|S|^2}\right) \\ &+ \frac{4n(n-1)(n-2)l^2Var(L)}{\pi^2|S|^2} \end{aligned}$$

For those situations where $|S|/|C| \approx 1$, it can be written as

$$Var(Y) \approx \frac{n(n-1)l^2}{\pi|S|}\left(1 - \frac{2l^2}{\pi|S|}\right) + \frac{4n(n-1)(n-2)l^2Var(L)}{\pi^2|S|^2}$$

where $Var(L) = E(L^2) - E^2(L)$.

For a square counting region, it has been shown [11] for $Pr(L \geq l)$,

$$Pr(L \geq t) = \frac{|C| + \{6t^2 - 2t(P + l) + Pl\}/\pi}{|S|}, \ t > 0$$

where P is a perimeter of the counting region, and thus omitting the algebraic details,

$$E\left(L^2\right) = \int_0^1 2tPr(L \geq t)dt = l^2\frac{|C|}{|S|}\left(1 - \frac{Pl}{3\pi|C|}\right) + \frac{l^4}{3\pi|S|}$$

Fiber Intersections in a Poisson field

The results above on the mean and variance of the number of fiber intersections were obtained with a fixed number of fibers n of equal length l. Considering the number of fibers as a random variable following a Poisson distribution, the unconditional mean and variance can be obtained. This concept is similar to that presented by Stoyan and Stoyan [12] or Stoyan et al. [13] based on a Poisson field. If a given fiber is thrown randomly onto a very large field F that contains the seeding area S, the probability that the midpoint of the fiber will be found within S would be $p = |S|/|F|$ as limiting form of a binomial probability. For a large n and $|F|$, we may write $n/|F| = \lambda$, a Poisson fiber density parameter per unit area. Then, N, the number of fibers to be found within S, follows a Poisson distribution with the mean density $\mu = \lambda|S|$. Therefore,

$$Pr[N = k] = \frac{\mu^k e^{-\mu}}{k!}.$$

and

$$E(N) = \mu, \ E(N^2) = \mu^2 + \mu. \ E(N^3) = \mu^3 + 3\mu^2 + \mu \quad (9)$$

Rewriting equation (4), the conditional expectation for the number of fiber intersections for a given N is

$$E(Y|N) = \frac{N(N-1)l^2|C|}{\pi|S|^2}$$

Taking the expectation with respect to N and by substituting the two moments of N given by equation (9),

$$\begin{aligned} E(Y) &= EE(Y|N) = \frac{l^2|C|}{\pi|S|^2}\left(E(N^2) - E(N)\right) \\ &= \frac{\mu^2l^2|C|}{\pi|S|^2} = \frac{\lambda^2l^2|C|}{\pi} \end{aligned}$$

From equation (8), the conditional variance for the number of fiber intersections for a given N is

$$Var(Y|N) = \frac{N(N-1)l^2|C|}{\pi|S|^2}\left(1-\frac{2l^2|C|}{\pi|S|^2}\right) + \frac{4N(N-1)(N-2)l^2Var(L)}{\pi^2|S|^2}$$

It is well known that

$$Var(Y) = EVar(Y|N) + VarE(Y|N) \tag{10}$$

After taking the expectation and by applying the first three moments of N shown in equation (9), the first part of equation (10) becomes

$$\begin{aligned} EVar(Y|N) &= \frac{l^2|C|}{\pi|S|^2}\left(1-\frac{2l^2|C|}{\pi|S|^2}\right)\left(E(N^2)-E(N)\right) \\ &+ \frac{4l^2Var(L)}{\pi^2|S|^2}\left(E(N^3)-3E(N^2)+2E(N)\right) \quad (11) \\ &= \frac{\mu^2l^2|C|}{\pi|S|^2}\left(1-\frac{2l^2|C|}{\pi|S|^2}\right)+\frac{4\mu^3l^2Var(L)}{\pi^2|S|^2} \end{aligned}$$

The second part of equation (10) is

$$\begin{aligned} VarE(Y|N) &= \frac{l^2|C|^2}{\pi^2|S|^4}Var(N^2-N) \\ &= \frac{2\mu^2(2\mu+1)l^4|C|^2}{\pi^2|S|^4} \end{aligned} \tag{12}$$

since $Var(N^2 - N) = 2\mu^2(2\mu + 1)$.

Finally, combining equations (11) and (12) together, we obtain

$$Var(Y) = \frac{\mu^2l^2|C|}{\pi|S|^2} + \frac{4\mu^3l^2}{\pi^2|S|^2}\left(Var(L) + l^2\frac{|C|^2}{|S|^2}\right)$$

Using $E(L) = l|C|/|S|$ and replacing μ by $\lambda|S|$,

$$Var(Y) = \frac{\lambda^2l^2}{\pi^2}\left\{\pi|C| + 4\lambda|S|E(L^2)\right\} \tag{13}$$

Note that by writing $Var(L) = E(L^2) - E^2(L)$ and by replacing n by N, equation (8) can be shown to be equivalent to equation (13) since $E(L) = l|C|/|S|$ and $N/|S| \to \lambda$ in probability when $|S| \to \infty$.

Simulations and Discussions

Although the foregoing probability theories on fiber intersections are proven to be quite complex as they are imbedded in the non-woven random web geometry, their validity can only be established through a proper Monte Carlo simulation method that matches the theories. In fact, the difficulty in assessing the validity of the prior published works was due to our inability to explain the systemic gap between the theories and simulation results. Simply, it was not possible to explain if the gap between the two was due to a wrong theory, or an inappropriate simulation method, or both. Needless to say, the following simulation methods and the theories presented in the prior section underwent repeated revisions after repeated failures without enumerating the wrong paths we had taken and abandoned subsequently. It is quite heartening to point out that the Monte Carlo simulation we present in the following was never possible until the late 1990s due to the limitation in the computing speed and the magnitudes of simulation required for the validation of the theories.

Omitting the details, the following procedure, as illustrated by Figure 4, was applied for the Monte Carlo simulation of the mean and variance of the number of fiber intersections for fibers of equal lengths = l. It is assumed that the numbers of fibers are fixed and that the orientation angles of fibers with respect to horizontal axis are random.

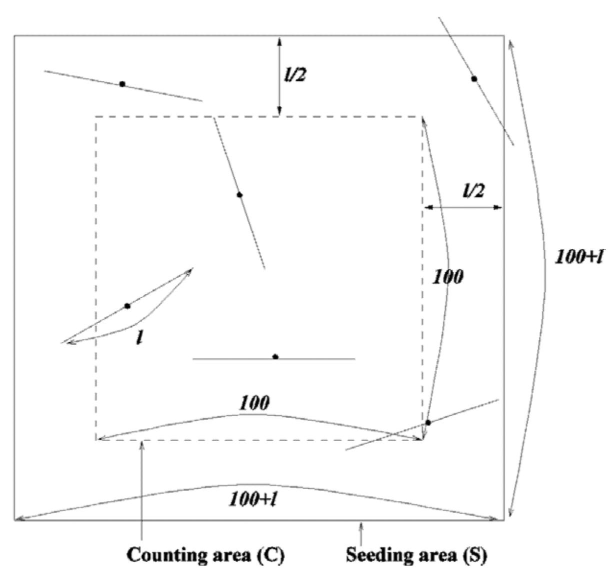

Figure 4 *Seeding Area* and *Counting Area* for simulation – an illustration.

1. Generate n fiber midpoints on the “seeding area,” $(100 + l) \times (100 + l)$ square, at random.
2. Obtain two fiber endpoints by assigning an orientation angle θ $[0, \pi]$ at random against the horizontal axis for every fiber.
3. Count the number of fiber intersections found only within the “counting area,” 100×100 square, located within $0.5\, l$ distance from the “seeding area.” All possible pairs formed from n fibers are checked. 3.
4. Calculate $E(L^2)$ and $Var(L)$.
5. Finally obtain $E(Y)$ and $Var(Y)$.

The sizes of the “seeding area” and the “counting area” shown in Figure 4 can be modified at will relative to the fiber length l and the density of the web to be examined.

Figure 5 and Figure 6 show the comparisons between theoretical values and the simulation averages for the mean and variance, respectively, of the number of fiber

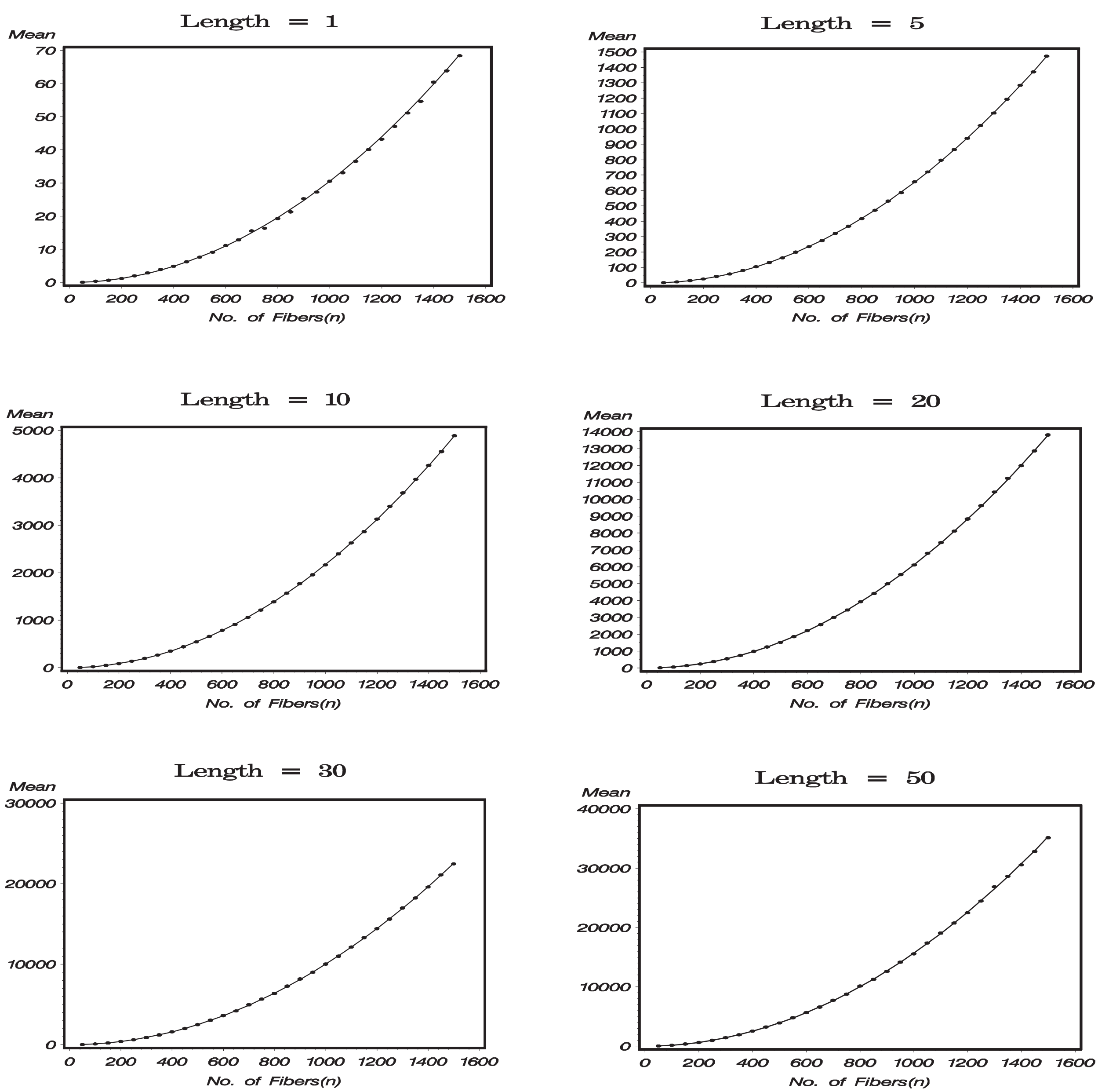

Figure 5 Comparison of theoretical results and simulations for $E(Y)$.

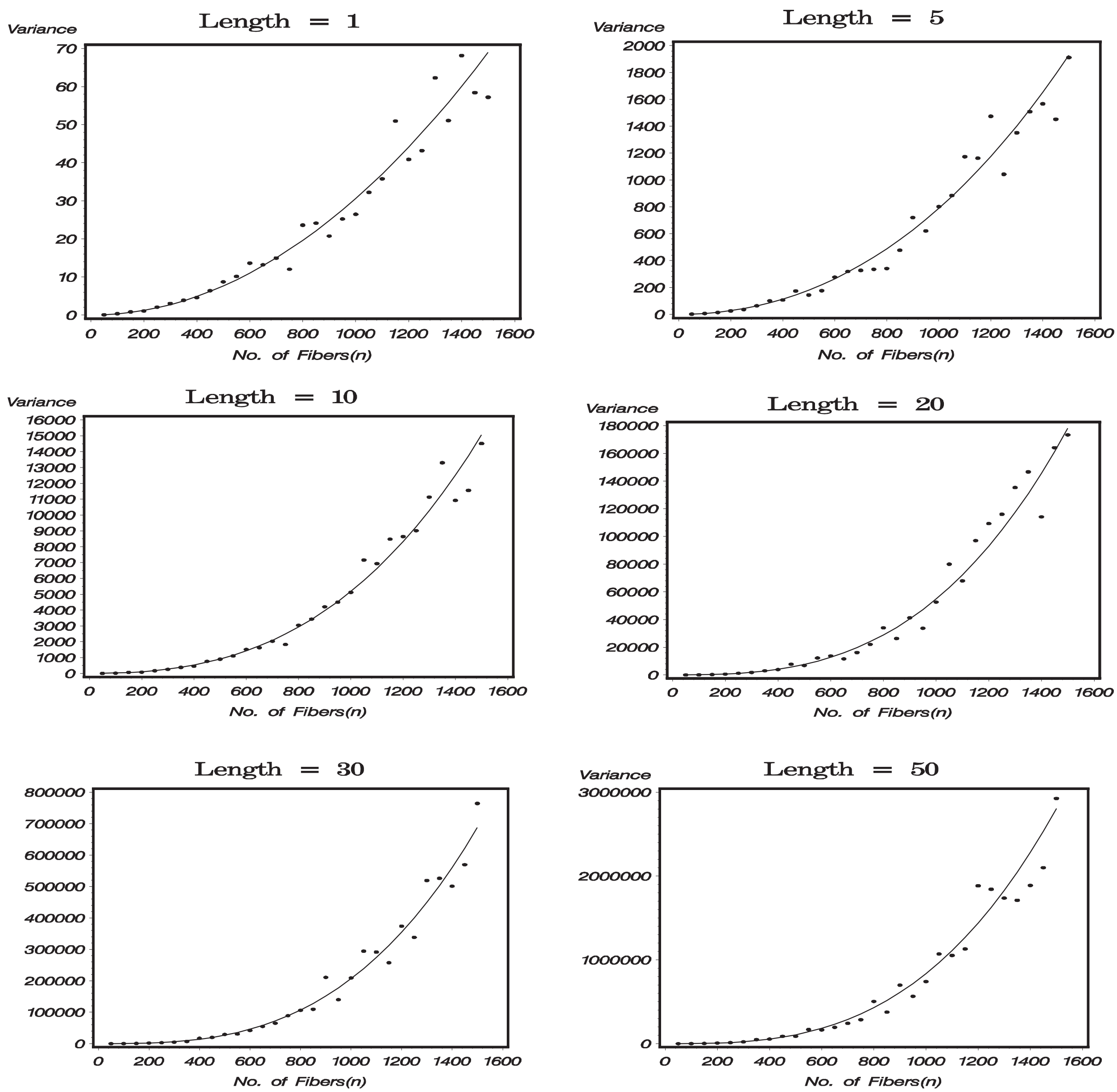

Figure 6 Comparison of theoretical results and simulations for *Var*(*Y*).

intersections. For both the mean and variance, the continuous lines denote the theoretical values by incorporating the edge effects, and each dot represents the average of 100 simulation replicates. For each fiber length l = 1, 5, 10, 20, 30 and 50, the simulations were run with the number of fibers n = 100, 150, …, 1500 with increments of 50 in such a way that the fiber midpoints are designated at random within the seeding area $(100 + l) \times (100 + l)$ and the number of intersections was counted within (100×100) counting area for each simulation run. For any given combination of l and n in Figures 4 and 5, therefore, a total of 2900 simulations were run to show the twenty-nine averages. Due to the large number of comparisons involved in the computational algorithms, the time requirements for computation used to be a huge bottleneck. Thanks to progress in computing science this task has now become manageable, enabling us to obtain Figures 5 and 6 with a moderate amount of effort.

As shown clearly in Figure 5, the simulated means are almost identical to the theoretical means based on the introduction of seeding area and counting area in order to eliminate the edge effects, thus validating the new theories imbedded in the edge effects. On the variances, Figure 6 also shows that all the simulated variances agree with the theoretical variances based on the new theories and the new simulation algorithms that match the theories developed. While the variance data are plotted in actual numbers here, they should be evaluated in terms of coefficient of variation, or the ratio of the standard deviation and the mean, for the right perspective. Based on the variances figures alone, the gaps between the theories and the simulated results are quite small as shown by the correlation coefficients of 0.977, 0.982, 0.987, 0.985, 0.987, 0.981, 0.978, respectively, for fiber lengths $l = 1, 5, 10, 20, 30$ and 50. However, when the variances are converted into CV%, or the ratio between the standard deviation and the mean, the gaps become uniformly negligible for small as well as large numbers of fibers and across all fiber lengths.

Concluding Remarks

New theories and the matching simulation algorithms derived for the mean and variance of the number of fiber intersections in a two-dimensional Poisson field produced, for the first time, near perfect agreements when the size of the seeding area, fiber length and number of fibers are fixed for equal fiber length case. Geometrical probability of fiber intersection and the ensuing first and second moments of the number of intersections were found deeply hidden in the covariance structure of the events with which any two fibers can intersect each other in the presence of another fiber that may intersect with them. While an absolute and lasting correctness in science has to be borne out by time, it is quite clear that the simulation results presented in this paper point to insurmountable evidence that the theories are valid. Equally significant is the fact that some of the theories and the simulation methods used by other researchers should be reexamined in this light. The implications of these results for uniformity and physical properties of random webs are many, and should be handled as a separate paper.

The current work no doubt leads to cases where the fiber lengths are equal and/or the orientation angles of the fibers are not random but partially directional. In addition, the equations of the mean and variance formula could be applied for design of non-woven structures that impart certain required geometrical, structural or mechanical properties. Some of these are being explored at the moment will be reported as a sequel.

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Finally, the senior author has to thank his wife Chisook for her loaning the entire inventory of chopsticks during many dinners in order to facilitate the "visual demonstrations" as to how the chopsticks land on each other under semi-random throws. Ironically, the "edge effects" reported here and the correction methods owe a lot to her patience and interest in the then-mysterious chopstick experiments conducted during the many *probabilistic dinners*!

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