USING COMPUTER ALGEBRA SYSTEMS TO TEACH GRADUATE MATHEMATICAL STATISTICS: POTENTIAL AND PITFALLS

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Computer algebra systems (CASs), e.g., Maple or Mathematica, are potentially useful tools in the teaching of mathematical statistics. Calculus techniques used to solve mathematical statistics problems may be difficult or tedious to perform and rarely shed any statistical understanding on the problem. For masters students, CASs hold the potential for helping them understand the concepts without the calculus drudgery. But for doctoral students, we should ask whether CASs are powerful enough or whether these students will need more traditional calculus skills to do useful research. Several examples show that a CAS can provide completely wrong answers to relatively simple problems. This leads to the question of whether it is valuable to encourage any students to use a CAS. It is concluded that, at best, a CAS can be viewed only as one among many useful tools.

1. Introduction

Computer algebra systems (CASs) are computer programs that perform symbolic algebra and calculus manipulations. These mathematical tools are often used in masters level mathematical statistics courses. Thus, CASs are potentially very useful in the teaching of these courses. The algebra and calculus techniques are only tools for obtaining solutions. In themselves, they do not illustrate any *statistical* concepts. If the CASs can help the students learn the statistical concepts, without stumbling over integration-by-parts or trigonometric substitutions, then the teaching of the statistics could be enhanced.

The two most widely used CASs are Mathematica and Maple. All my examples will be using Maple V, Release 4. But, the same potential and pitfalls I will discuss apply to other CASs as well.

The masters level mathematical statistics courses I will discuss usually are one-year courses that include an introduction to probability theory and mathematical statistics. Textbooks that are commonly used at this level are Hogg and Craig (1995), Mood, Graybill, and Boes (1974), and Casella and Berger (1990). The students in these courses are typically statistics, economics, engineering, and mathematics students. They include students seeking a terminal masters degree, as well as students planning to continue to pursue a doctoral degree.

Several authors have discussed the use of CASs in mathematical statistics courses.

Glen, Leemis, and Drew (1997), Cook and Broemeling (1995) and Baglivo (1995) give examples of the types of problems that can be solved. Karian and Tannis (1995) provide a library of Maple functions specifically useful for mathematical statistics. In this paper I will give examples of problems that can arise in using a CAS and discuss the impact of these problems on the usefulness of CASs in such courses.

2. Good Results

Students in these classes often spend a lot of time learning or remembering calculus and algebra techniques that are tedious, if not difficult. These techniques, in themselves, really do not teach any statistical concepts. The time spent struggling with these techniques could be better spend learning statistics. If a CAS can remove the difficulty of the techniques, to get more quickly to the statistical result, it will be a very helpful tool.

A typical problem in the introductory probability portion of this kind of course is to find the moment generating function of a normal distribution and use it to calculate several moments. The maple code

```
> assume(sigma>0);
> pdf:=exp(-(x-mu)^2/(2*sigma^2))/(sqrt(2*Pi)*sigma);
> mgf:=int(exp(t*x)*pdf,x=-infinity..infinity);
> seq(simplify(subs(t=0,diff(mgf,t$n))),n=1..4);
```

quickly returns the result that the mgf is

$$e^{\left(\frac{1}{2}t(t\sigma^2+2\mu)\right)}$$

and that the first four moments are

$$\mu, \quad \sigma^2+\mu^2, \quad 3\sigma^2\mu+\mu^3, \quad \text{and} \quad 3\sigma^4+6\mu^2\sigma^2+\mu^4.$$

I think this is a very good illustration of the usefulness of a CAS. “Completing the square” is typically used to calculate the mgf. Students need to be reminded of this technique, if they have not used it in a few years, and it involves some messy algebra. But more importantly, completing the square does not illustrate any statistical concept. The subsequent differentiation to obtain the moments is also messy, and students are prone to make mistakes. But the CAS does the calculation quickly and correctly. And there is no need to stop at the fourth moment. The CAS just as easily calculates the tenth moment, for example.

3. Questionable Results

However, a CAS is not so successful on all problems. Consider again the calculation of a mgf, this time for a negative binomial distribution. The Maple code

```
> pmf:=binomial(r+x-1,x)*p^r*(1-p)^x;
> mgf:=sum(exp(t*x)*pmf,x=0..infinity);
```

returns the result that the mgf is $p^r(1-e^t+e^tp)^{-r}$. This is not exactly correct. This is the formula for the mgf if $t < -\ln(1-p)$. But, the sum does not converge and the mgf does not exist for $t \geq -\ln(1-p)$. There is no mention of this domain of convergence in the Maple output.

Some may argue that this is not too important. The above formula is the correct formula at $t=0$. Evaluating the derivatives of this function at $t=0$ does give the correct moments of the negative binomial distribution, and the CAS can be used to do this calculation easily.

But, in a graduate level course, we should expect our students to exhibit mathematical rigor and worry about things like domains of convergence. Especially the doctoral bound students in the course need to exercise proper care for details which may be important in their later research. At the least, this example illustrates that some care must be taken in accepting the output of a CAS.

4. No Results

There are, of course, problems that a CAS cannot solve. Unfortunately, some of these problems are not too difficult and do arise in a masters level course.

Consider the problem of finding the pdf of the sum of two independent Cauchy$(0,1)$ random variables. A standard result that a student should learn in this course is that this pdf is given by the convolution formula,

$$f(x) = \int_{-\infty}^{\infty} g(z)g(x-z)\,dz,$$

where $g(z)$ is the Cauchy$(0,1)$ pdf. This is a rather difficult integration. One technique for solving it involves expressing the integrand using a partial fraction decomposition. Again, this integration technique does not teach any statistical concepts. One would hope that a CAS could help avoid this work. Unfortunately, the Maple code

```
> int((1/(Pi*(1+z^2)))*(1/(Pi*(1+(x-z)^2))),
> z=-infinity..infinity);
```

simply returns the unevaluated integral. That is, Maple is unable to do this integration. Thus, hope that students could learn interesting statistical facts by doing challenging calculus problems using a CAS is not always realized.

Curiously, Maple can solve this problem if it is expressed as

$$f(x) = \int_{-\infty}^{0} g(z)g(x-z)\,dz + \int_{0}^{\infty} g(z)g(x-z)\,dz.$$

That is

```
> int((1/(Pi*(1+z^2)))*(1/(Pi*(1+(x-z)^2))),
> z=-infinity..0)+
> int((1/(Pi*(1+z^2)))*(1/(Pi*(1+(x-z)^2))),
> z=0..infinity);
```

returns the correct result that the pdf is

$$\frac{2}{\pi(x^2+4)},$$

a Cauchy$(0, 2)$ pdf. But, I consider this a curiosity rather than a useful feature of Maple. There seems to be no reason why the student should think that the range of integration needs to be broken into two parts. (I cannot remember how I discovered this trick.) Teaching the student to try this does not appear to be a useful skill. There is no reason to believe this trick will work on a different problem. Thus, students who are going to tackle challenging calculus problems, either as classwork or in later doctoral research, will need to have the full range of calculus skills. Encouraging them to rely on a CAS to solve difficult problems is probably unwise.

5. Wrong Results

Worse than not being able to solve a problem, however, is when the CAS gives an answer to a problem but it is completely wrong. Unfortunately, this too can happen for typical problems in a masters level course, as the following example illustrates.

Consider a game between two players, A and B. A coin is tossed until the first Head appears. If the H appears on the xth toss, and x is odd, A pays B 2^x dollars. But, if x is even, B pays A 2^x dollars. Let X denote the random toss on which the first H appears.

Then, X has a geometric(p) distribution, where the probability of a H is p on each toss. A's winnings are $(-2)^X$, and A's expected winnings are

$$\mathrm{E}\,(-2)^X = \sum_{x=1}^{\infty}(-2)^x p(1-p)^{x-1}.$$

The following Maple code to compute the expected winnings, if the coin is fair,

```
> pmf:=p*(1-p)^(x-1);win:=(-2)^x;p:=1/2;
> sum(pmf*win,x=1..infinity);
```

returns the expected winnings of A as $-1/2$ dollar. This seems like a reasonable answer and would probably be accepted by the student. But closer examination reveals that the successive terms in the sum are $-1, 1, -1, 1, -1, \ldots$, and the sum does not converge at all! The expected winnings do not exist.

6. Conclusions

The examples in this paper illustrate that CASs, in their present level of development, are not very reliable. They sometimes give the correct answer only when the problem is posed in an odd form, like in Section 4, or they can give completely erroneous results with no warning that there is any difficulty, like in Section 5. If students are encouraged to use a CAS in a graduate level course, they must also be encouraged to check their answers by independent means. Thus, students will probably need the same algebra and calculus skills they need now, just to verify that answers given by a CAS are, in fact, correct.

So, is there any advantage to the use of a CAS? I can see a use in a masters level course that is taken only by students who are to learn an appreciation of the mathematical theory behind statistical procedures, but are never expected to do these kinds of calculations in the future. For these students, freedom from the calculus details might help them understand the theory. But the problems that they are shown or are assigned need to be carefully chosen so that they were not exposed to misleading or erroneous results. And, it would be an odd situation in which the students would not be encouraged to explore other problems on their own, lest they stumble upon a problem for which the CAS gave the wrong answer.

There seems to be much less advantage for a doctoral bound student in such a class. Such a student will be expected to produce original results in a dissertation. Relying on a

CAS as a tool would be unwise. Any result from a CAS would need to be verified by independent means, that is, by using the skills presently taught in a non-CAS course. For these students, the use of a CAS appears to be simply an additional, not extremely useful, skill to be mastered.

Nevertheless, I think that CASs will play ever more important roles in mathematical statistics courses in the future. Mathematica and Maple both seem to be very successful and are continuously being improved. They are used in many undergraduate mathematics courses. If those undergraduate students are expected to use these tools in graduate level courses, the CASs will need to be continuously improved. I believe that his improvement will continue and that, five, ten, or fifteen years from now, the solution from a CAS will be accepted as true in masters level mathematical statistics courses.

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