TESTING WHETHER ONE REGRESSION FUNCTION IS LARGER THAN ANOTHER

Roger L. Berger

Department of Statistics
North Carolina State University
Raleigh, North Carolina 27695-8203

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ABSTRACT

The problem of testing whether one regression function is larger than another on a specified set R is considered. The regression functions must be linear functions of the parameters but need not be linear functions of the independent variables. The proposed test has an exactly specified size in typical situations. The test's critical value is a standard t percentile. The power function of the test is investigated.

1. INTRODUCTION

In this paper we consider testing whether the regression function from one population is above the regression function from another population for all values of the independent variable in a specified set.

Tsutakawa and Hewett (1978), Hewett and Lababidi (1980) and Spurrier, Hewett and Lababidi (1982) have previously considered
this testing problem. Their tests assume that the regression functions are linear functions of the independent variable. Their tests cannot be used if some form other than linear is assumed for the regression functions. The model considered herein generalizes the models of Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980) in three ways. First the regression functions may be functions other than linear functions of the independent variables. For example, the regression functions may be quadratic or higher degree polynomials. Second, the two regression functions need not be of the same functional form. For example, one may be a linear function and the other a quadratic function. Third, the set of values of the independent variable of interest to the experimenter need not be a hypercube as required by these two models. The test proposed herein reduces to the tests of Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980) for the special models they consider.

In Section 3 an example is considered. It concerns comparison of particulate emissions (dependent variable) over a range of flue openings (independent variable) for large and small flues (populations). The models and tests of the previous authors are inappropriate for this data because a linear regression provides a poor fit for the data. But the model and test proposed in Section 2 may be used because a quadratic regression provides a good fit. Computational aspects of the test are discussed in Section 4. The results of a simulation study investigating the power of the proposed test are discussed in Section 5. Theoretical results regarding the size of the test and equivalence with other tests are proved in Section 6.

2. MODEL AND TEST

Let \( \{X_{1j}, Y_{1j}\}, j = 1, \ldots, n_1 \) and \( \{X_{2j}, Y_{2j}\}, j = 1, \ldots, n_2 \) denote two independent sets of observations where \( X_{1j} = (X_{1j1}, \ldots, X_{1jk}) \). The independent variables \( X_{ij} \) may be observed random vectors or design variables fixed by the experimenter.
The entire analysis is conditioned on the observed values of \( X_{ij} \).
Let \( R \) denote a set of values of \( X_{ij} \) of interest to the experimenter. We assume that given the \( X_{ij} = x_{ij} \), the dependent variables \( Y_{ij} \) are independent normal random variables with mean and variance given by

\[
E(Y_{ij} | X_{ij} = x_{ij}) = \sum_{m=1}^{p_i} \beta_{im} f_{im}(x_{ij}) = f_i(x_{ij}) \beta_i
\]

and

\[
\text{Var}(Y_{ij}) = \sigma^2.
\]

\( \beta_i = (\beta_{i1}, \ldots, \beta_{ip_i})' \), \( i = 1, 2 \), and \( \sigma^2 \) are unknown parameters. The \( f_i(x) = (f_{i1}(x), \ldots, f_{ip_i}(x)) \), \( i = 1, 2 \), are known vectors of functions which define the functional form of the regression functions.

By allowing \( p_1 \neq p_2 \) and \( f_{1m}(x) \neq f_{2m}(x) \), this model allows the two regression functions to have different functional forms. The first might be a linear function and the second a quadratic function. As mentioned in Section 1, previous models for this problem required both regression functions to be linear functions of the independent variable.

We wish to compare the regression functions \( f_1(x) \beta_1 \) and \( f_2(x) \beta_2 \). In particular we are interested in whether \( f_1(x) \beta_1 \) is greater than \( f_2(x) \beta_2 \) on \( R \). The test we will propose is a size \( \alpha \) test of

\[
H_0: f_1(x) \beta_1 \leq f_2(x) \beta_2 \quad \text{for at least one } x \in R
\]

versus

\[
H_A: f_1(x) \beta_1 > f_2(x) \beta_2 \quad \text{for every } x \in R.
\]

Let \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) denote the least squares estimates of \( \beta_1 \) and \( \beta_2 \) and let

\[
s^2 = \frac{2}{\nu} \sum_{i=1}^{n_1} \sum_{j=1}^{p_i} (Y_{ij} - f_i(x_{ij}) \hat{b}_i)^2 / \nu
\]

denote the pooled estimate of \( \sigma^2 \) where \( \nu = n_1 - p_1 + n_2 - p_2 \). The
estimate \( b_i \) has a multivariate normal distribution with mean \( \beta_i \) and covariance matrix \( \sigma^2 D^{-1} \) where the \((m, n)\) element of \( D \) is
\[
\sum_{j=1}^{n_i} f_{im}(x_{ij}) f_{in}(x_{ij}).
\]
Let \( e(x) = f_1(x)D^{-1} f_1'(x) + f_2(x)D^{-1} f_2'(x) \). Then the variance of 
\( f_1(x)b_1 - f_2(x)b_2 \) is \( \sigma^2 e(x) \).

The test we propose for testing \( H_0 \) versus \( H_A \) is based on the test statistic \( T \) defined by
\[
T = \min_{x \in \mathbb{R}} T(x)
\]
where
\[
T(x) = \frac{f_1(x)b_1 - f_2(x)b_2}{\sqrt{e(x)}}.
\]
The test rejects \( H_0 \) in favor of \( H_A \) if and only if \( T > t_{1-\alpha, v} \) where
\( t_{1-\alpha, v} \) is the \( 1 - \alpha \) percentile of a \( t \) distribution with \( v \) degrees of freedom.

This test is always a level \( \alpha \) test in that the probability of a type one error is always less than or equal to \( \alpha \). This test has size exactly equal to \( \alpha \) if (2.3), (2.4), and (2.5) are true.

\( R \) is a closed and bounded, i.e., compact, subset of \( \mathbb{R}^k \).
The functions \( f_{ij}(x) \) are continuous functions on \( R \).

There are values of \( \beta_1 \) and \( \beta_2 \) such that 
\( f_1(x)\beta_1 = f_2(x)\beta_2 \) for one value of \( x \in \mathbb{R} \) and 
\( f_1(x)\beta_1 > f_2(x)\beta_2 \) for all other \( x \in \mathbb{R} \).

These facts are proved in Section 6. Two examples of when (2.3), (2.4), and (2.5) are satisfied and the size is exactly \( \alpha \) are also given in Section 6.

The test we have proposed may be motivated in this way. It rejects \( H_0 \) if and only if for each \( x \in \mathbb{R} \), the individual test of
\( H_{0x}^*: f_1(x)\beta_1 \leq f_2(x)\beta_2 \) versus \( H_{A}^*: f_1(x)\beta_1 > f_2(x)\beta_2 \) based on the test statistic \( T(x) \) rejects \( H_{0x}^* \). Such tests are called
intersection-union tests. They have been discussed by Gleser (1973), Berger (1982) and Berger and Sinclair (1984).

3. APPLICATION

As an application of the test proposed in Section 2, consider the air pollution data for wood stoves found in Table I. The independent variable $X$ is the air intake setting; values of .25, .50, .75 and 1.00 = fully open were used in the study. Table I contains data from two populations corresponding to large (population 1) and small (population 2) flue sizes.

Suppose the experimenter wishes to compare the average particulate matter vented over the range of air intake settings $R = \{x: .50 \leq x \leq 1.00\}$. Perhaps this range is of interest because the settings people use most often are in this range. In particular suppose the experimenter wishes to determine if the average emissions from the small flue are lower than the average emissions from the large flue. If this were true then the small flue could be considered better than the large flue in controlling emissions.

The data from Table I are plotted in Figure 1. The plot indicates that a linear regression will provide a poor fit to both the large and small flue data. Indeed, $R^2 = .16$ for the large

<table>
<thead>
<tr>
<th>Setting (fraction open)</th>
<th>Emissions (percent light blocked)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Large flue</td>
</tr>
<tr>
<td>.25</td>
<td>42, 40, 39</td>
</tr>
<tr>
<td>.50</td>
<td>26, 26, 30</td>
</tr>
<tr>
<td>.75</td>
<td>29, 27, 29</td>
</tr>
<tr>
<td>1.00</td>
<td>34, 33, 34</td>
</tr>
</tbody>
</table>

flue and $R^2 = .23$ for the small flue when simple linear regressions are performed. Since linear regressions provide such a poor fit, the tests of Tsutakawa and Hewett (1978), Hewett and Lababidi (1980), and Spurrier, Hewett and Lababidi (1982) are inappropriate for this problem. But quadratic regressions fit the data from both populations well. For quadratic regressions, $R^2 = .90$ for the large flue and $R^2 = .96$ for the small flue.

In an effort to determine if the small flue is better than
the large flue for \(0.50 \leq x \leq 1.00\), consider testing

\[
H_0: \quad \beta_{11} + \beta_{12}x + \beta_{13}x^2 \leq \beta_{21} + \beta_{22}x + \beta_{23}x^2
\]

for some \(0.50 \leq x \leq 1.00\)

versus

\[
H_A: \quad \beta_{11} + \beta_{12}x + \beta_{13}x^2 > \beta_{21} + \beta_{22}x + \beta_{23}x^2
\]

for all \(0.50 \leq x \leq 1.00\).

The estimated regression functions are \(60.08 - 99.27x + 73.33x^2\)
for the large flue and \(68.00 - 124.80x + 90.67x^2\) for the small flue. These regression functions are shown in Figure 1.

The pooled estimate of \(\sigma^2\) is \(s^2 = 53.08/18 = 2.95\). The correlation matrices are

\[
D^{-1}_1 = D^{-1}_2 = \begin{pmatrix}
2.58 & -9.00 & 6.67 \\
-9.00 & 34.40 & -26.67 \\
6.67 & -26.67 & 21.33
\end{pmatrix}.
\]

Thus the test statistic \(T\) for testing \(H_0\) versus \(H_A\) is

\[
T = \min_{0.5 \leq x \leq 1} T(x)
\]

\[
= \min_{0.5 \leq x \leq 1} \frac{-7.92 + 25.53x - 17.34x^2}{1.72\sqrt{5.16 - 36.00x + 95.40x^2 - 106.68x^3 + 42.66x^4}}.
\]

The function \(T(x)\) is graphed in Figure 2 and is clearly minimized for \(0.5 \leq x \leq 1\) at \(x = 1\). Thus \(T = T(1) = 0.21\). Using \(\alpha = 0.10\), we find \(t_{0.10, 18} = 1.333\). The test does not reject \(H_0\) since \(T < 1.333\). We cannot conclude that the average emissions for large flues is greater than the average emissions for small flues for all settings between 0.5 and 1.
In (2.1) the test statistic $T$ was defined as the minimum of $T(x)$ over the set $R$. Typically the computation of the test statistic will be accomplished by a numerical minimization of $T(x)$. If $x$ is univariate the evaluation of this minimum might be accomplished by means of a graph of $T(x)$, such as Figure 2. The problem of minimizing a function such as $T(x)$ which is the ratio of two functions of $x$ has been studied extensively in the
mathematical programming literature by Charnes and Cooper (1962), Swarup (1965), Sharma (1967) and Craven and Mond (1973, 1975a, and 1975b). These authors have found that this nonlinear programming problem is equivalent to other nonlinear programming problems which do not involve fractions. These results could simplify the numerical minimization of $T(x)$.

But to perform the test the actual value of $T$ need not be computed. One only needs to know whether $T > t_{1-\alpha,\nu}$ or $T \leq t_{1-\alpha,\nu}$. In this section we describe two shortcuts for making this determination without the exact computation of $T$.

4.1 Shortcut for determining if $H_0$ is accepted

Let $X^*$ denote an arbitrary finite subset of $R$. For example, if $R = \{x_i: x_i^* \leq x_i \leq x_i^*, i = 1,\ldots,k\}$, $X^*$ might be the set of $2^k$ extreme points $(x_1,\ldots,x_k)$ where $x_i = x_i^*$ or $x_i^*$. Let $T' = \min_{x \in X^*} T(x)$. Since $T \leq T'$, if $T' \leq t_{1-\alpha,\nu}$ accept $H_0$.

The results of a simulation study are given in Tables I and III. This study is described in Section 5. In Tables I and III the second (middle) number for each entry is the proportion of the acceptances which were determined by the shortcut method. $X^* = \{1, -1\}$ was used. These values indicate that the usefulness of this shortcut depends on the actual regression functions. But, in many cases, a large proportion of the acceptances were determined by this shortcut. In 18 out of the 62 cases in which there were some acceptances, all of the acceptances were determined by the shortcut.

4.2 Shortcut for determining if $H_0$ is rejected

For this shortcut to be valid, $\alpha$ must be no more than .5. Since $\alpha$ is usually $\leq .1$, this restriction is not practically important. Let $m$ denote the number of distinct nonconstant functions in $\{f_{ij}(x): i = 1,2; j = 1,\ldots,p_i\}$. Let $Z^*$ denote the set of $2^m$ points $(z_1^*, z_2^*) = ((z_{11},\ldots,z_{1p_1}),(z_{21},\ldots,z_{2p_2}))$ formed by replacing $f_{ij}(x)$ by either $\max_{x \in R} f_{ij}(x)$ or $\min_{x \in R} f_{ij}(x)$ in $(f_1(x), f_2(x))$. Note that if $f_{1r}(x) = f_{2s}(x)$ then $z_{1r} = z_{2s}$, i.e.,
<table>
<thead>
<tr>
<th>( \delta(x) = c(1-x^2) )</th>
<th>( \delta(x) = cx^2 )</th>
<th>( \delta(x) = c(x+1)^2/4 )</th>
<th>( \delta(x) = c(x+1)/2 )</th>
<th>( \delta(x) = c(-x^2+2x+3)/4 )</th>
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<td>( c )</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>25</td>
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<td>0.0003</td>
<td>0.0027</td>
<td>0.0053</td>
<td>0.0060</td>
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<td>99</td>
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<td>100</td>
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<td>( \delta(x) = cx^2 )</td>
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<td>0.0080</td>
<td>0.0287</td>
<td>0.0483</td>
<td>0.0510*</td>
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<td>( \delta(x) = c(x+1)^2/4 )</td>
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<td>0.0003</td>
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<td>0.0267</td>
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<td>0.0067</td>
<td>0.0237</td>
<td>0.0493</td>
<td>0.0523*</td>
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<tr>
<td>( \delta(x) = c(-x^2+2x+3)/4 )</td>
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<td>0.0003</td>
<td>0.0087</td>
<td>0.0327</td>
<td>0.0523*</td>
<td>0.0523*</td>
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<tr>
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</tbody>
</table>

1. First (top) entry: estimated power of the test
2. Second (middle) entry: percentage of acceptances detected by shortcut in Section 4.1
3. Third (bottom) entry: percentage of rejections detected by shortcut in Section 4.2

*Theoretically these values are \( \leq .05 \). These estimates are \( > .05 \) due to sampling error.
### TABLE III

Power of the Test and Percentage of Acceptances and Rejections by Shortcuts for Selected Points $\delta(x) = f_1(x)\beta_1 - f_2(x)\beta_2$ in $H_A$.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>.5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
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<tr>
<td></td>
<td>98</td>
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<tr>
<td></td>
<td>38</td>
<td>45</td>
<td>65</td>
<td>90</td>
<td>100</td>
</tr>
<tr>
<td>$\delta(x) = (x+1)^2/4 + c$</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>94</td>
<td>87</td>
<td>81</td>
<td>93</td>
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<tr>
<td></td>
<td>33</td>
<td>44</td>
<td>65</td>
<td>90</td>
<td>100</td>
</tr>
<tr>
<td>$\delta(x) = (-x^2 + 2x + 3)/4 + c$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>96</td>
<td>94</td>
<td>96</td>
<td>99</td>
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<tr>
<td></td>
<td>51</td>
<td>63</td>
<td>81</td>
<td>97</td>
<td>100</td>
</tr>
<tr>
<td>$\delta(x) = x^2 + 1 + c$</td>
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<td>99</td>
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<td></td>
<td>76</td>
<td>78</td>
<td>92</td>
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<td>$\delta(x) = x^2 + c$</td>
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<td></td>
<td>73</td>
<td>91</td>
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<td>21</td>
<td>36</td>
<td>59</td>
<td>88</td>
<td>100</td>
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<tr>
<td>$\delta(x) = x + 1 + c$</td>
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<td>99</td>
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<td>11</td>
<td>15</td>
<td>32</td>
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<tr>
<td>$\delta(x) = x^2 + 2x + 3 + c$</td>
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<td></td>
<td>79</td>
<td>87</td>
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<td>100</td>
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<tr>
<td>$\delta(x) = 4x^2 + 4 + c$</td>
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<td>$\delta(x) = 4x^2 + c$</td>
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<td>17</td>
<td>28</td>
<td>57</td>
<td>88</td>
<td>100</td>
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</tbody>
</table>

*See Table II footnote.*
the maximum or minimum is used in both $z_1^*$ and $z_2^*$. Let

$$T^* = \min_{z^* \in Z^*} T(z^*)$$

where

$$T(z^*) = \frac{(z_1^* b_1 - z_2^* b_2)/s}{\sqrt{z_1^* D_1 z_1^* + z_2^* D_2 z_2^*}}.$$  

If $T^* > t_{1-\alpha,\nu}$ then $T > t_{1-\alpha,\nu}$ and $H_0$ can be rejected. Furthermore, if $T^* = T(z^*)$ where $z^* = (f_1(x), f_2(x))$ for some $x \in R$ then, in fact $T^* = T$.

For example, suppose $f_1(x) = (1, x, x^2)$, and $f_2(x) = (1, x)$, and $R = \{x: -1 < x < 2\}$. There are two distinct nonconstant functions, $x$ and $x^2$, and the $2^2 = 4$ values of $(z_1^*, z_2^*)$ in $Z^*$ are $((1, -1, 0), (1, -1), ((1, -1, 4), (1, -1), ((1, 2, 0), (1, 2))$ and $((1, 2, 4), (1, 2))$. Points like $((1, -1, 0), (1, 2))$ where $x$ has been replaced by its minimum in $z_1^*$ and its maximum in $z_2^*$ are not in $Z^*$.

The validity of this shortcut is based on two facts, the fact about functions which are the ratios of linear functions and square roots of positive quadratic functions mentioned in the proof of Theorem 2, Section 6, and the fact that $A = \{f_1(x), f_2(x): x \in R\}$ is a subset of

$$B = \{(z_1, z_2): \min_{x \in R} f_{ij}(x) \leq z_{ij} \leq \max_{x \in R} f_{ij}(x) \quad \text{and} \quad z_{1r} = z_{2s} \quad \text{if} \quad f_{1r}(x) = f_{2s}(x)\}$$

so a minimum over $A$ is not less than a minimum over $B$.

This shortcut was used in the simulation study of Section 5. In this case $R = \{x: -1 < x < 2\}$. $f_1(x) = f_2(x) = (1, x, x^2)$ so there are $m = 2$ distinct nonconstant functions. The $2^m = 4$ points in $Z^*$ are $((1, -1, 0), (1, -1, 0)), ((1, -1, 1), (1, -1, 1)), ((1, 1, 0), (1, 1, 0))$ and $((1, 1, 1), (1, 1, 1))$.

In Tables II and III, the third (bottom) number for each entry is the proportion of the rejections which were detected by this shortcut. The usefulness of this shortcut is seen to depend on the actual value of the regression function but in many cases the
The proportion is fairly high. In 17 out of 71 cases all of the rejections were detected by this method, avoiding numerical minimization of $T(x)$.

## 5. Power Function

A simulation study was conducted to investigate the power function of the test. In this study the regression functions were $f_1(x)\beta_1 = \beta_{i1} + \beta_{i2}x + \beta_{i3}x^2$, $i = 1, 2$. The variance $\sigma^2$ was set equal to one. The sample sizes $n_1$ and $n_2$ were both 10 with 3 observations at each of $x = 1$ and $x = -1$ and 4 observations at $x = 0$. $R = \{x: -1 \leq x \leq 1\}$. The size of the test was fixed at $\alpha = .05$ by using $t_{.95,14} = 1.761$. The International Mathematics and Statistics Library programs GGNSM and GGCHS were used to generate the random vector $b_1 - b_2$ and the random variable $s^2$. A total of 3000 repetitions were used to obtain each of the estimates in Tables II and III.

The maximum probability of a Type I error takes place when $f_1(x)\beta_1 = f_2(x)\beta_2$ for one $x$ and $f_1(x)\beta_1 - f_2(x)\beta_2$ becomes large for all other $x$. This can be observed in Table II where the probability of a Type I error is given for various values of $\beta_1$ and $\beta_2$ in the null hypothesis. As one proceeds across rows II, III, IV or V of Table II, $f_1(x)\beta_1 = f_2(x)\beta_2$ for one value of $x$ (for Rows III, IV and V and $x = 0$ for Row II) and $f_1(x)\beta_1 - f_2(x)\beta_2$ becomes large for all other values of $x$. The probability of a Type I error increases to $\alpha = .05$ as one proceeds across any row. The estimates slightly exceed .05 in a few cases due to sampling error.

The power function of the proposed test exhibits the following monotonicity property. If $(\beta_1, \beta_2)$ and $(\beta_1^+, \beta_2^+)$ are two parameter vectors which satisfy

$$f_1(x)\beta_1^+ - f_2(x)\beta_2^+ \geq f_1(x)\beta_1 - f_2(x)\beta_2$$

for every $x$ with strict inequality for some $x$, then the power at
is greater than the power at \((\beta_1, \beta_2)\). This property is apparent as one proceeds across any row in Tables II or III.

The power of the test is near one only if \(f_1(x)\beta_1 - f_2(x)\beta_2\) is large for all \(x\). This is the case for the rightmost entries in Table III. In Table III, the minimum distance between the regression functions is \(c\). The power nears one only as the minimum distance \(c\) becomes large.

The test we propose is biased in that the probability of rejecting \(H_0\) is less than \(\alpha\) for some \((\beta_1, \beta_2)\) in \(H_A\). The feature was noted by Tsutakawa and Hewett (1978) for the special model they considered and it continues to exist for the more general models we consider. This biasedness can be observed in the entries for \(c = .5\) which are less than .05 in Table III. But as noted by Tsutakawa and Hewett (1978) for their special case, the test we propose is consistent in that, for any fixed point \((\beta_1, \beta_2)\) in \(H_A\), the power can be made arbitrarily near one by choosing the sample sizes sufficiently large. Although we do not feel this bias is serious, it should be noted that the power of the test we propose may be small if \(f_1(x)\beta_1\) exceeds \(f_2(x)\beta_2\) by only a small amount over most of \(R\).

The power function properties we have described in this section are true in general, not just for the case of quadratic regression we considered in the simulation experiment. The proofs of these facts can be accomplished using the methods employed in proofs in Section 6.

6. SIZE AND EQUIVALENCE RESULTS

Results regarding the size of the test and the equivalence of the test to the test proposed by Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980) are given in this section.

Theorem 1: Under the assumptions of our model the test has level \(\alpha\), i.e.,

\[
\sup_{(\beta_1, \beta_2) \in H_0} P_{\beta_1, \beta_2} (T > t_{1-\alpha, \nu}) \leq \alpha.
\]  
(6.1)
If in addition (2.3), (2.4), and (2.5) are true then the test has size exactly \( \alpha \), i.e., (6.1) is true with \( \leq \) replacing \( < \).

The proof of Theorem 1 will use Lemma 1 which can be proved using standard analysis methods.

**Lemma 1:** Let \( g_n(x), n = 1, 2, \ldots, \) be continuous functions on a compact set \( R \). Suppose there exists an \( x_0 \in R \) such that \( g_n(x_0) \) is constant (say \( c \)) for all \( n \). Suppose \( g_n(x) \) increases to infinity as \( n \to \infty \) for all \( x \neq x_0 \). Then

\[
\lim_{n \to \infty} \min_{x \in R} g_n(x) = c. \tag{6.2}
\]

**Proof of Theorem 1:** Fix \( (\beta_1', \beta_2') \in H_0 \). There is an \( x_0 \in R \) such that \( f_1(x_0) \beta_1' \leq f_2(x_0) \beta_2' \). Consider testing

\[
H_{0x_0} : f_1(x_0) \beta_1' \leq f_2(x_0) \beta_2' \quad \text{versus} \quad H_{A_{x_0}} : f_1(x_0) \beta_1' > f_2(x_0) \beta_2'.
\]

The test which rejects \( H_{0x_0} \) if \( T(x_0) > t_{1-\alpha, \nu} \) is a level \( \alpha \) test of \( H_{0x_0} \). Since \( T \leq T(x_0) \) and \( (\beta_1', \beta_2') \in H_{0x_0} \), then

\[
P_{\beta_1', \beta_2'}(T > t_{1-\alpha, \nu}) \leq P_{\beta_1', \beta_2'}(T(x_0) > t_{1-\alpha, \nu}) \leq \alpha.
\]

Since \( (\beta_1', \beta_2') \) was arbitrary, (6.1) is true.

Since (6.1) is true, to prove the second part of the theorem it suffices to show there exists a sequence \( (\beta_1^n, \beta_2^n), n = 1, 2, \ldots, \) such that \( (\beta_1^n, \beta_2^n) \in H_0 \) for \( n = 1, 2, \ldots, \) and

\[
\lim_{n \to \infty} P_{\beta_1^n, \beta_2^n}(T > t_{1-\alpha, \nu}) \geq \alpha. \tag{6.3}
\]

The estimates \( b_i, i = 1, 2, \) can be written as \( b_i = Z_i + \beta_i \) where \( Z_1, Z_2 \) and \( s \) are independent, and \( Z_i \) has a \( p_i \)-variate normal distribution with mean 0 and variance-covariance matrix \( \sigma^2D_i^{-1} \). In terms of these quantities, the statistics \( T \) and \( T(x) \) can be written as

\[
T = T(Z_1, Z_2, s, \beta_1, \beta_2) = \min_{x \in R} T(x; Z_1, Z_2, s, \beta_1, \beta_2)
\]

and

\[
T(x; Z_1, Z_2, s, \beta_1, \beta_2)
\]
Consider the sequence \((\beta_1^n, \beta_2^n)\) defined by \(\beta_1^n = n\beta_1^*\) where the \(\beta_1^*\) are the parameters identified in (2.5). For a fixed value of \(z_1^* \in \mathbb{R}^p_1, z_2^* \in \mathbb{R}^p_2\) and \(s^* > 0\), define

\[ g_n(x) = T(x; z_1^*, z_2^*, s^*, \beta_1^n, \beta_2^n) \]

The \(g_n(x)\) satisfy the conditions of Lemma 1 since 1) \(f_{ij}\) are continuous, 2) \(s^* \sqrt{v(x)} > 0\), 3) \(f_1(x_0)\beta_1^n = f_2(x_0)\beta_2^n\) and 4) \(f_1(x)\beta_1^n - f_2(x)\beta_2^n\) increases to infinity as \(n \to \infty\) for all \(x \neq x_0\). By Lemma 1, \(\lim_{n \to \infty} T(z_1^*, z_2^*, s^*, \beta_1^n, \beta_2^n) = T(x_0; z_1^*, z_2^*, s^*, \beta_1^*, \beta_2^*)\). Since \(z_1^*, z_2^*, s^*, \beta_1^*, \beta_2^*\), were arbitrary, this implies that the random variables \(T(Z_1, Z_2, s, \beta_1^n, \beta_2^n)\) converge to \(T(x_0; Z_1, Z_2, s, \beta_1^*, \beta_2^*)\) with probability one and hence in distribution. Thus

\[ \lim_{n \to \infty} P_{\beta_1^n, \beta_2^n}(T > t_{1-\alpha, \nu}) = \lim_{n \to \infty} P(T(Z_1, Z_2, s, \beta_1^n, \beta_2^n) > t_{1-\alpha, \nu}) = P(T(x_0; Z_1, Z_2, s, \beta_1^*, \beta_2^*) > t_{1-\alpha, \nu}) = \alpha. \]

Conditions (2.3), (2.4), and (2.5) are satisfied in these two simple cases. These conditions are satisfied if the \(f_{ij}(x)\) include the constant 1, the linear functions \(x_i, i = 1, \ldots, k\), and the quadratic functions \(x_i x_j, i = 1, \ldots, k, j = 1, \ldots, i\). Then \(\beta_1\) and \(\beta_2\) can be chosen so that \(f_1(x)\beta_1 - f_2(x)\beta_2 = (x - x_0)(x - x_0)'\). Another situation in which the condition is satisfied is if

\[ f_1(x)\beta_i = \beta_{10} + \sum_{j=1}^{k} \beta_{ij} x_j \]

and \(R = \{x: x_j^* \leq x_j \leq x_j^*, j = 1, \ldots, k\}\), the model considered by Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980). Then \(\beta_1\) and \(\beta_2\) can be chosen so that
which is zero for \( x = (x_1^*, \ldots, x_k^*) \) and positive for all other \( x \in \mathbb{R} \).

**Theorem 2:** Suppose

\[
f_i(x) = \beta_i 0 + \sum_{j=1}^{k} \beta_{ij} x_j
\]

and \( R \) has the form \( R = \{ x: x_j^* \leq x_j \leq x_j^*, j = 1, \ldots, k \} \). Consider the test which rejects \( H_0 \) if \( T^* > t_{1-\alpha, v} \) where \( T^* = \min_{x \in X^*} T(x) \) and \( X^* \) is the set of \( 2^k \) points for which \( x_j \) is either \( x_j^* \) or \( x_j^* \). Suppose \( \alpha \leq .5 \). Then the tests based on \( T^* \) and \( T \) are equivalent.

**Proof.** For any \( k + 1 \) dimensional vectors \( b_1 \) and \( b_2 \) and \( s > 0 \), \( T(x) \) is a linear function of \( (x_1, \ldots, x_k) \) divided by the square root of a quadratic function of \( (x_1, \ldots, x_k) \) which is positive for all \( (x_1, \ldots, x_k) \in \mathbb{R}^k \). Such a function has the property that \( T^* = \min_{x \in X^*} T(x) \geq 0 \) implies \( T = \min_{x \in \mathbb{R}^k} T(x) = T \). (This is easily proved for \( k = 1 \) and can be proved for general \( k \) by induction.)

Suppose \( b_1, b_2 \) and \( s \) are such that \( T^* \) rejects \( H_0 \). Then \( T^* > t_{1-\alpha, v} \), since \( \alpha \leq .5 \), so \( T = T^* \) and \( T \) also rejects \( H_0 \). For any \( b_1, b_2 \) and \( s > 0 \), \( T \leq T^* \) so if \( T \) rejects \( H_0 \), so does \( T^* \).

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Recommended by S. S. Gupta, Purdue University, W. Lafayette, IN

Refereed by John E. Hewett, University of Missouri, Columbia, MO