Testing Hypotheses Concerning Unions of Linear Subspaces

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The likelihood ratio test (LRT) for hypotheses concerning unions of linear subspaces is derived for the normal theory linear model. A more powerful test, an intersection-union test, is proposed for the case in which the subspaces are not all of the same dimension. A theorem is proved that may be used to identify hypotheses that concern unions of linear subspaces. Some hypotheses about the spacings between normal means are shown to concern unions of linear subspaces and therefore can be tested using the LRT. Finally, the computation of the LRT statistic is discussed.

KEY WORDS: Linear model; Likelihood ratio test; Intersection-union test; Ordered means; Spacings between means.

1. INTRODUCTION

Let $X_1, \ldots, X_K$ denote independent normal random variables with means $\xi_1, \ldots, \xi_K$ and common variance σ^2. We assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_K)'$ lies in ω, a subspace of $\mathcal{R}^K$ of dimension $J < K$. For example, $X_1, \ldots, X_K$ may be comprised of independent samples from J populations ($J < K$). We discuss testing hypotheses about $\boldsymbol{\xi}$ concerning unions of subspaces of ω. (Throughout this article, *subspace* refers to a *linear* subspace.) We derive the likelihood ratio test (LRT) of

$$H_0\colon \boldsymbol{\xi} \in \omega_0 \quad \text{versus} \quad H_1\colon \boldsymbol{\xi} \in \omega - \omega_0, \tag{1.1}$$

where $\omega_0 = \cup_{i=1}^{m} \omega_i$ and each ω_i is a q_i-dimensional subspace of ω. We show that the critical value for the LRT is a multiple of a percentile from an F distribution, even though ω_0 is not necessarily a subspace of ω.

We also consider an intersection-union test (IUT) of H_0 versus H_1. We show that the IUT is more powerful than the LRT if at least two of the ω_i have different dimensions. Throughout we assume that (1.1) has been expressed in such a way that the ω_i are all distinct; that is, there do not exist i and j, $1 \leqq i, j \leqq m$, $i \neq j$, such that $\omega_i \subset \omega_j$.

An example of a hypothesis concerning a union of subspaces that falls into the framework of (1.1) is the following. Let $\mu_1, \ldots, \mu_J$ denote the means of J normal populations. Let $\mu_1^* \leqq \cdots \leqq \mu_J^*$ denote the ordered values of the means. The hypothesis

$$H_0^1\colon \mu_{j+1}^* - \mu_j^* = \mu_{J+1-j}^* - \mu_{J-j}^*, \qquad j = 1, \ldots, [(J-1)/2],$$

where $[s]$ denotes greatest integer less than or equal to s, states that the symmetric spacings between the means are equal. Although it is perhaps not obvious, H_0^1 concerns a union of subspaces and may be tested using the LRT and IUT we discuss in this article.

In (1.1), the null hypothesis H_0 is expressed in terms of a union. The IUT of H_0 (Gleser 1973; Berger 1982) is constructed in the following way. Test each of the hypotheses $H_{0i}\colon \boldsymbol{\xi} \in \omega_i$ individually and reject H_0 if and only if each of the individual tests rejects H_{0i}. This situation, in which H_0 concerns a union, is the opposite of the situation addressed by Roy's (1953) union-intersection method of test construction. The null hypothesis for a union-intersection test concerns an intersection, say $\cap_{i=1}^{n} \eta_i$. A union-intersection test tests each of the hypotheses $H_{0i}\colon \boldsymbol{\xi} \in \eta_i$ individually and rejects H_0 if and only if at least one of the individual hypotheses H_{0i} is rejected.

A convenient feature of some IUT's is that if the individual tests are exact size-α tests then the IUT is an exact size-α test, which is the case for the tests in this article. The critical values for the LRT and IUT we discuss are percentiles (or multiples of percentiles) from standard F distributions. No new tables are required to implement these tests.

In Section 2 we derive the LRT and IUT for hypotheses concerning unions of subspaces. In Section 3 we prove a theorem that may be used to identify hypotheses with this property. We use the theorem to show that four hypotheses about ordered means, including H_0^1, concern unions of subspaces and thus may be tested with the LRT of Section 2. Finally, in Section 4 we discuss how the test statistic may be computed.

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2. LIKELIHOOD RATIO AND INTERSECTION-UNION TESTS

For the model described in Section 1, let the density of $\mathbf{X} = (X_1, \ldots, X_K)'$ be denoted by

$$p(\mathbf{x}; \xi, \sigma) = (2\pi\sigma^2)^{-K/2}\exp\left(-\sum_{i=1}^{K}(x_i - \xi_i)^2/2\sigma^2\right). \tag{2.1}$$

Let $\Theta_i = \{(\xi, \sigma): \xi \in \omega_i, \sigma > 0\}$, $\Theta_0 = \{(\xi, \sigma): \xi \in \omega_0, \sigma > 0\} = \cup_{i=1}^{m} \Theta_i$, and $\Theta = \{(\xi, \sigma): \xi \in \omega, \sigma > 0\}$. The LRT statistic for testing H_0 is defined as

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} p(\mathbf{x}; \xi, \sigma)}{\sup_{\Theta} p(\mathbf{x}; \xi, \sigma)}. \tag{2.2}$$

If we let $\hat{\xi}$ denote the projection of $\mathbf{X}$ on ω and $\hat{\xi}_i$ denote the projection of $\mathbf{X}$ on ω_i, then $\lambda(\mathbf{x})$ is

$$\begin{aligned}\lambda(\mathbf{x}) &= \frac{\max_{1\leq i\leq m} \sup_{\Theta_i} p(\mathbf{x}; \xi, \sigma)}{\sup_{\Theta} p(\mathbf{x}; \xi, \sigma)} \\ &= \max_{1\leq i\leq m} \frac{\sup_{\Theta_i} p(\mathbf{x}; \xi, \sigma)}{\sup_{\Theta} p(\mathbf{x}; \xi, \sigma)} \\ &= \max_{1\leq i\leq m}\left[\frac{\| \mathbf{x} - \hat{\xi} \|^2}{\| \mathbf{x} - \hat{\xi}_i \|^2}\right]^{K/2} \\ &= \left[\frac{\| \mathbf{x} - \hat{\xi} \|^2}{\min_{1\leq i\leq m} \| x - \hat{\xi}_i \|^2}\right]^{K/2},\end{aligned} \tag{2.3}$$

where $\| \mathbf{y} \|^2 = \mathbf{y}'\mathbf{y}$. The third equality in (2.3) is a standard result from linear model theory. The last expression in (2.3) reflects the fact that the maximum likelihood estimate of ξ under H_0 is the projection of $\mathbf{X}$ on the nearest subspace ω_i. Since

$$\| \mathbf{x} - \hat{\xi}_i \|^2 = \| \mathbf{x} - \hat{\xi} \|^2 + \| \hat{\xi} - \hat{\xi}_i \|^2,$$

rejecting H_0 if $\lambda(\mathbf{x}) < c$ is equivalent to rejecting H_0 if $\lambda^*(\mathbf{x}) > c^*$, where

$$\lambda^*(\mathbf{x}) = \frac{\min_{1\leq i\leq m} \| \hat{\xi} - \hat{\xi}_i \|^2}{\| \mathbf{x} - \hat{\xi} \|^2}. \tag{2.4}$$

The value of c^*, which produces an exact size-α test, is given in Theorem 1. The value of c^* is a multiple of a percentile of an F distribution Let $F_{\alpha,a,b}$ denote the upper 100α percentile of a central F distribution with a and b degrees of freedom (df). Lemmas 1 and 2 (proved in the Appendix) are used in the proof of Theorem 1.

Lemma 1. For any values of α and t, the quantity $(s/t)F_{\alpha,s,t}$ is an increasing function of s.

Lemma 2. Let Y_δ have a noncentral F distribution with noncentrality parameter δ and df s and t (arbitrary but fixed). For any $y < \infty$, $P(Y_\delta > y) \to 1$ as $\delta \to \infty$.

Theorem 1. Let $q^* = \min_{1\leq i\leq m} q_i$. Let $c_\alpha^* = (J - q^*)F_{\alpha,J-q^*,K-J}/(K - J)$. The test that rejects H_0 if $\lambda^*(\mathbf{X}) > c_\alpha^*$ is an exact size-α test; that is, the test satisfies

$$\sup_{\Theta_0} P_{\xi,\sigma}(\lambda^*(\mathbf{X}) > c_\alpha^*) = \alpha. \tag{2.5}$$

Proof. By Lemma 1, $c_\alpha^* = \max_{1\leq i\leq m} (J - q_i) F_{\alpha,J-q_i,K-J}/(K - J)$. For any $\xi \in \omega_j$ and $\sigma > 0$,

$$\begin{aligned}&P_{\xi,\sigma}(\lambda^*(\mathbf{X}) > c_\alpha^*) \\ &\leq P_{\xi,\sigma}(\| \hat{\xi} - \hat{\xi}_j \|^2/\| \mathbf{x} - \hat{\xi} \|^2 > c_a^*) \\ &\leq P_{\xi,\sigma}(\| \hat{\xi} - \hat{\xi}_j \|^2/\| \mathbf{x} - \hat{\xi} \|^2 \\ &> (J - q_j)F_{\alpha,J-q_j,K-J}/(K - J)) \\ &= \alpha.\end{aligned}$$

The last equality is true since a standard result from linear model theory states that for any $\xi \in \omega_j$ and $\sigma > 0$,

$$[(K - J)/(J - q_j)] \| \hat{\xi} - \hat{\xi}_j \|^2/\| \mathbf{x} - \hat{\xi} \|^2$$

has an F distribution with $J - q_j$ and $K - J$ df. Thus

$$\sup_{\Theta_0} P_{\xi,\sigma}(\lambda^*(\mathbf{X}) > c_\alpha^*) \leq \alpha. \tag{2.6}$$

To prove the reverse inequality, let j be such that $q_j = q^*$. We have assumed that ω_j is not a subset of ω_i for any $i \neq j$. So for every $i = 1, \ldots, m$, $i \neq j$, $\omega_i \cap \omega_j$ is a subspace of dimension at most $q_j - 1$. The set,

$$\bigcup_{\substack{i=1\\ i\neq j}}^{m} \{\omega_i \cap \omega_j\},$$

cannot contain the q_j-dimensional set ω_j since each set in the union is at most $(q_j - 1)$-dimensional. Thus there exists ξ^*, such that $\xi^* \in \omega_j$ and $\xi^* \notin \omega_i$ for any $i = 1, \ldots, m$, $i \neq j$. Let ξ_i^* denote the projection of ξ^* on ω_i. Then $\| \xi^* - \xi_j^* \| = 0$ and $\| \xi^* - \xi_i^* \| > 0$ for $i = 1, \ldots, m$, $i \neq j$.

For $i = 1, \ldots, m$, let

$$R_i = \{\mathbf{x}: \| \hat{\xi} - \hat{\xi}_i \|^2/\| \mathbf{x} - \hat{\xi} \|^2 > c_\alpha^*\}.$$

At (ξ^*, σ) $[(K - J)/(J - q_i)] \| \hat{\xi} - \hat{\xi}_i \|^2/\| \mathbf{x} - \hat{\xi} \|^2$ has a noncentral F distribution with $J - q_i$ and $K - J$ df and noncentrality parameter $\delta_{\sigma,i} = \| \xi^* - \xi_i^* \|^2/\sigma^2$. For $i = 1, \ldots, m$, $i \neq j$, $\delta_{\sigma,i} \to \infty$ as $\sigma \to 0$. By Lemma 2, for $i = 1, \ldots, m$, $i \neq j$,

$$P_{\xi^*,\sigma}(R_i) \to 1 \text{ as } \sigma \to 0. \tag{2.7}$$

On the other hand

$$[(K - J)/(J - q_j)] \| \hat{\xi} - \hat{\xi}_j \|^2/\| \mathbf{x} - \hat{\xi} \|^2$$

has a central F distribution with $J - q_j$ and $K - J$ df at (ξ^*, σ). Furthermore,

$$c_\alpha^* = (J - q_j)F_{\alpha,J-q_j,K-J}/(K - J).$$

Thus

$$P_{\xi^*,\sigma}(R_j) = \alpha \quad \text{for every} \quad \sigma > 0. \tag{2.8}$$

Using (2.7) and (2.8) we obtain

$$\lim_{\sigma\to 0} P_{\xi^*,\sigma}(\lambda^*(\mathbf{X}) > c_\alpha^*)$$

$$= \lim_{\sigma\to 0} P_{\xi^*,\sigma}\left(\bigcap_{i=1}^{m} R_i\right)$$

$$= \lim_{\sigma\to 0}\left[1 - P_{\xi^*,\sigma}\left(\bigcup_{i=1}^{m} R_i^c\right)\right]$$

$$\geqq 1 - \lim_{\sigma\to 0}\sum_{i=1}^{m} P_{\xi^*,\sigma}(R_i^c)$$

$$= 1 - (1-\alpha) - \lim_{\sigma\to 0}\sum_{\substack{i=1\\ i\neq j}}^{m} P_{\xi^*,\sigma}(R_i^c)$$

$$= \alpha - 0 = \alpha.$$

Since $(\xi^*, \sigma) \in \Theta_j \subset \Theta_0$ for every σ,

$$\sup_{\Theta_0} P_{\xi,\sigma}(\lambda^*(\mathbf{X}) > c_\alpha^*)$$

$$\geqq \lim_{\sigma\to 0} P_{\xi^*,\sigma}(\lambda^*(\mathbf{X}) > c_\alpha^*) \geqq \alpha. \quad (2.9)$$

Combining (2.6) and (2.9) yields (2.5).

We believe that in most applications all of the ω_i, $i = 1, \ldots, m$, will have the same dimension q, in which case $c_\alpha^* = (J - q)F_{\alpha,J-q,K-J}/(K - J)$. This is the case for all the examples we discuss in Section 3. But if the dimensions of some of the ω_i differ, there is an IUT that is also an exact size-α test and has higher power than the LRT. This test is described in Theorem 2. The IUT and LRT are the same if all the ω_i have the same dimension.

Theorem 2. Let $F_i = [(K - J)/(J - q_i)] \,\|\, \hat{\xi} - \hat{\xi}_i \,\|^2 / \|\, \mathbf{x} - \hat{\xi} \,\|^2$, $i = 1, \ldots, m$. The test λ^{**} that rejects H_0 if and only if $F_i > F_{\alpha,J-q_i,K-J}$ for every $i = 1, \ldots, m$ is an exact size-α test. The test λ^{**} has a power that is greater than or equal to the power of the LRT for every $(\xi, \sigma) \in \Theta$.

Proof. The proof that λ^{**} is an exact size-α test is almost identical to the proof given in Theorem 1 that λ^* is an exact size-α test. In this case any ω_i, $i = 1, \ldots, m$, can play the special role played by ω_j in the second half of the proof of Theorem 1.

The set $\{\mathbf{x}: \lambda^*(\mathbf{x}) > c_\alpha^*\} \subset \{\mathbf{x}: F_i > F_{\alpha,J-q_i,K-J},\ i = 1, \ldots, m\}$. That is, the rejection region for the LRT is a subset of the rejection region for λ^{**}. Thus the power of λ^{**} is greater than or equal to the power of the LRT.

Unless all of the quantities $(J - q_i)F_{\alpha,J-q_i,K-J}/(K - J)$, $i = 1, \ldots, m$, are equal, the rejection region for the LRT is a proper subset of the rejection region for λ^{**}, and the power for the LRT is strictly smaller than the power of λ^{**} for every parameter in H_1. This provides an example, like that of Stein (Bickel and Doksum 1977, p. 239), of an LRT whose power is everywhere dominated by the power of another test.

3. HYPOTHESES ABOUT ORDERED MEANS

In the remaining sections, we discuss some specific problems that fall into the general framework described in Section 2. These problems involve hypotheses about ordered normal means.

We will consider the following special case of the model presented in Section 2. Let X_{ijk}, $i = 1, \ldots, I$, $j = 1, \ldots, J_i$, $k = 1, \ldots, K_{ij}$ denote $K = \sum_{i,j} K_{ij}$ independent normal observations. The mean of X_{ijk} is μ_{ij} and all the X_{ijk} have a common variance of σ^2. Let $\boldsymbol{\mu} = (\mu_{11}, \ldots, \mu_{1J_1}, \mu_{21}, \ldots, \mu_{IJ_I})'$ and

$$\xi = (\mu_{11}\mathbf{1}_{K_{11}}', \ldots, \mu_{1J_1}\mathbf{1}_{K_{1J_1}}', \mu_{21}\mathbf{1}_{K_{21}}', \ldots, \mu_{IJ_I}\mathbf{1}_{K_{IJ_I}}')',$$

where $\mathbf{1}_b$ is a column vector of b ones. In the formulation of Section 2 we would consider $\xi \in \omega \subset \Re^K$. Since here there is a one-to-one correspondence between $\boldsymbol{\mu}$ and ξ, we may equivalently consider $\boldsymbol{\mu} \in \Re^J$, where $J = \sum_{i=1}^I J_i$. We now consider the subspaces ω_i, $i = 1, \ldots, m$, as subspaces in $\Re^J$.

We will be concerned with permutations of $(\mu_{i1}, \ldots, \mu_{iJ_i})'$ for each $i = 1, \ldots, I$. Let $\boldsymbol{\mu}_i = (\mu_{i1}, \ldots, \mu_{iJ_i})'$, so that $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \ldots, \boldsymbol{\mu}_I')'$. A map $\pi: \Re^J \to \Re^J$ is called a *subpermutation* if $\pi(\boldsymbol{\mu}) = (\pi_1(\boldsymbol{\mu}_1)', \ldots, \pi_I(\boldsymbol{\mu}_I)')'$ where $\pi_i(\boldsymbol{\mu}_i)$ is a permutation of $\boldsymbol{\mu}_i$, $i = 1, \ldots, I$. There are $\prod_{i=1}^I J_i!$ subpermutations. A vector $\boldsymbol{\nu} \in \Re^J$ is called a subpermutation of a vector $\boldsymbol{\mu}$ if $\pi(\boldsymbol{\mu}) = \boldsymbol{\nu}$ for some subpermutation π. A set B is called a subpermutation of a set A if B is the image of A under some subpermutation π. It is easily verified that if A is a subspace then any subpermutation of A is also a subspace. For any $\boldsymbol{\mu} \in \Re^J$, let $\boldsymbol{\mu}^*$ denote the subpermutation of $\boldsymbol{\mu}$ such that $\mu_{i1}^* \leqq \cdots \leqq \mu_{iJ_i}^*$, $i = 1, \ldots, I$.

The following theorem may be used to identify hypotheses that concern unions of subspaces. We shall use this theorem to show that four hypotheses about ordered means concern unions of subspaces and hence are testable using the LRT of Section 2.

Theorem 3. Let $\mathbf{H}$ denote a $(J - q) \times J$ matrix of rank $J - q$. Let $\omega_0 = \{\boldsymbol{\mu} \in \Re^J: \mathbf{H}\boldsymbol{\mu}^* = \mathbf{0}\}$. If $\mathbf{H}$ is such that $\mathbf{H}\boldsymbol{\mu} = \mathbf{0}$ implies $\mathbf{H}\boldsymbol{\mu}^* = \mathbf{0}$, then $\omega_0 = \bigcup_{i=1}^m \omega_i$, where $m = \prod_{i=1}^I J_i!$ and $\omega_1, \ldots, \omega_m$ are the subpermutations of the q-dimensional subspace $N = \{\boldsymbol{\mu} \in \Re^J: \mathbf{H}\boldsymbol{\mu} = \mathbf{0}\}$. Thus ω_0 is the union of subspaces, each of dimension q.

Proof. Let $\boldsymbol{\mu} \in \omega_0$. Then $\boldsymbol{\mu}$ is a subpermutation of $\boldsymbol{\mu}^*$ and $\boldsymbol{\mu}^* \in N$ so $\boldsymbol{\mu} \in \omega_i$ for some i, $i = 1, \ldots, m$. Thus $\omega_0 \subset \bigcup_{i=1}^m \omega_i$.

Now let $\boldsymbol{\mu} \in \bigcup_{i=1}^m \omega_i$. Then $\boldsymbol{\mu} = \pi(\boldsymbol{\nu})$ for some subpermutation π and some $\boldsymbol{\nu} \in N$. Since $\boldsymbol{\nu} \in N$, $\boldsymbol{\nu}^* \in N$. But $\boldsymbol{\nu}^* = \boldsymbol{\mu}^*$. Therefore, $\boldsymbol{\mu}^* \in N$ and $\boldsymbol{\mu} \in \omega_0$. So $\bigcup_{i=1}^m \omega_i \subset \omega_0$.

Finally, since N is a q-dimensional subspace of $\Re^J$, each of the subpermutations of N, $\omega_1, \ldots, \omega_m$, is also a q-dimensional subspace of $\Re^J$.

The subspaces $\omega_1, \ldots, \omega_m$ defined in Theorem 3 will not be distinct. The number of distinct subspaces is at most $m/2$ and can be much smaller, in fact, as Example 1 shows. Recognizing this fact results in a saving of effort in the computation of the test statistic λ^* for which the minimum needs to be taken only over distinct subspaces. Taking the minimum over all m subspaces in Theorem 3 will, of course, give the same value of λ^*. It would just be inefficient if many of the subspaces are equal.

We now consider four hypotheses about the spacings between normal means. Hypothesis H_0^1 states that the means are arranged in a symmetric fashion. Hypothesis H_0^2 specifies the relative sizes of the spacings between the means in the symmetric pattern. Hypothesis H_0^3 states that the corresponding spacings in two groups are equal, and the relative sizes of the spacings are specified in H_0^4.

Example 1. (symmetric spacings). For this example, $I = 1$, so we will denote the means of $\mu_1, \ldots, \mu_J$ and the ordered means by $\mu_1^* \leqq \cdots \leqq \mu_J^*$. By the *symmetric spacings hypothesis* we mean

$$H_0^1\colon \mu_{j+1}^* - \mu_j^* = \mu_{J+1-j}^* - \mu_{J-j}^* \qquad j = 1, \ldots [(J-1)/2]. \quad (3.1)$$

We shall use Theorem 3 to verify that H_0^1 is the union of subspaces. Let $\mathbf{H}$ be a $[(J-1)/2] \times J$ matrix such that $\mathbf{H\mu} = \mathbf{0}$ is equivalent to the conditions $\mu_{j+1} - \mu_j = \mu_{J+1-j} - \mu_{J-j}, j = 1, \ldots, [(J-1)/2]$. The hypotheses H_0^1 can be written as $H_0^1\colon \boldsymbol{\mu} \in \omega_0 = \{\boldsymbol{\mu} \in \mathcal{R}^J\colon \mathbf{H\mu}^* = \mathbf{0}\}$. To show that ω_0 is a union of subspaces, by Theorem 3 it suffices to show that $\mathbf{H\mu} = \mathbf{0}$ implies $\mathbf{H\mu}^* = \mathbf{0}$. Let $\boldsymbol{\mu}$ satisfy $\mathbf{H\mu} = \mathbf{0}$. Let $\bar{\mu} = (\mu_1 + \mu_J)/2$. For any $j = 1, \ldots, [(J-1)/2]$,

$$\mu_j - \bar{\mu} = \mu_1 - \bar{\mu} + \sum_{i=1}^{j-1} (\mu_{i+1} - \mu_i)$$
$$= \bar{\mu} - \mu_J + \sum_{i=1}^{j-1} (\mu_{J+1-i} - \mu_{J-i}) = \bar{\mu} - \mu_{J+1-j}.$$

So each pair μ_j and μ_{J+1-j} is symmetrically placed about $\bar{\mu}$. (If J is odd, $[(J-1)/2] + 1 = J - [(J-1)/2]$, and $\mu_{[(J-1)/2]+1} = \mu_{J-[(J-1)/2]} = \bar{\mu}$.) If $\mu_r = \mu_j^*$ and $\mu_s = \mu_{j+1}^*$ then $\mu_{J+1-r} = \mu_{J+1-j}^*$ and $\mu_{J+1-s} = \mu_{J-j}^*$. Thus $\mu_{j+1}^* - \mu_j^* = \mu_s - \bar{\mu} - (\mu_r - \bar{\mu}) = \bar{\mu} - \mu_{J+1-s} - (\bar{\mu} - \mu_{J+1-r}) = \mu_{J+1-j}^* - \mu_{J-j}^*$. Therefore, $\mathbf{H\mu}^* = \mathbf{0}$. By Theorem 3, the ω_0 for the symmetric spacings hypothesis is the union of subspaces of dimension $J - [(J-1)/2]$ and λ^* can be used to test H_0^1.

This example gives a good illustration of the fact that the m subspaces defined in Theorem 3 are not distinct. Let $J = 4$. In this example, $m = 4! = 24$ but actually H_0^1 consists of only 3 distinct subspaces. These are the subspaces defined by $(1, -1, -1, 1)\boldsymbol{\mu} = 0$ (this is N in Theorem 3), $(1, -1, 1, -1)\boldsymbol{\mu} = 0$, and $(1, 1, -1, -1)\boldsymbol{\mu} = 0$. For example, the subpermutation of N defined by $\pi(\boldsymbol{\mu}) = (\mu_1, \mu_3, \mu_2, \mu_4)'$ is just N itself. But the subpermutation of N defined by $\pi(\boldsymbol{\mu}) = (\mu_1, \mu_2, \mu_4, \mu_3)'$ is the different subspace defined by $(1, -1, 1, -1)\boldsymbol{\mu} = 0$.

Example 2 (symmetric spacings with specified ratios). Again $I = 1$ so the notation of Example 1 is used. Now the hypothesis of interest is a subhypothesis of H_0^1, namely,

$$H_0^2\colon \mu_{j+1}^* - \mu_j^* = c_j(\mu_2^* - \mu_1^*), \qquad j = 2, \ldots, J-1, \quad (3.2)$$

where $c_2, \ldots, c_{J-1}$ are specified positive constants with $c_j = c_{J-j}, j = 2, \ldots, J-2$, and $c_{J-1} = 1$. The restrictions on the c_j imply that the symmetric spacings are equal as in H_0^1. The hypothesis H_0^2 can be used to test whether the means are spaced like the expected values of order statistics from some symmetric distribution. For example, if $c_j = 1, j = 2, \ldots, J-1$, the distribution is the uniform. If $J = 5$, $c_2 = c_3 = .74111$ and $c_4 = 1$, the distribution is the normal. Let $\mathbf{H}$ be a $(J-2) \times J$ matrix such that $\mathbf{H\mu} = \mathbf{0}$ is equivalent to the conditions $\mu_{j+1} - \mu_j = c_j(\mu_2 - \mu_1), j = 2, \ldots, J-1$. The hypothesis H_0^2 can be written as $H_0^2\colon \boldsymbol{\mu} \in \omega_0 = \{\boldsymbol{\mu} \in \mathcal{R}^J\colon \mathbf{H\mu}^* = \mathbf{0}\}$. By Theorem 3, ω_0 is a union of subspaces if $\mathbf{H\mu} = \mathbf{0}$ implies $\mathbf{H\mu}^* = \mathbf{0}$. Let $\boldsymbol{\mu}$ satisfy $\mathbf{H\mu} = \mathbf{0}$. If $\mu_1 \leqq \mu_2$, then $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_J$ since $c_j > 0, j = 1, \ldots, J-1$. Thus, $\boldsymbol{\mu} = \boldsymbol{\mu}^*$ so $\mathbf{H\mu}^* = \mathbf{0}$. If $\mu_1 \geqq \mu_2$, then $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_J$. For any $j = 2, \ldots, J-1$,

$$\mu_{j+1}^* - \mu_j^* = \mu_{J-j} - \mu_{J+1-j} = -c_{J-j}(\mu_2 - \mu_1)$$
$$= c_{J-j}(\mu_{J-1} - \mu_J) = c_j(\mu_{J-1} - \mu_J) = c_j(\mu_2^* - \mu_1^*).$$

The second, third, and fourth equalities are true since $\mathbf{H\mu} = \mathbf{0}$, $c_{J-1} = 1$, and $c_j = c_{J-j}$. Thus $\mathbf{H\mu}^* = \mathbf{0}$. By Theorem 3, ω_0 is the union of two-dimensional subspaces and λ^* can be used to test H_0^2.

Example 3 (equal spacings in two sets of means). In this example there are two sets of means of interest, $I = 2$, and $J_1 = J_2 = J' = J/2$. We are interested in testing the hypothesis that the spacings in the first set, $\mu_{11}, \ldots, \mu_{1J'}$, are equal to the spacings in the second set, $\mu_{21}, \ldots, \mu_{2J'}$. That is, we wish to test

$$H_0^3\colon \mu_{1,j+1}^* - \mu_{1j}^* = \mu_{2,j+1}^* - \mu_{2j}^*, \quad j = 1, \ldots, J'-1.$$

Let $\mathbf{H}$ be a $(J'-1) \times J$ matrix such that $\mathbf{H\mu} = \mathbf{0}$ is equivalent to the conditions $\mu_{1,j+1} - \mu_{1j} = \mu_{2,j+1} - \mu_{2j}, j = 1, \ldots, J'-1$. Then H_0^3 can be written as $H_0^3\colon \boldsymbol{\mu} \in \omega_0 = \{\boldsymbol{\mu}\colon \mathbf{H\mu}^* = \mathbf{0}\}$. To verify the assumptions of Theorem 3, let $\boldsymbol{\mu}$ satisfy $\mathbf{H\mu} = \mathbf{0}$. For any $j = 1, \ldots, J'$, $\mu_{2j} = \sum_{r=1}^{j-1} (\mu_{2,r+1} - \mu_{2r}) + \mu_{21} = \sum_{r=1}^{j-1} (\mu_{1,r+1} - \mu_{1r}) + \mu_{21} = \mu_{1j} + (\mu_{21} - \mu_{11})$. Thus the set of means, $\mu_{21}, \ldots, \mu_{2J'}$, is a translation of the set of means, $\mu_{11}, \ldots, \mu_{1J'}$, the amount of translation being $(\mu_{21} - \mu_{11})$. Thus the spacings among $\mu_{21}, \ldots, \mu_{2J'}$, are all equal to the spacings among $\mu_{11}, \ldots, \mu_{1J'}$. That is, $\mu_{1,j+1}^* - \mu_{1j}^* = \mu_{2,j+1}^* - \mu_{2j}^*, j = 1, \ldots, J'-1$, and $\mathbf{H\mu}^* = \mathbf{0}$. By Theorem 3, ω_0 is the union of subspaces of dimension $J - J' + 1 = J/2 + 1$ and λ^* can be used to test H_0^3.

This argument can be easily extended to the situation in which one wishes to test for equal spacings in I ($I > 2$) sets of means. In this case H is an $(I - 1)(J' - 1) \times J$ matrix and the subspaces are of dimension $J - (I - 1)(J' - 1) = I + J' - 1 = 1 = I + J/I - 1$.

Example 4 (equal spacings in two sets of means with specified ratios). For this example, the notation is the same as in Example 3. We combine the ideas in Examples 2 and 3 to consider testing

$$H_0^4: \begin{array}{l} \mu_{1,j+1}^* - \mu_{1j}^* = \mu_{2,j+1}^* - \mu_{2j}^*, \\ j = 1, \ldots, J' - 1, \\ \mu_{1,j+1}^* - \mu_{1j}^* = c_j(\mu_{12}^* - \mu_{11}^*), \\ j = 2, \ldots, J' - 1, \end{array} \tag{3.4}$$

where the c_j satisfy the same conditions as in Example 2. If we let ω_r denote the subspaces in Example 2 (now considered as subspaces of $\mathcal{R}^J$ with $\mu_{21}, \ldots, \mu_{2J'}$ unrestricted) and let η_s denote the subspaces in Example 3, we see that for H_0^4, $\omega_0 = (\cup_r \omega_r) \cap (\cup_s \eta_s) = \cup_{r,s} (\omega_r \cap \eta_s)$. But for every r and s, $\omega_r \cap \eta_s$ is a linear subspace of Dimension 3. (The subspace ω_r involves $J' - 2$ restrictions, and η_s involves an additional $J' - 1$ restrictions. This results in $\omega_r \cap \eta_s$ having Dimension 3.) Thus λ^* can be used to test H_0^4.

The hypotheses H_0^1, H_0^2, H_0^3, and H_0^4 might be of interest in the study of the ecological theory of character displacement (see, e.g., Grant 1972). In this context the μ_{ij}'s would be means of some characteristic of different species. The spacings between the means could reflect the adaptation to competition among the species for scarce resources. Sinclair, Mosimann, and Meeter (1982) discuss this application of these hypotheses.

4. COMPUTATION OF THE TEST STATISTIC

As in standard linear model theory, the LRT statistic λ^* can be expressed in terms of products of matrices. This will simplify the computation of λ^*. Furthermore, if all the sample sizes within each group are equal and the hypothesis ω_0 satisfies the conditions of Theorem 3, only two, not $m + 1$, sums of squares are needed to compute λ^*. These points will now be discussed.

Let the J-dimensional subspace ω be defined as $\omega = \{\boldsymbol{\xi}: \boldsymbol{\xi} = \mathbf{W}\boldsymbol{\beta}, \boldsymbol{\beta} \in \mathcal{R}^J\}$, where $\mathbf{W}$ is a known $K \times J$ design matrix of rank J. Let the subspaces ω_i be defined by $\omega_i = \{\boldsymbol{\xi}: \boldsymbol{\xi} = \mathbf{W}\boldsymbol{\beta}, \mathbf{H}_i\boldsymbol{\beta} = \mathbf{0}\}$, where $\mathbf{H}_i$ is a known $(J - q_i) \times J$ matrix of rank $J - q_i$, $i = 1, \ldots, m$. Let $\hat{\boldsymbol{\beta}} = (\mathbf{W}'\mathbf{W})^{-1}\mathbf{W}'\mathbf{X}$. Then analogous to standard linear model theory we can write

$$\lambda^*(\mathbf{X}) = \frac{\mathrm{SSH}_0}{\mathrm{SSR}}, \tag{4.1}$$

where

$$\mathrm{SSH}_0 = \min_{1 \leq i \leq m} \hat{\boldsymbol{\beta}}'\mathbf{H}_i'(\mathbf{H}_i(\mathbf{W}'\mathbf{W})^{-1}\mathbf{H}_i')^{-1}\mathbf{H}_i\hat{\boldsymbol{\beta}}, \tag{4.2}$$

and

$$\mathrm{SSR} = (\mathbf{X} - \mathbf{W}\hat{\boldsymbol{\beta}})'(\mathbf{X} - \mathbf{W}\hat{\boldsymbol{\beta}}). \tag{4.3}$$

Expression (4.2) is true since the numerator of λ^* in (2.4) is the minimum of the sums of squares associated with each of the hypotheses H_{0i}: $\boldsymbol{\xi} \in \omega_i$. By standard linear model theory these sums of squares are the expressions given in (4.2).

For the remainder of Section 4, assume the model defined in Section 3, assume ω_0 is a hypothesis about ordered means that satisfies the condition of Theorem 3, and assume all the sample sizes within each group are equal; that is, $K_{i1} = \cdots = K_{iJ_i}$ ($= K_i$, say) $i = 1, \ldots, I$. Under these assumptions, the m sums of squares in (4.2) do not need to be computed. It is possible to determine which sum of squares will be the minimum by examining the order of the sample means, as shown in Theorem 4. Thus only one sum of squares needs to be computed to calculate SSH_0 for these models.

For this model $\hat{\boldsymbol{\beta}} = \bar{\mathbf{X}} = (\bar{\mathbf{X}}_1', \ldots, \bar{\mathbf{X}}_I')'$, where $\bar{\mathbf{X}}_i = (\bar{X}_{i1}, \ldots, \bar{X}_{iJ_i})'$, and $\bar{X}_{ij} = \sum_{k=1}^{K_i} X_{ijk}/K_i$. We will call a vector $\boldsymbol{\nu} \in \mathcal{R}^J$ *correctly ordered* if $\nu_{i1} \leqq \cdots \leqq \nu_{iJ_i}$, $i = 1, \ldots, I$. As in Section 3, $\boldsymbol{\mu}^*$ is the correctly ordered subpermutation of $\boldsymbol{\mu}$ and $\bar{\mathbf{X}}^*$ is the correctly ordered subpermutation of $\bar{\mathbf{X}}$. For any $\boldsymbol{\nu} \in \mathcal{R}^J$ let $\boldsymbol{\Pi}^{\boldsymbol{\nu}}$ be a $J \times J$ subpermutation matrix such that $\boldsymbol{\Pi}^{\boldsymbol{\nu}}\boldsymbol{\nu} = \boldsymbol{\nu}^*$. (If all the coordinates of $\boldsymbol{\nu}$ are distinct, $\boldsymbol{\Pi}^{\boldsymbol{\nu}}$ is unique. But $\boldsymbol{\Pi}^{\boldsymbol{\nu}}$ may not be unique if some of the coordinates of $\boldsymbol{\nu}$ are equal.) The LRT statistic for hypotheses satisfying the conditions of Theorem 3 is given in Theorem 4.

Theorem 4. Assume $K_{i1} = \cdots = K_{iJ_i}$, $i = 1, \ldots, I$. Let $\mathbf{H}$ denote a $(J - q) \times J$ matrix of rank $J - q$. Let $\omega_0 = \{\boldsymbol{\mu} \in \mathcal{R}^J: \mathbf{H}\boldsymbol{\mu}^* = \mathbf{0}\}$. If $\mathbf{H}$ is such that $\mathbf{H}\boldsymbol{\mu} = \mathbf{0}$ implies $\mathbf{H}\boldsymbol{\mu}^* = \mathbf{0}$, then the LRT statistic for testing H_0: $\boldsymbol{\mu} \in \omega_0$ is $\lambda^*(\mathbf{X}) = \mathrm{SSH}_0/\mathrm{SSR}$. The SSR is given by (4.3) and

$$\mathrm{SSH}_0 = \bar{\mathbf{X}}'\mathbf{H}^{*\prime}(\mathbf{H}^*(\mathbf{W}'\mathbf{W})^{-1}\mathbf{H}^{*\prime})^{-1}\mathbf{H}^*\bar{\mathbf{X}}, \tag{4.4}$$

where $\mathbf{H}^* = \mathbf{H}\boldsymbol{\Pi}^{\bar{\mathbf{X}}}$

Theorem 4 will be proved using Lemmas 3 and 4 (proved in the Appendix).

Lemma 3. Assume $\mathbf{H}$, $\mathbf{H}^*$, and ω_0 are as defined in Theorem 4. (a) $N = \{\boldsymbol{\mu} \in \mathcal{R}^J: \mathbf{H}\boldsymbol{\mu} = \mathbf{0}\}$ contains all the values of $\boldsymbol{\mu} \in \omega_0$ which are correctly ordered. (b) $\omega^{\bar{\mathbf{X}}} = \{\boldsymbol{\mu} \in \mathcal{R}^J: \mathbf{H}^*\boldsymbol{\mu} = \mathbf{0}\}$ is a subset of ω_0 and $\omega^{\bar{\mathbf{X}}}$ contains all the values of $\boldsymbol{\mu} \in \omega_0$ such that $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} = \boldsymbol{\mu}^*$.

Lemma 4. For any $\boldsymbol{\mu} \in \mathcal{R}^J$,

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i} K_i(\bar{X}_{ij} - \mu_{ij})^2 \geqq \sum_{i=1}^{I} \sum_{j=1}^{J_i} K_i\,(\bar{X}_{ij} - \nu_{ij})^2, \tag{4.5}$$

where $\boldsymbol{\nu}$ is a subpermutation of $\boldsymbol{\mu}$ such that $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\nu} = \boldsymbol{\nu}^*$.

Proof of Theorem 4. By Theorem 3, if $\boldsymbol{\mu} \in \omega_0$, every subpermutation of $\boldsymbol{\mu}$ is in ω_0. Thus by Lemma 4, $g(\bar{\mathbf{X}}, \boldsymbol{\mu}) = \sum_{i=1}^{I} \sum_{j=1}^{J_i} K_i(\bar{X}_{ij} - \mu_{ij})^2$ is minimized, for $\boldsymbol{\mu} \in \omega_0$, by a $\boldsymbol{\mu}$ such that $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} = \boldsymbol{\mu}^*$. By Lemma 3b, all such $\boldsymbol{\mu}$ are in $\omega^{\bar{\mathbf{X}}} \subset \omega_0$. Thus,

$$\mathrm{SSH}_0 = \inf_{\boldsymbol{\mu} \in \omega_0} g(\bar{\mathbf{X}}, \boldsymbol{\mu}) = \inf_{\boldsymbol{\mu} \in \omega^{\bar{\mathbf{X}}}} g(\bar{\mathbf{X}}, \boldsymbol{\mu})$$

$$= \bar{\mathbf{X}}'\mathbf{H}^{*\prime}(\mathbf{H}^*(\mathbf{W}'\mathbf{W})^{-1}\mathbf{H}^{*\prime})^{-1}\mathbf{H}^*\bar{\mathbf{X}},$$

the last equality being the standard linear model theory result.

APPENDIX: PROOFS OF LEMMAS

Proof of Lemma 1. Let U, V, and W be independent with χ_{s-1}^2, χ_1^2, and χ_t^2 distributions, respectively. Then

$$\begin{aligned} P(U/W > ((s-1)/t)F_{\alpha,s-1,t} &= \alpha \\ &= P((U+V)/W > (s/t)\,F\alpha,\text{s,t}) \\ &> P(U/W > (s/t)F\alpha,\text{s.t}). \end{aligned}$$

Therefore $((s-1)/t)F_{\alpha,s-1,t} < (s/t)F_{\alpha,s,t}$.

Proof of Lemma 2. Let U, V, and Z be independent with χ_{s-1}^2, χ_t^2, and $N(0, 1)$ distributions, respectively. Then

$$\begin{aligned} P(Y_\delta > y) &= P((U + (Z + \sqrt{\delta})^2)/V > sy/t) \\ &> P((Z + \sqrt{\delta})^2/V > sy/t). \end{aligned}$$

But for every $(z, v) \in \{(z, v): -\infty < z < \infty, v > 0\}$, a set with probability one, $(z + \sqrt{\delta})^2/v \to \infty$ as $\delta \to \infty$. Thus, by the dominated convergence theorem, $P((Z + \sqrt{\delta})^2/V > sy/t) \to 1$ as $\delta \to \infty$. Hence, $P(Y_\delta > y) \to 1$ as $\delta \to \infty$.

Proof of Lemma 3. (a) Let $\boldsymbol{\mu} \in \omega_0$, such that $\boldsymbol{\mu}$ is correctly ordered. By Theorem 3, there exists a subpermutation $\boldsymbol{\nu}$ of $\boldsymbol{\mu}$, such that $\boldsymbol{\nu} \in N$. By hypothesis $\boldsymbol{\nu}^* \in N$. But $\boldsymbol{\mu} = \boldsymbol{\nu}^*$ since $\boldsymbol{\mu}$ is correctly ordered. Therefore $\boldsymbol{\mu} \in N$. (b) $\omega^{\bar{\mathbf{X}}} = \{\boldsymbol{\mu}: \mathbf{H}\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} = \mathbf{0}\} = (\boldsymbol{\Pi}^{\bar{\mathbf{X}}})^{-1}N$. So $\omega^{\bar{\mathbf{X}}}$ is one of the $\omega_i \subset \omega_0$ from Theorem 3. Suppose $\boldsymbol{\mu} \in \omega_0$ is such that $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} = \boldsymbol{\mu}^*$. By Theorem 3, $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} \in \omega_0$ and, by Part (a), $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} \in N$. From above we have $\boldsymbol{\mu} = (\boldsymbol{\Pi}^{\bar{\mathbf{X}}})^{-1}\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\mu} \in \omega^{\bar{\mathbf{X}}}$.

Proof of Lemma 4. Since $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\bar{\mathbf{X}}$ and $\boldsymbol{\Pi}^{\bar{\mathbf{X}}}\boldsymbol{\nu}$ are both correctly ordered, $\bar{\mathbf{X}}$ and $\boldsymbol{\nu}$ are in the same order. Let $g(\bar{\mathbf{X}}_i, \boldsymbol{\mu}_i) = -\sum_{j=1}^{J_i} K_i(\bar{X}_{ij} - \mu_{ij})^2 = G(\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i)$. It is easily shown (Marshall and Olkin 1979, p. 57) that G is Schur concave. It follows by Lemma 2.2 of Hollander, Proschan, and Sethuraman (1977) that g is decreasing in transposition and, hence,

$$-\sum_{j=1}^{J_i} K_i(\bar{X}_{ij} - \nu_{ij})^2 \geqq -\sum_{j=1}^{J_i} K_i(\bar{X}_{ij} - \mu_{ij})^2.$$

Inequality (4.5) follows since the above inequality holds for each $i = 1, \ldots, I$.

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