Journal of Theoretical Biology 321 (2013) 40–43

Contents lists available at SciVerse ScienceDirect

Journal of Theoretical Biology

journal homepage: www.elsevier.com/locate/yjtbi

Towards a rigorous framework for studying 2-player continuous games

Shade T. Shutters *

Center for Social Dynamics and Complexity and School of Sustainability, P.O. Box 875402, Arizona State University, Tempe, AZ 85287-5402, USA

HIGHLIGHTS

- Continuous games are increasingly popular to study evolutionary phenomena.
- Leading models of continuous games have led to confusion and errors in the literature.
- A favorite model of the continuous prisoners dilemma actually models other games.
- A favorite model of the continuous snow drift game also models other games.

ARTICLE INFO

Article history:
Received 15 October 2012
Received in revised form
5 December 2012
Accepted 29 December 2012
Available online 7 January 2013

Keywords:
Continuous prisoners dilemma
Continuous snow drift game
Game theory
Evolution

ABSTRACT

The use of 2-player strategic games is one of the most common frameworks for studying the evolution of economic and social behavior. Games are typically played between two players, each given two choices that lie at the extremes of possible behavior (e.g. completely cooperate or completely defect). Recently there has been much interest in studying the outcome of games in which players may choose a strategy from the continuous interval between extremes, requiring the set of two possible choices be replaced by a single continuous equation. This has led to confusion and even errors in the classification of the game being played. The issue is described here specifically in relation to the continuous prisoners dilemma and the continuous snowdrift game. A case study is then presented demonstrating the misclassification that can result from the extension of discrete games into continuous space. The paper ends with a call for a more rigorous and clear framework for working with continuous games.

1. Introduction

Economic game theory is a central framework for addressing questions in several disciplines ranging from evolutionary biology to international relations. Researchers may use a variety of games to study social dilemmas and to suggest resolutions, so it is important to understand exactly what game best represents the system under study (Kummerli et al., 2007; Ostrom et al., 1994; Skyrms, 1996). Policy makers, in particular, often use diplomatic tools in an attempt to change one game into another (Barrett, 2003; Sandler, 2004) and so it is equally important to understand the transitional effects of moving between games.

Fig. 1 presents a comprehensive framework for the simultaneous study of multiple games and for the transitions between those games. However, this framework is limited to binary-choice games—those games in which participants are limited to two choices (e.g. yes or no). Many researchers have begun to move to continuous versions of traditional games because the rich diversity of choices it allows each player is more reflective of the complexities of real world strategic situations (Doebeli and Knowlton, 1998; Grim, 1996; Le and Boyd, 2007; Wahl and Nowak, 1999a, 1999b). Unfortunately there is no framework analogous to Fig. 1 for the systematic exploration of continuous games. A step towards developing such a framework is to first address existing confusing and misconceptions regarding continuous 2-player games.

This paper highlights some of the pitfalls of moving from discrete games to their continuous counterparts. It demonstrates how confusion can arise when translating the qualitative description of a game into a formal representation. Directly translating the qualitative game description into a formula can create an overly simple and overgeneralized formula that may represent much more than is intended. Finally, it illustrates the consequences of this confusion with a case from the literature in which overgeneralizing the formal representation of certain continuous games leads to errors in classifying which game is actually being modeled.

2. Discussion

2.1. Discrete 2-player games

2×2 discrete games are those in which two players i and j play strategies x and y respectively. Each must choose its strategy from

* Tel.: +1 480 278 6098; fax: +1 480 965 2519.
E-mail address: shade.shutters@asu.edu

0022-5193/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.jtbi.2012.12.026

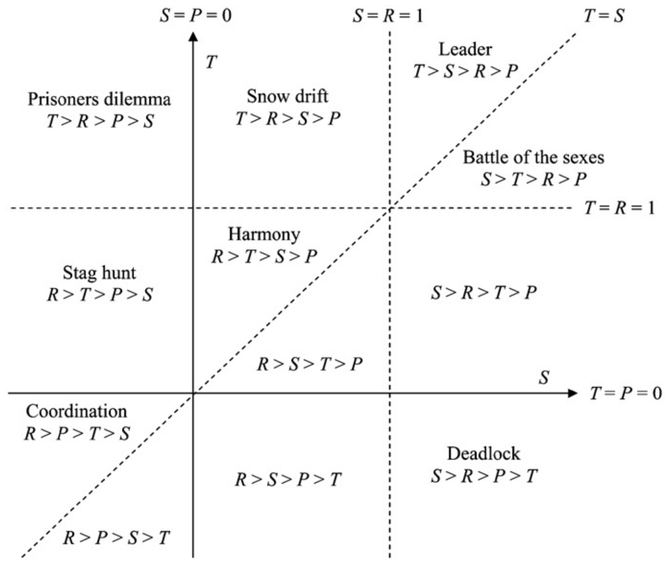

Fig. 1. Partition of the S–T plane into different 2×2 games, which are determined by the ranking of R, T, S, and P. For example the prisoners dilemma is defined as $T > R > P > S$. Here, two payoffs are fixed at $R=1$ and $P=0$ so that 12 different games arise by varying the remaining payoffs S and T, many of which have been named and studied. Adapted from Hauert (2001) and Stark (2010).

two options $\{m, n\}$, so that $x \in \{m, n\}$ and $y \in \{m, n\}$. This leads to four possible outcomes, each of which is denoted by a different payoff to player i in the following payoff matrix:

$$\Pi(x) = \begin{array}{c c} & \begin{array}{cc} y=m & y=n \end{array} \\ \begin{array}{c} x=m \\ x=n \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array} \tag{1}$$

By convention, the payoffs associated with the four possible outcomes are denoted by R, T, P, and S and different games arise depending on the ordering of the four payoffs. For instance, when $R > T > P > S$, the players are engaged in the stag hunt game (Fig. 1).

However, the usefulness of binary choice models may be limited when studying complex behavior and continuous choice models offer a more realistic representation of real-world strategic dilemmas. This shift requires that a game where players choose between two discrete alternatives $x \in \{m, n\}$ be converted to a game where players choose from continuous $x \in [m, n]$. This likewise replaces the set of four possible outcomes with an infinite set of potential outcomes.

2.2. The continuous prisoners dilemma and snow drift games

We now address the confusion that may arise when attempting to extend discrete games to continuous space. Building on earlier work (Killingback and Doebeli, 2002; Killingback et al., 1999; Doebeli, Hauert and Killingback (2004), hereafter DHK, present continuous versions for two of the most well-studied 2-player games, the prisoners dilemma ($T > R > P > S$) and the snowdrift game ($T > R > S > P$). DHK define the 2-person continuous prisoners dilemma as

$$\Pi(x) = B(y) - C(x) \tag{2}$$

and the 2-person continuous snowdrift game as

$$\Pi(x) = B(x+y) - C(x) \tag{3}$$

where B is a benefit function and C is a cost function[1] . These definitions arise from qualitative descriptions of each game, particularly that a player in the prisoners dilemma does not share in his own donation but that a player in the snowdrift game does. The use of (2) and (3) to represent the prisoners dilemma and snow drift games respectively has been widely adopted and several studies rely on the assumption that the above formulae describe their respective games as intended (e.g. Anne-Ly et al., 2010; Doebeli and Hauert, 2005; Jiménez et al., 2009; McNamara et al., 2008; Zhong et al., 2008).

However, as demonstrated below, (2) and (3) can represent not only several different games but they may also represent the exact same game. Though an important step towards developing a comprehensive framework, these continuous game definitions are too generalized to uniquely define one specific game. As the case study in this paper shows, experimental results using either (2) or (3) may have actually been derived using games other than those intended by the experimenter.

2.3. The public goods game: where does it fit?

To better examine DHK's definitions, we turn to a continuous game known as the N-person continuous public goods game (Deng and Chu, 2011), or PGG, which is an instance of the N-person prisoners dilemma (Schofield, 1977). In the PGG, each player is given an endowment e and plays a strategy $x \in [0, e]$, representing the player's contribution to a public good pool. The pool is then multiplied by a cooperative enhancement factor $r \in (1, N)$ and distributed evenly among all N players, regardless of their level of contribution. Thus, the payoff to player i in a PGG is

$$\Pi(x_i) = \frac{1}{N} r \sum_{j=1}^{N} x_j + (e - x_i); \quad r \in (1, N) \tag{4}$$

Because the endowment is somewhat arbitrary, it may be ignored (let $e=0$) for the sake of clarity so that

$$\Pi(x_i) = \frac{1}{N} r \sum_{j=1}^{N} x_j - x_i \tag{5}$$

To correctly determine which game is actually represented by (5) we further simplify the analysis by reducing the game to two players, i and j, playing strategies x and y respectively, and by restricting the strategy choices to $x \in [0, 1]$ and $y \in [0, 1]$. The payoff to i then is

$$\Pi(x) = \frac{r}{2}(x+y) - x; \quad x \in [0,1], \quad y \in [0,1] \tag{6}$$

Because discrete 2×2 games are classified by the ordering of payoffs in (1), a difficulty arises when attempting to classify a continuous function in the same manner. How does one use (1) to rank the possible outcomes of a single function of two continuous choices presented in (6)? One method is to recognize that the discrete version of a game is a special case of its continuous version (Zhong et al., 2012), and to evaluate the four points at which the continuous and discrete versions of a game intersect. We may then verify that the ordering of payoffs at these endpoints is consistent between the discrete and continuous versions of a game. This is accomplished by calculating payoffs at the endpoints of the allowable range of strategy choices for each player (see also Verhoeff, 1998). For example, when $x \in [0, 1]$, we determine the game represented by (6) by evaluating i's 2×2 payoff matrix at full cooperation ($x=1$) and full defection ($x=0$)

[1] Though DHK use P to denote payoffs, Π is used in this paper to avoid confusion with the ordinal payoff matrix in Eq. (1). Otherwise, formulae correspond exactly to those presented in DHK.

Table 1
Determination of which game is being played using DHK's quadratic benefit and cost functions and several coefficient sets. Payoffs in the 2-player PGG (6) are calculated using r=1.5.

DKH simulation	Coefficients used				Payoffs				Payoff order	Game
	b_2	b_1	c_2	c_1	R	T	S	P		
A	−1.4	6	−1.6	4.56	3.4	4.6	1.6	0.0	$T>R>S>P$	Snow drift
B	−1.5	7	−1	4.6	4.4	5.5	1.9	0.0	$T>R>S>P$	Snow drift
C	−0.5	3.4	−1.5	4	2.3	2.9	0.4	0.0	$T>R>S>P$	Snow drift
D	−1.5	7	−1	8	1.0	5.5	−1.5	0.0	$T>R>P>S$	Prisoners dilemma
E	−1.5	7	−1	2	7.0	5.5	4.5	0.0	$R>T>S>P$	Harmony
Additional models										
PGG (6)	0	0.75	0	1	0.5	0.8	−0.3	0.0	$T>R>P>S$	Prisoners dilemma
Stag hunt	1.5	−1	1	1	2.0	0.5	−0.3	0.0	$R>T>P>S$	Stag hunt

for each player

$$\Pi(x)=\begin{array}{c|cc} & y=1 & y=0 \\ \begin{matrix} x=1 \\ x=0\end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} r-1 & \frac{r}{2}-1 \\ \frac{r}{2} & 0 \end{pmatrix}}\end{array}, \tag{7}$$

and ordering the resulting payoffs. When $r\in(1, 2)$, as required by the definition of a PGG, the payoffs in (7) satisfy the inequality

$$\frac{r}{2}>r-1>0>\frac{r}{2}-1 \tag{8}$$

Or using the payoff representations from (1) $T>R>P>S$. Recall that this particular ordering of payoffs defines a prisoners dilemma and therefore the PGG in (6) is correctly classified as a continuous prisoners dilemma, as is its N-person version in (4).

Having confirmed that (6) is a prisoners dilemma we seek to confirm that it adheres to the prisoners dilemma as defined by (2). Through simple rearrangement (6) may be stated as

$$\Pi(x)=\frac{r}{2}y-\left(1-\frac{r}{2}\right)x \tag{9}$$

This is indeed an instance of (2) in which $B(y)=ry/2$ and $C(x)=(1-(r/2))x$. Therefore, (6) is an instance of the continuous prisoners dilemma, as defined by DHK, with linear cost and benefit functions. However, as written, (6) is an instance of (3) in which $B(x+y)=(r/2)(x+y)$ and $C(x)=x$. Thus (6) is also the continuous snowdrift game as defined by DHK. This is clearly inconsistent as the PGG cannot simultaneously be a prisoners dilemma and a snowdrift game. In fact, not only can both (2) and (3) represent the prisoners dilemma, in truth they are so general that each can represent many different 2-player games.

This confusion is not restricted to the 2-player versions of the games in question. When the N-player PGG in (5) is presented as the difference between a benefit function $B(x)$ and a cost function $C(x)$, it may be stated as

$$\Pi(x_i)=\frac{1}{N}B\left(\sum_{j=1}^{N}x_j\right)-C(x_i), \tag{10}$$

where $B(x)=rx$ and $C(x)=x$. This exact formula is identified by DHK (SOM, p. 4) as the N-person continuous snowdrift game. Yet we have already established that (4) is a prisoners dilemma. Potential inconsistencies like this must be addressed in developing a generalized continuous function for simultaneously analyzing multiple 2-player games. Such a function is critical in order to move from an analysis of discrete game space with binary decisions to a continuous analysis of the same space.

2.4. Confusion in practice: a case study

To examine the implications such misclassifications may have, we revisit the five simulations of the continuous snowdrift g ame presented in DHK. Using the validation method for continuous games presented above we determine whether the simulated games are actually the snowdrift game. DHK define their snowdrift game by implementing (3) with quadratic benefit and cost functions

$$B(x)=b_2x^2+b_1x \quad \text{and} \quad C(x)=c_2x^2+c_1x \tag{11}$$

When played between two players using strategies x and y respectively, (3) expands to

$$\Pi(x)=b_2x^2+2b_2xy+b_2y^2+b_1x+b_1y-c_2x^2-c_1x \tag{12}$$

Evaluating this function as in (7) at full cooperation and full defection yields

$$\Pi(x)=\begin{array}{c|cc} & y=1 & y=0 \\ \begin{matrix} x=1 \\ x=0\end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} 4b_2+2b_1-c_2-c_1 & b_2+b_1-c_2-c_1 \\ b_2+b_1 & 0 \end{pmatrix}}\end{array}, \tag{13}$$

To determine the nature of the game represented by (12) we evaluate (13) for a given set of coefficients $\{b_2, b_1, c_2, c_1\}$ and examine the order of payoffs. Table 1 presents results of this technique for DHK's five simulations of their continuous snowdrift game (p. 860, Fig. 1), each of which used a different set of coefficients. Additionally, coefficients are listed that recreate the 2-person PGG in (6) and confirm that it is a prisoners dilemma. Finally, a set of coefficients is introduced that results in the stag hunt game, demonstrating how easily an entirely different game may arise by simply manipulating the model's parameters.

Results presented in Table 1 reveal that DHK's simulations D and E do not model the snowdrift game as the authors intended. In experiment D, $P>S$ indicates their parameters actually created a prisoners dilemma. Not surprisingly, simulation D evolved to 100% defection. In simulation E, $R>T$ indicates a harmony game (Hauert, 2001) and again the result of 100% cooperation is expected. Thus the authors have presented a continuous function which is only a snowdrift game for certain coefficient values; at other values their function models other games. This discrepancy is representative of the potential problems when attempting to extend 2×2 discrete games to continuous functions and highlights the need for a rigorous framework for their study.

3. Summary and future directions

The aim of this paper has not been to point out errors in any single study but to draw attention to the shortcomings of a particular continuous game framework before it becomes widely adopted. The broader goal is to focus efforts on developing a continuous version of the discrete framework presented in Fig. 1. Such a framework is imperative for studying settings where multiple continuous games may be in play and for understanding the effects of transitioning from one game to another.

Acknowledgments

I thank Charles Perrings for helpful feedback on this study.

References

Anne-Ly, D., Lars, R., Thilo, G., 2010. Patterns of cooperation: fairness and coordination in networks of interacting agents. New J. Phys. 12, 063023.
Barrett, S., 2003. Environment and Statecraft: The Strategy of Environmental Treaty-Making. Oxford University Press, Oxford.
Deng, K., Chu, T., 2011. Adaptive evolution of cooperation through Darwinian dynamics in public goods games. PLoS One 6, e25496.
Doebeli, M., Knowlton, N., 1998. The evolution of interspecific mutualisms. Proc. Natl. Acad. Sci. USA 95, 8676–8680.
Doebeli, M., Hauert, C., 2005. Models of cooperation based on the Prisoner's dilemma and the snowdrift game. Ecol. Lett. 8, 748–766.
Doebeli, M., Hauert, C., Killingback, T., 2004. The evolutionary origin of cooperators and defectors. Science 306, 859–862.
Grim, P., 1996. Spatialization and greater generosity in the stochastic prisoner's dilemma. Biosystems 37, 3–17.
Hauert, C., 2001. Fundamental clusters in spatial 2×2 games. Proc. R. Soc. London B: Biol. Sci. 268, 761–769.
Jiménez, R., Lugo, H., San Miguel, M., 2009. Gradual learning and the evolution of cooperation in the spatial continuous prisoner's dilemma. Eur. Phys. J. B—Condens. Matter 71, 273–280.
Killingback, T., Doebeli, M., 2002. The continuous prisoner's dilemma and the evolution of cooperation through reciprocal altruism with variable investment. Am. Nat. 160, 421–438.
Killingback, T., Doebeli, M., Knowlton, N., 1999. Variable investment, the continuous prisoner's dilemma, and the origin of cooperation. Proc. R. Soc. B: Biol. Sci. 266, 1723–1728.
Kummerli, R., Colliard, C., Fiechter, N., Petitpierre, B., Russier, F., Keller, L., 2007. Human cooperation in social dilemmas: comparing the snowdrift game with the prisoner's dilemma. Proc. R. Soc. London B: Biol. Sci. 274, 2965–2970.
Le, S., Boyd, R., 2007. Evolutionary dynamics of the continuous iterated prisoner's dilemma. J. Theor. Biol. 245, 258–267.
McNamara, J.M., Barta, Z., Fromhage, L., Houston, A.I., 2008. The coevolution of choosiness and cooperation. Nature 451, 189–192.
Ostrom, E., Gardner, R., Walker, J., 1994. Rules, Games, and Common-Pool Resources. University of Michigan Press, Ann Arbor, Michigan.
Sandler, T., 2004. Global Collective Action. Cambridge University Press, Cambridge.
Schofield, N., 1977. Dynamic games of collective action. Public Choice 30 77–105
Skyrms, B., 1996. Evolution of the Social Contract. Press Syndicate of the University of Cambridge, Cambridge, UK.
Stark, H.-U., 2010. Dilemmas of partial cooperation. Evolution 64, 2458–2465.
Verhoeff, T., 1998. The Trader's Dilemma: A Continuous Version of the Prisoner's Dilemma. Eindhoven University of Technology, Eindhoven, Netherlands.
Wahl, L.M., Nowak, M.A., 1999a. The continuous prisoner's dilemma: I. Linear reactive strategies. J. Theor. Biol. 200, 307–321.
Wahl, L.M., Nowak, M.A., 1999b. The continuous prisoner's dilemma: II. Linear reactive strategies with noise. J. Theor. Biol. 200, 323–338.
Zhong, L.-X., Qiu, T., Xu, J.-R., 2008. Heterogeneity improves cooperation in continuous snowdrift game. Chin. Phys. Lett. 25, 2315.
Zhong, W., Kokubo, S., Tanimoto, J., 2012. How is the equilibrium of continuous strategy game different from that of discrete strategy game? Biosystems 107, 88–94.