Consider the following magic squares\(^1\) with lines connecting the values in ascending order.

\[
\begin{array}{ccc}
4 & 9 & 2 \\
3 & 5 & 7 \\
8 & 1 & 6 \\
\end{array}
\quad
\begin{array}{ccc}
4 & 9 & 8 \\
11 & 7 & 3 \\
6 & 5 & 10 \\
\end{array}
\quad
\begin{array}{ccc}
8 & 19 & 18 \\
25 & 15 & 5 \\
12 & 11 & 22 \\
\end{array}
\]

A curious pattern appears, which raises the question of whether the same polygonal pattern appears in all magic squares.\(^2\) The following magic square tells us that that the answer to this question is no.

\[
\begin{array}{ccc}
8 & 1 & 9 \\
7 & 6 & 5 \\
3 & 11 & 4 \\
\end{array}
\]

In fact, there are only 2 possible patterns. In order to verify this claim, we define requisite notation, introduce several preliminary properties, and present the main result in Property 3. Let the 9 unique values of a \(3 \times 3\) magic square be denoted as \(a_1 < a_2 < \ldots < a_9\). The position values are denoted as \(x_{ij}\), i.e.,

\[
\begin{array}{ccc}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33} \\
\end{array}
\]

and \(S = x_{11} + x_{12} + x_{13}\) denotes the magic constant.

\[^{1}\text{Nine unique positive numbers arranged in a square matrix such that all row sums, column sums, and diagonal sums are equal.}\]

\[^{2}\text{This question was raised in the August 1997 issue of Scientific American (see p. 88).}\]
Property 1. $a_5 = x_{22} = S/3$.

Proof.\textsuperscript{3}  
\begin{align*}
x_{11} + x_{21} + x_{31} &= S \Rightarrow x_{21} + x_{31} = S - x_{11} \\
x_{11} + x_{22} + x_{33} &= S \Rightarrow x_{22} + x_{33} = S - x_{11} \\
x_{13} + x_{23} + x_{33} &= S \Rightarrow x_{23} + x_{33} = S - x_{13} \\
x_{13} + x_{22} + x_{31} &= S \Rightarrow x_{22} + x_{31} = S - x_{13}
\end{align*}

Therefore, $x_{21} + x_{31} = x_{22} + x_{13}$ and $x_{23} + x_{33} = x_{22} + x_{31}$. Adding these equations:
\[x_{21} + x_{31} + x_{23} + x_{33} = 2x_{22} + x_{33} + x_{31} \text{ or } x_{21} + x_{23} = 2x_{22}.
\]
Adding $x_{22}$ to each side of the equation gives,
\[x_{21} + x_{22} + x_{23} = 3x_{22}.
\]
But $x_{21} + x_{22} + x_{23} = S$, which implies $x_{22} = S/3$.

In order to show that $a_5 = x_{22}$, note that $x_{11} + x_{22} + x_{33} = S$ and $x_{22} = S/3$ implies that either $x_{11} < x_{22} < x_{33}$ or $x_{11} > x_{22} > x_{33}$. Similarly, $x_{13} < x_{22} < x_{31}$ or $x_{13} > x_{22} > x_{31}$, $x_{21} < x_{22} < x_{23}$ or $x_{21} > x_{22} > x_{23}$, and $x_{12} < x_{22} < x_{32}$ or $x_{12} > x_{22} > x_{32}$. Therefore, there are 4 values smaller than $x_{22}$ and 4 values larger than $x_{22}$, which implies $a_5 = x_{22}$. \(\square\)

Property 2. $a_1 + a_9 = a_2 + a_8 = a_3 + a_7 = a_4 + a_6 = 2a_5$.

Proof. Due to Property 1, for any $i \in \{1,2,3,4,\ldots,9\}$ there exists some $j \in \{1,2,3,4,\ldots,9\}\setminus\{i\}$ such that $a_i + a_5 + a_j = S$. Consider $i = 1$ and suppose $j \neq 9$. Then $S = a_1 + a_5 + a_j \leq a_1 + a_5 + a_9 < a_1 + a_5 + a_9 < a_9 + a_5 + a_k$ for $k \neq 1$. But when $i = 9$, there exits some $k \in \{2,3,4,\ldots,8\}$ such that $a_9 + a_5 + a_k = S$, which contradicts the possibility $j \neq 9$. Therefore, $a_1 + a_5 + a_9 = S = 3a_5$, or $a_1 + a_9 = 2a_5$. The same arguments can be applied for other values of $i$ (i.e., consider $i = 2$ and suppose $j \notin \{8,9\}$, consider $i = 3$ and suppose $j \notin \{7,8,9\}$, etc.). \(\square\)

Corollary 1. The values of $a_1, a_2, a_3, a_4, a_6, a_7, a_8,$ and $a_9$ are symmetric with respect to $a_5$, i.e., $a_5 - a_1 = a_9 - a_5$, $a_5 - a_2 = a_8 - a_5$, $a_5 - a_3 = a_7 - a_5$, and $a_5 - a_4 = a_6 - a_5$.

Proof. Follows directly from Property 2 (e.g., $a_1 + a_9 = 2a_5$ can be rewritten as $a_5 - a_1 = a_9 - a_5$). \(\square\)

\textsuperscript{3} Property 1 is a slight extension of the property $x_{22} = S/3$; the proof of $x_{22} = S/3$ appears in The Moscow Puzzles by B.A. Kordemsky (Dover Publications, 1992, p. 292).
**Property 3.** A 3 × 3 magic square is 1 of 2 possible forms.

**Form 1:**

\[
\begin{array}{ccc}
    a_7 & a_1 & a_8 \\
    a_6 & a_5 & a_4 \\
    a_2 & a_9 & a_3 \\
\end{array}
\]

**Form 2:**

\[
\begin{array}{ccc}
    a_6 & a_1 & a_8 \\
    a_7 & a_5 & a_3 \\
    a_2 & a_9 & a_4 \\
\end{array}
\]

Specific magic squares can be composed according to the following rules.

**Form 1:**
1. Arbitrarily select values for $D_1$ and $D_2$.
2. Arbitrarily select a value for $a_5$ that satisfies $a_5 > 3D_1 + 2D_2$.
3. Set the remaining values according to:
   \[
   \begin{align*}
   a_6 &= a_5 + D_1, \\
   a_7 &= a_6 + D_2, \\
   a_8 &= a_7 + D_1, \\
   a_9 &= a_8 + D_1 + D_2, \\
   a_1 &= a_2 - (D_1 + D_2).
   \end{align*}
   \]

Accordingly, form 1 can be rewritten as:

\[
\begin{align*}
   a_5 + (D_1 + D_2) & \quad a_5 - (3D_1 + 2D_2) \quad a_5 + (2D_1 + D_2) \\
   a_5 + D_1 & \quad a_5 \quad a_5 - D_1 \\
   a_5 - (2D_1 + D_2) & \quad a_5 + (3D_1 + 2D_2) \quad a_5 - (D_1 + D_2)
\end{align*}
\]

**Form 2:**
1. Arbitrarily select values for $D_1$ and $D_2$.
2. Arbitrarily select a value for $a_5$ that satisfies $a_5 > 3D_1 + 2D_2$.
3. Set the remaining values according to:
   \[
   \begin{align*}
   a_6 &= a_5 + D_1, \\
   a_7 &= a_6 + D_2, \\
   a_8 &= a_7 + D_1, \\
   a_9 &= a_8 + D_1, \\
   a_1 &= a_2 - D_1.
   \end{align*}
   \]

Accordingly, form 2 can be rewritten as:

\[
\begin{align*}
   a_5 + D_1 & \quad a_5 - (3D_1 + D_2) \quad a_5 + (2D_1 + D_2) \\
   a_5 + (D_1 + D_2) & \quad a_5 \quad a_5 - (D_1 + D_2) \\
   a_5 - (2D_1 + D_2) & \quad a_5 + (3D_1 + D_2) \quad a_5 - D_1
\end{align*}
\]

**Proof.** We may limit our consideration to the possibilities of $a_1 = x_{11}$ and $a_1 = x_{12}$ (cases 1 and 2 below). This is because the form of a magic square does not substantively change when it is transposed (i.e., columns become rows and rows become columns), columns 1 and 3 are interchanged, or rows 1 and 3 are interchanged.
Case 1: $a_1 = x_{11}$
From Property 2, it follows that $a_0 = x_{33}$. This means that $x_{31} + x_{32} = x_{13} + x_{23} = a_1 + a_5$. But this implies that $x_{31}, x_{32}, x_{13}, x_{23}$ all must be between $a_1$ and $a_5$. This is impossible because there are only 3 such values. Hence $a_1 \neq x_{11}$, and a form corresponding to case 1 does not exist.

Case 2: $a_1 = x_{12}$
From Property 2, it follows that $a_0 = x_{32}$. This means that $x_{31} + x_{33} = a_1 + a_5$, which implies that $x_{31}$ and $x_{33}$ are between $a_1$ and $a_5$. As a form is invariant when columns are interchanged, we may assume without loss of generality that $x_{31} < x_{33}$. This leads to 3 possibilities that we consider in turn.

Case 2a: $a_1 = x_{12}, x_{31} = a_3, x_{33} = a_4$
From Property 2, $x_{31} = a_3$ implies $x_{13} = a_7$, and $x_{33} = a_4$ implies $x_{11} = a_6$. Up to this point, the magic square appears as:

\[
\begin{array}{ccc}
    a_6 & a_1 & a_7 \\
    a_5 & & \\
    a_3 & a_9 & a_4 \\
\end{array}
\]

From $a_6 + x_{21} + a_3 = a_7 + x_{23} + a_4$, it follows that $x_{21} = a_8$ and $x_{23} = a_2$. But $a_6 + a_8 + a_3 < a_7 + a_2 + a_4$. Hence, a form corresponding to case 2a does not exist.

Case 2b: $a_1 = x_{12}, x_{31} = a_2, x_{33} = a_3$
From Property 2, $x_{31} = a_2$ implies $x_{13} = a_8$, and $x_{33} = a_3$ implies $x_{11} = a_7$. From $a_7 + x_{21} + a_2 = a_8 + x_{23} + a_3$, it follows that $x_{21} = a_6$ and $x_{23} = a_4$, and the magic square appears as:

\[
\begin{array}{ccc}
    a_7 & a_1 & a_8 \\
    a_6 & a_5 & a_4 \\
    a_2 & a_9 & a_3 \\
\end{array}
\]

Note that $a_1 + a_7 + a_8 = a_2 + a_6 + a_7$ implies

\[
a_8 - a_6 = a_2 - a_1
\]

and $a_1 + a_7 + a_8 = a_2 + a_3 + a_9$ implies

\[
(a_9 - a_8) + (a_2 - a_1) = (a_7 - a_3).
\]

But from Corollary 1, it follows that $a_2 - a_1 = a_0 - a_8$ and $(a_7 - a_3) = 2(a_7 - a_5)$. Therefore,

\[
a_8 - a_6 = a_9 - a_8 = a_7 - a_5.
\]

Furthermore, $a_8 - a_6 = a_7 - a_5$ implies $a_8 - a_7 = a_6 - a_5$. Letting $\Delta_1 = a_6 - a_5$ and $\Delta_2 = a_7 - a_6$ we find $a_8 - a_7 = \Delta_1$ and $a_9 - a_8 = \Delta_1 + \Delta_2$, which leads to the rules for composing a magic square that matches form 1.

Case 2c: $a_1 = x_{12}, x_{31} = a_2, x_{33} = a_4$
From Property 2, $x_{31} = a_2$ implies $x_{13} = a_8$, and $x_{33} = a_4$ implies $x_{11} = a_6$. From $a_6 + x_{21} + a_2 = a_8 + x_{23} + a_4$, it follows that $x_{21} = a_7$ and $x_{23} = a_3$, and the magic square appears as:

\[
\begin{array}{ccc}
    a_6 & a_1 & a_8 \\
    a_7 & a_5 & a_3 \\
    a_2 & a_9 & a_4 \\
\end{array}
\]
Note that $a_2 + a_4 + a_9 = a_3 + a_4 + a_8$ implies
\[a_9 - a_8 = a_3 - a_2\]
and $a_1 + a_6 + a_8 = a_3 + a_4 + a_8$ implies
\[a_6 - a_4 = a_3 - a_1.\]
But from Corollary 1, it follows that $a_3 - a_2 = a_8 - a_7$, $a_6 - a_4 = 2(a_6 - a_5)$, and $a_3 - a_1 = a_9 - a_7$. Therefore,
\[a_9 - a_8 = a_8 - a_7\] and
\[a_9 - a_7 = 2(a_6 - a_5).\]
Furthermore, $a_9 - a_8 = a_8 - a_7$ implies $a_9 - a_7 = 2(a_9 - a_8) = 2(a_6 - a_5)$, or $a_9 - a_8 = a_6 - a_5$.
Letting $\Delta_1 = a_6 - a_5$ and $\Delta_2 = a_7 - a_6$ we find $a_8 - a_7 = \Delta_1$ and $a_9 - a_8 = \Delta_1$, which leads to the rules for composing a magic square that matches form 2. \(\square\)

**Two Examples**

<table>
<thead>
<tr>
<th>Form 1: $\Delta_1 = 1$, $\Delta_2 = 2$, $a_5 = 10$</th>
<th>Form 2: $\Delta_1 = 1$, $\Delta_2 = 2$, $a_5 = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13 3 14</td>
<td>7 1 10</td>
</tr>
<tr>
<td>11 10 9</td>
<td>9 6 3</td>
</tr>
<tr>
<td>6 17 7</td>
<td>2 11 5</td>
</tr>
</tbody>
</table>