Supplementary Materials

PROOF OF THEOREM 1

Lemma 3: The maximum stability region of the F-framed policies can be characterized by

$$\overline{\Lambda_{\mathrm{NC}}(F)} = \left\{ \lambda \succeq 0 \colon \lambda F \preceq \sum_{J\in\mathcal{J}(F)} \pi(J)\eta_J, \eta_J \in \mathcal{CH}(M_J) \right\},$$

where $\overline{A}$ denotes the closure of A and $\mathcal{CH}(M_J)$ is the convex hull over the set of columns of M_J.

Proof of Lemma 3: If λ is strictly outside the maximum stability region, it can be proved that the total amount of deficits increase to infinity with probability one using the strictly separating hyperplane theorem [1] and Lyapunov drift arguments. If λ is strictly inside the maximum stability region, then we can find $\eta = (\eta_J \colon J \in \mathcal{J}(F))$ that dominates λ and make the long-term-average of the scheduling process be at least $\eta \succ \lambda$, where $\succ$ denotes strict pairwise greater than. So the system can be stabilized. ■

We then present the next lemma.

Lemma 4: For any F,

$$\Lambda_{\mathrm{NC}}(F) \supseteq \mathrm{int}\left(\Lambda \cap \left(\Lambda - \frac{\tau_{\max}}{F}\mathbf{1}\right)\right),$$

where $\Lambda - \frac{\tau_{\max}}{F}\mathbf{1} = \{\lambda - \frac{\tau_{\max}}{F}\mathbf{1} \colon \lambda \in \Lambda\}$.

Proof of Lemma 4: Note that given the schedules of any causal policy, we can convert them into valid schedules under the F-framed policy by removing those transmissions that serve those packets whose arrival times and transmission times are not in the same frame. Since at most one packet can be scheduled on a link at each time slot and the maximum delay bound is $\tau_{\max}$, the number of packets across frame (i.e., those arriving in one frame with deadlines in another) scheduled by the causal policy on a link is at most $\tau_{\max}$ at the end of a frame. As a result, we only need to remove at most $\tau_{\max}$ transmissions for each frame on each link, which is equivalent to at most $\tau_{\max}/F$ packets per time slot. So the lemma holds. ■

Lemma 5: If $F > \tau_{\max}$, then

$$\Lambda_{\mathrm{LDF}} \supseteq \sigma^* \cdot \mathrm{int}(\Lambda_{\mathrm{NC}}(F)).$$

Proof of Lemma 5: Let $\lambda' \in \mathrm{int}(\Lambda_{\mathrm{NC}}(F))$ and let $\lambda = \sigma^*\lambda'$. Then by the definition of interior point and the characterization of $\Lambda_{\mathrm{NC}}(F)$ in Lemma 3, there exist $(\xi_J \colon J \in \mathcal{J}(F))$ with $\xi_J \in \mathcal{CH}(M_J)$ for each $J \in \mathcal{J}(F)$ and $\delta > 0$ such that

$$\lambda' + \delta\mathbf{1} \preceq \frac{1}{F}\sum_{J\in\mathcal{J}(F)} \pi(J)\xi_J, \tag{5}$$

where $\mathbf{1}$ is a vector with all 1's.

We now establish the fluid limits for the system sampled every F time slots. Let $\{D(t), t \in \mathbb{N}_+\}$ and $\{S(t), t \in \mathbb{N}_+\}$ be the cumulative deficit and service processes under LDF (without frame). Let $\Psi_{l,\tau}(t)$ be the number of packets with deadline $t+\tau-1$ on link l at time slot t, and let $\Psi(t) = (\Psi_{l,\tau}(t) \colon l \in \mathcal{K}, 1 \le \tau \le \tau_{\max})$. Then under LDF $\{(\Psi(t), D(t)), t \in \mathbb{N}_+\}$ is a Markov chain. Let $\{(\Psi^{(n)}(t), D^{(n)}(t)), t \in \mathbb{N}_+\}$ be the system with arbitrary initial state $(\Psi^{(n)}(0), D^{(n)}(0))$ associated with the requirement that $\|(\Psi^{(n)}(0), D^{(n)}(0))\| := \sum_{l\in\mathcal{K}}\sum_{\tau=1}^{\tau_{\max}} \Psi_{l,\tau}^{(n)}(0) + \sum_{l\in\mathcal{K}} D_l^{(n)}(0) = n$ for any $n \in \mathbb{N}_+$, and let $S^{(n)}(t)$ be the corresponding cumulative service process. We sample $\Psi^{(n)}$, $D^{(n)}$ and $S^{(n)}$ every F time slots to get $\Psi^{(n)(F)}(t) = \Psi^{(n)}(Ft)$, $D^{(n)(F)}(t) = D^{(n)}(Ft)$ and $S^{(n)(F)}(t) = S^{(n)}(Ft)$ for $t \in \mathbb{N}_+$. Define the *scaled* deficit and service processes to be

$$\bar{D}^{(n)(F)}(t) = \frac{1}{n} D^{(n)(F)}(\lfloor nt \rfloor)$$

and

$$\bar{S}^{(n)(F)}(t) = \frac{1}{n} S^{(n)(F)}(\lfloor nt \rfloor).$$

Note that the scaled processes are defined for any nonnegative real number t rather than just integers, and can take values in vectors of multiples of $\frac{1}{n}$ rather than vectors of integers. Following Lemma 1 in Andrews et al. [2], for almost all sample paths and any sequence of initial states there exists a subsequence (n_j) such that for any $l \in \mathcal{K}$

$$\bar{D}_l^{(n_j)(F)} \to \bar{D}_l^{(F)} \quad \text{u.o.c.} \tag{6}$$

and

$$\bar{S}_l^{(n_j)(F)} \to \bar{S}_l^{(F)} \quad \text{u.o.c.}$$

as $j \to \infty$, where u.o.c. denotes uniform convergence over compact sets, and $\bar{D}_l^{(F)}$ and $\bar{S}_l^{(F)}$ are nonnegative nondecreasing Lipschitz-continuous functions with domain $\mathbb{R}_+$. The limiting functions are called the fluid limits.

Let $L_0^{(F)}(t)$ be the set of links with the largest deficit fluids at time t, and let $L^{(F)}(t) \subseteq L_0^{(F)}(t)$ be the set of links in $L_0^{(F)}(t)$ with largest derivatives at time t; i.e.,

$$L_0^{(F)}(t) = \left\{ l \in \mathcal{K} \colon \bar{D}_l^{(F)}(t) = \max_{i\in\mathcal{K}} \bar{D}_i^{(F)}(t) \right\}$$

and

$$L^{(F)}(t) = \left\{ l \in L_0^{(F)}(t) \colon \frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_l^{(F)}(t) = \max_{i\in L_0(t)} \frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_i^{(F)}(t) \right\},$$

where we assume t is a regular point; i.e., the derivatives of the fluid limits exist at t. Then we can construct $(\eta_J \in \mathcal{CH}(M_J) \colon J \in \mathcal{J}(F))$ such that for any $l \in L^{(F)}(t)$, the service fluids satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{S}_l^{(F)}(t) \ge \sum_{J\in\mathcal{J}(F)} \pi(J)\eta_{J,l} - \tau_{\max}, \tag{7}$$

where $\eta_{J,l}$ is the lth entry of the vector η_J.

To understand (7), note that $\bar{S}^{(F)}(t)$ is the fluid limit of the service process sampled every F time slots, so the derivative of $\bar{S}^{(F)}(t)$ is the average service over F time slots under LDF. Now consider a frame of F time slots with arrival and maximum delay pattern J, and denote by s_J^{LDF} the schedule under LDF during the F time slots. We next construct another schedule s_J^F, which is a maximal schedule under the F-framed policy. The construction is by removing those transmissions in s_J^{LDF} which serve packets that arrived before the frame started, and then add more transmissions to make it a maximal schedule. So for link l,

$$W(s_J^{\mathrm{LDF}})_l - o_{J,l} + n_{J,l} = W(s_J^F)_l,$$

where $o_{J,l}$ is the number of removed transmissions on link l, $n_{J,l}$ is the number of added transmissions on link l, $W(s_J^F) \in \mathcal{M}_J$, and $(\cdot)_l$ denotes the lth component of the vector. Note that those removed transmissions must occur at the first $\tau_{\max}$ time slots because the maximum delay is $\tau_{\max}$ so none of the packets that arrived before the frame can be transmitted after the first $\tau_{\max}$ time slots. This also implies that the added transmissions must be in the first $\tau_{\max}$ time slots as well. Therefore, $n_{J,l} \le \tau_{\max}$, and

$$W(s_J^{\mathrm{LDF}})_l \ge W(s_J^F)_l - \tau_{\max}$$

holds for any J.

Now assuming $\bar{D}_l^{(F)}(t) > 0$ for $l \in L^{(F)}(t)$, the derivative of $\bar{D}_l^{(F)}(t)$ is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_l^{(F)}(t) &= \lambda_l F - \frac{\mathrm{d}}{\mathrm{d}t}\bar{S}_l^{(F)}(t) && (8)\\
&\le \lambda_l F - \sum_{J\in\mathcal{J}(F)} \pi(J)\eta_{J,l} + \tau_{\max} && (9)\\
&\le \sigma^*\left(\sum_{J\in\mathcal{J}(F)} \pi(J)\xi_{J,l} - \delta F\right) \\
&\quad - \sum_{J\in\mathcal{J}(F)} \pi(J)\eta_{J,l} + \tau_{\max} && (10)\\
&= \left[\sigma^*\left(\sum_{J\in\mathcal{J}(F)} \pi(J)\xi_{J,l}\right)\right. && (11)\\
&\quad \left. - \left(\sum_{J\in\mathcal{J}(F)} \pi(J)\eta_{J,l}\right)\right] && (12)\\
&\quad - \sigma^*\delta F + \tau_{\max}, && (13)
\end{aligned}$$

where (9) comes from (7), and (10) holds because $\lambda = \sigma^*\lambda'$ and inequality (5). By the definition of the R-LPF and the fact that $L^{(F)}(t)$ has higher scheduling priority over $\mathcal{K}\backslash L^{(F)}(t)$, there exists $i \in L^{(F)}(t)$ such that

$$\sigma^*\left(\sum_{J\in\mathcal{J}(F)} \pi(J)\xi_{J,i}\right) \le \left(\sum_{J\in\mathcal{J}(F)} \pi(J)\eta_{J,i}\right).$$

Thus by definition of $L^{(F)}(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_l^{(F)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_i^{(F)}(t) \le \tau_{\max} - \sigma^*\delta F.$$

We note that for any positive integer k,

$$\Lambda_{\mathrm{NC}}(F) \subseteq \Lambda_{\mathrm{NC}}(kF),$$

since any F-framed policy is a valid but more restrictive kF-framed policy. Then (5) holds with the same δ for any frame size kF. Thus for large enough integer k, the deficit fluid limits associated with the frame size kF satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{D}_l^{(kF)}(t) \le \tau_{\max} - \sigma^*\delta kF \le -\epsilon < 0$$

for some $\epsilon > 0$, as long as $\max_l \bar{D}_l^{(kF)}(t) > 0$ and t is regular. Since $\|\bar{D}^{(kF)}(0)\| = 1$, we have $\|\bar{D}^{(kF)}(t)\| = 0$ for any $t \ge 1/\epsilon$. By the convergence in (6) and the arbitrary choice of initial states of the systems with the prescribed requirements, we have that $\|\bar{D}^{(n)(kF)}(t)\| \to 0$ almost surely as $n \to \infty$ for any $t \ge 1/\epsilon$. Since $\{\bar{D}^{(n)(kF)}(t), n \in \mathbb{N}_+\}$ is uniformly integrable (see, e.g., Dai [3]), we get that $\mathbb{E}\|\bar{D}^{(n)(kF)}(t)\| \to 0$ as $n \to \infty$ for $t \ge 1/\epsilon$. Note that $\sum_{l\in\mathcal{K}}\sum_{\tau=1}^{\tau_{\max}} \Psi_{l,\tau}^{(n)(kF)}(nt) \le K a_{\max}\tau_{\max}^2$. We then have

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\|(\Psi^{(n)(kF)}(nt), D^{(n)(kF)}(nt))\|\right] = 0$$

for any $t \ge 1/\epsilon$. Then by Theorem 4 in Andrews et al. [2] we get that the sampled deficit process of the original system $\{D^{(kF)}(t), t \in \mathbb{N}_+\}$ is stable as defined in Andrews et al. [2], which implies the existence of a stationary distribution of $\{D^{(kF)}(t), t \in \mathbb{N}_+\}$ and in turn implies the stability of $\{D^{(kF)}(t), t \in \mathbb{N}_+\}$ as defined in Definition 1. Finally,

$$\begin{aligned}
&\lim_{C\to\infty}\limsup_{t\to\infty} \Pr\left(\sum_{l\in\mathcal{K}} D_l(t) \ge C\right)\\
&\le \lim_{C\to\infty}\limsup_{t\to\infty} \Pr\left(\sum_{l\in\mathcal{K}} D_l^{(kF)}\left(\left\lfloor \frac{t}{kF}\right\rfloor\right) + kKFa_{\max} \ge C\right)\\
&= \lim_{C\to\infty}\limsup_{t\to\infty} \Pr\left(\sum_{l\in\mathcal{K}} D_l^{(kF)}(t) \ge C - kKFa_{\max}\right)\\
&= 0.
\end{aligned}$$

So the original unsampled deficit process $\{D(t), t \in \mathbb{N}_+\}$ is stable, and therefore the deficit arrival rate $\lambda = \sigma^*\lambda' \in \Lambda_{\mathrm{LDF}}$. ■

We can now proceed to prove Theorem 1.

Proof of Theorem 1: By Lemma 5 and Lemma 4, we have

$$\Lambda_{\mathrm{LDF}} \supseteq \sigma^* \cdot \mathrm{int}\left(\Lambda \cap \left(\Lambda - \frac{\tau_{\max}}{F}\mathbf{1}\right)\right).$$

The theorem is obtained by letting $F \to \infty$. ■

References

[1] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY: Cambridge Univ. Press, 2004.

[2] M. Andrews, K. Kumaran, K. Ramanan, A. Stolyar, R. Vijayakumar, and P. Whiting, "Scheduling in a queuing system with asynchronously varying service rates," *Prob. Eng. and Informational Sciences*, vol. 18, no. 2, pp. 191–217, Apr. 2004.

[3] J. G. Dai, "On positive Harris recurrence of multiclass queueing networks: A unified approach via fluid limit models," *Ann. Appl. Probab.*, vol. 5, no. 1, pp. 49–77, Feb. 1995.