Proof of lemma A1.

We first write the second stage revenue function.

\[
\Pi(y_0, y_1) = H_0(y_0 + y_1) + S_0(y_0 - y_0)^+ + S_1(y_1 - y_1)^+ - \alpha_0 y_0 - \beta_1 y_1
\]

Taking first order derivative w.r.t \( y_0 \) and \( y_1 \),

\[
\frac{\partial \Pi(y_0, y_1)}{\partial y_0} = (r + p - s_2) \bar{F}(y_0 + y_1) + s_2 - s_0 - \alpha_0 ,
\]

\[
\frac{\partial \Pi(y_0, y_1)}{\partial y_1} = (r + p - s_2) \bar{F}(y_0 + y_1) + s_2 - s_1 - \beta_1 .
\]

Hence, the "interior" solution satisfies

\[
\begin{aligned}
y_0^* + y_1^* &= \bar{F}(\frac{r + p - s_0 - \alpha_0}{r + p - s_2}) \quad (A) \\
y_0^* + y_1^* &= \bar{F}(\frac{r + p - s_1 - \beta_1}{r + p - s_2}) \quad (B)
\end{aligned}
\]

We need to incorporate the following constraints

\[0 \leq y_0^* \leq q_0 \quad , \quad 0 \leq y_1^* \leq q_1 .\]

To do this, we need to consider the regions of \( q_0 \) and \( q_1 \).

Consider the case where \( q_1 \leq -(s_1 - s_0) + \beta_0 \) (i.e., \( (A) \leq (B) \)).
Note that in this case
\[
F'(\frac{r+p-s_0-\bar{z}_1}{r+p-s_2}) \leq F'\left(\frac{r+p-s_1-\bar{z}_2}{r+p-s_2}\right)
\]
This implies that \( y_0^* = 0 \), \( y_1^* = F'(\frac{r+p-s_1-\bar{z}_2}{r+p-s_2}) \), if
\[
F'(\frac{r+p-s_1-\bar{z}_2}{r+p-s_2}) \leq \bar{z}_1 \iff \bar{z}_1 \geq (r+p-s_1)-(r+p-s_2)F(\bar{z}_1)
\]
But we also need to make sure that \( y_1^* \geq 0 \), which requires that \( r+p-s_1-\bar{z}_1 \geq 0 \) \( \Rightarrow \bar{z}_1 \leq r+p-s_1 \).
So summarize the above analysis so far, we have
\[
\begin{cases}
  y_0^* = 0 \\
y_1^* = F'(\frac{r+p-s_1-\bar{z}_2}{r+p-s_2}) , \text{ if } (r+p-s_1)-(r+p-s_2)F(\bar{z}_1) \\
\end{cases} \leq \bar{z}_1 \leq r+p-s_1
\]
and
\[
\begin{cases}
  y_0^* = 0 \\
y_1^* = 0 , \text{ if } \bar{z}_1 > r+p-s_1
\end{cases}
\]
Next consider the situation where
\[
F'(\frac{r+p-s_1-\bar{z}_2}{r+p-s_2}) > \bar{z}_1 \iff \bar{z}_1 < (r+p-s_1)-(r+p-s_2)F(\bar{z}_1)
\]
In this case, we know \( y_1^* = \bar{z}_1 \).
Using FOC (A), we know
\[
y_0^* = F'(\frac{r+p-s_0-\bar{z}_0}{r+p-s_2}) - \bar{z}_1
\]
we need to ensure that $y^*_0 > 0$ and $y^*_0 \leq y_0$.

\[
y^*_0 > 0 \iff f\left(\frac{r+p-s_0-z_0}{r+p-s_z}\right) - y_1 > 0
\]
\[
\iff r+p-s_0-z_0 > (r+p-s_z) F(\delta_1)
\]
\[
\iff z_0 \leq (r+p-s_0) - (r+p-s_z) F(y_1)
\]

\[
y^*_0 \leq y_0 \iff f\left(\frac{r+p-s_0-z_0}{r+p-s_z}\right) - y_1 \leq y_0
\]
\[
\iff r+p-s_0-z_0 \leq (r+p-s_z) F(y_0+y_1)
\]
\[
\iff z_0 \geq (r+p-s_0) - (r+p-s_z) F(y_0+y_1)
\]

Hence, to summarize above, we have

\[
\begin{cases}
    y^*_0 = f\left(\frac{r+p-s_0-z_0}{r+p-s_z}\right) - y_1, & \text{if } z_0 \leq (r+p-s_0) - (r+p-s_z) F(\delta_1) \\
    y^*_0 = y_0, & \text{if } z_0 > (r+p-s_0) - (r+p-s_z) F(y_0+y_1)
\end{cases}
\]

\[
\begin{cases}
    y^*_1 = y_1, & \text{if } z_0 < (r+p-s_0) - (r+p-s_z) F(y_0+y_1)
\end{cases}
\]

The above analysis completes the characterization of the lower-right part of Figure A1.

The case of $z_1 > -(s_1-s_0)+z_0$ can be analogously analyzed.