Outage analysis of Block-Fading Gaussian Interference Channels

Yang Weng and Daniela Tuninetti
University of Illinois at Chicago, IL (USA), Department of Electrical and Computer Engineering, Email: yweng3@uic.edu, danielat@uic.edu.

Abstract

This paper considers the asymptotic behavior of two-source block-fading single-antenna Gaussian interference channels in the high-SNR regime by means of the diversity-multiplexing tradeoff. We consider a general setting where the users and the average channel gains are not restricted to be symmetric. Our results are not just extensions of previous results for symmetric networks, as our setting covers scenarios that are not possible under the symmetric assumption, such as the case of “mixed” interference, i.e., when difference sources have different distances from their intended receivers. We derive upper and lower bounds on the diversity. We show that for a fairly large set of channel parameters the two bounds coincides.

I. INTRODUCTION

Wireless networks deal with two fundamental limits that make the communication problem challenging and interesting. On the one hand, simultaneous communications from uncoordinated users create undesired interference. On the other hand, fluctuations of the channel condition due to multi-path and mobility cause signals to fade randomly. In today’s cellular and ad-hoc networks orthogonalization techniques, such as F/T/C/SDMA, are employed to avoid interference. However, although leading to simple network architectures, interference avoidance techniques are suboptimal in terms of achievable rates. Moreover, the relative strength of the intended data signal and the interference signals changes over time due to fading. This makes fixed channel access strategies suboptimal. Thus, understanding how to deal simultaneously with interference and with fading holds the key to the deployment of future broadband wireless networks. The simplest model for analyzing these problems jointly is the two-source Block-Fading Gaussian InterFerence Channel (BF-GIFC).
It is well known that the Han-Kobayashi (HK) [1] scheme with superposition coding, rate splitting, and joint decoding, gives the largest known achievable rate region for GIFC without fading. Several outer bounds are known in the literature for GIFC without fading [2]–[7]. In particular, Etkin et al. [4] showed that a simple rate splitting strategy in the HK scheme is within one bit/sec/Hz of the capacity region of Gaussian unfaded GIFCs for any possible channel parameters. In [4], all interfering signals above the noise floor are decoded, that is, the private messages—which are treated as noise—are assigned a transmit power such that they are going to be received at, or below, the level of the noise. In doing so, roughly speaking, the effective noise power at the receiver is at most doubled, thus giving a rate penalty of at most 1 bit/sec/Hz.

Recently, GIFCs with fading were considered in [?], [9]–[15].

For ergodic channels, such as fast fading channels, the (Shannon) capacity is the performance measure of the ultimate system performance. In [16], it was showed that the sum-rate ergodic capacity of a $K$-source fading GIFC scales linearly with the number of sources. In [9], the sum-rate capacity of a two-source strong ergodic fading GIFCs was shown to be equal to that of the corresponding compound MAC. In [10], optimal power allocation policies for outer and inner bounds for ergodic fading GIFCs with perfect transmitter CSI were derived.

For slow fading channels, the proper measurement of performance is the outage capacity. In particular, the Diversity Multiplex Tradeoff (DMT) [17], quantifies the tradeoff between rate and outage probability as the Signal to Noise Ratio (SNR) grows to infinity. In [11] the DMT of symmetric two-source BF-GIFCs was studied based on the “within one bit” outer bound of [4]. The authors of [11] claimed that the derived DMT is actually achievable because the “one bit penalty” for using a simple HK strategy vanishes at high SNR. However, the achievability of the “within one bit” outer bound requires a very specific rate splitting in the HK achievable scheme that depends on the instantaneous fading values. Hence, as pointed out in [12]–[14], [18], [19] the DMT derived in [11] is achievable only if the transmitters know the instantaneous fading values perfectly. In the the case of no channel state information at the transmitter (TXCSI) the DMT of [11] is an upper-bound on the actual DMT.

The DMT of BF-GIFCs without TXCSI is the subject of investigation of [12]–[15], [18], [19] as well as of this work. In [12], it was proved that in strong interference joint decoding of all message at all destinations achieves the DMT outer-bound of [11]. In [13], it was showed that multilevel superposition coding achieves the DMT of any two-source BF-GIFC; however, no explicit formula was given for more than two levels of superposition. In [15], it was showed that the DMT of BF-GIFCs reduces to that of Multiple Access Channel (MAC) if transmitters are not aware of the channel gains. In [14], it was shown
that one bit of TXCSI suffices to achieve the optimum DMT for certain ranges of channel parameters.

The works [11], [12], [14], [15] focused on two-source symmetric networks, that is to say, networks for which the average SNR and the average Interference to Noise Ratio (INR) at all receivers are the same. In this work, we consider two-source asymmetric GIFCs as in our conference papers [13], [18], [19]. In [18], [19] we generalized the DMT outer-bound of [11] to asymmetric networks and studied HK achievable schemes with and without rate splitting. It should be point out that our results are not just a generalization of the symmetric network results. Our setting covers all possible classes of channels and includes channels not possible under the symmetric assumptions, such as the case of “mixed” interference. Mixed interference occurs in practice when sources have different distances from their intended receivers and is the most practical scenario for wireless networks.

We assume that the channel variations over time are sufficiently slow so that the fading coefficients may be considered as fixed for the whole codeword duration (i.e., block fading assumption). We assume that the receivers know perfectly the channel realization, but the transmitters do not. In this case, if the instantaneous fading realization is such that the transmission rates cannot be reliably decoded, the system is said to experience outage. In an outage setting without TXCSI, it is not clear that a fixed rate splitting strategy can actually achieve the DMT upper bound of [11]. Here we consider both HK achievable strategies with and without rate splitting. In the case of rate splitting, we consider the case where the average received power of the signals that are treated as noise is set below the noise floor, as done in [4] for the unfaded case. We also generalized the outer-bound of [11] to asymmetric networks. We show that for a very wide range of channel parameters, the inner and outer bound meet. In particular, rate splitting improves the achievable DMT in weak and mixed interference channel.

The rest of the paper is organized as follows: Section II presents the system model and the problem formulation; Section III and IV present DMT upper and lower bounds, respectively; Section V presents numerical results; Section VI concludes the paper.

II. Channel Model

A two-source single-antenna Rayleigh fading GIFC in standard form is defined as:

\[ Y_u = H_{u1}X_1 + H_{u2}X_2 + Z_u \in \mathbb{C}, \]  

(1)

where the noises \( Z_u \sim \mathcal{N}(0, 1) \) and the inputs are subject to the average power constraint \( \mathbb{E}||X_u||^2 \leq P_u \), \( u \in \{1, 2\} \). We assume the channel to be block-fading and that each codeword spans one fading block, i.e., no coding across multiple blocks is allowed. Moreover, we assume arbitrarily large block lengths. The
receivers are assumed to perfectly know the fading realization \((H_{11}, H_{12}, H_{21}, H_{22})\), while the transmitters are not. In the rest of the paper we parameterize the received SNR/INRs as

\[
\mathbb{E}[|H_{cu}X_u|^2] = \mathbb{E}[|H_{cu}|^2]P_u \overset{\Delta}{=} x^{\beta_{cu}}, \quad \beta_{cu} \in \mathbb{R}^+, \quad (c, u) \in \{1, 2\} \times \{1, 2\},
\]

for some \(x > 1\), and the transmission rates as

\[
R_u \overset{\Delta}{=} \log(1 + x^{r_u}), \quad r_u \in \mathbb{R}^+, \quad u \in \{1, 2\}.
\]

We focus our analysis on the high-SNR regime, that is, in the limit for \(x \to +\infty\). Notice that, although we impose that the channel gains \(\beta\)'s and the rates \(r\)'s to be non-negative, the results derived in the following can be extended to any \(\beta\)'s and \(r\)'s by replacing each \(\beta\) with \([\beta]^+ \overset{\Delta}{=} \max\{0, \beta\}\) and each \(r\) with \([r]^+ \overset{\Delta}{=} \max\{0, r\}\).

\(a)\) **Capacity outer bound:** The capacity region of GIFIC is not known in general. Recently, Etkin et al. [4] proposed a novel outer bound for the capacity region of unfaded GIFIC that is shown to be “within one bit” of a simplified version of the Han-Kobayashi [1] achievable region. More precisely, let

\[
H_{ij} = \sqrt{\mathbb{E}[|H_{ij}|^2]} e^{i \theta_{ij}}
\]

where \((\gamma_{ij}, \theta_{ij})\) are iid for all \((i, j)\) and with \(\theta_{ij}\) uniformly distributed on \([0, 2\pi]\) and independent of \(\gamma_{ij}\). In [17] it is shown that in the limit for \(x \to +\infty\), the random variables \(\gamma_{ij}\) are asymptotically iid negative exponential with mean \(\log(x)\). By using the parameterization (2), (3), and (4), for each fading...
realization \((\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22})\), the “within one bit” outer bound of [4] can be written as [11]:

\[
\mathcal{R}_{EW} = \left\{ (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbb{R}^4 : \right. \\
\log(1 + x^{r_1}) \leq \log(1 + x^{\beta_{11} - \gamma_{11}}) \tag{5a} \\
\log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{22} - \gamma_{22}}) \tag{5b} \\
\log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{11} - \gamma_{11}}) \\
\quad + \log\left(1 + x^{\beta_{22} - \gamma_{22}} + x^{\beta_{21} - \gamma_{21}}\right) \tag{5c} \\
\log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{22} - \gamma_{22}}) \\
\quad + \log\left(1 + x^{\beta_{11} - \gamma_{11}} + x^{\beta_{12} - \gamma_{12}}\right) \tag{5d} \\
\log(1 + x^{r_1}) + (1 + x^{r_2}) \leq \\
\log(1 + x^{\beta_{12} - \gamma_{12}} + x^{\beta_{11} - \gamma_{11}}) \\
\quad + \log\left(1 + x^{\beta_{21} - \gamma_{21}} + x^{\beta_{22} - \gamma_{22}}\right) \tag{5e} \\
2 \log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{11} - \gamma_{11}}) \\
\quad + \log\left(1 + x^{\beta_{11} - \gamma_{11}} + x^{\beta_{12} - \gamma_{12}}\right) \\
\quad + \log\left(1 + x^{\beta_{21} - \gamma_{21}} + x^{\beta_{22} - \gamma_{22}}\right) \tag{5f} \\
\log(1 + x^{r_1}) + 2 \log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{22} - \gamma_{22}}) \\
\quad + \log\left(1 + x^{\beta_{22} - \gamma_{22}} + x^{\beta_{21} - \gamma_{21}}\right) \\
\quad + \log\left(1 + x^{\beta_{12} - \gamma_{12}} + x^{\beta_{11} - \gamma_{11}}\right) \tag{5g} 
\]
A. Capacity inner bound

The HK achievable region, in a form that matches the rate bounds of (5), can be found in [20], and is given by:

\[
\mathcal{R}_{\text{HK,complete}} = \bigcup_{P(Q,W_1,W_2,X_1,X_2)} \left\{(R_1, R_2) \in \mathbb{R}^2_+ : \right. \\
R_1 \leq I(X_1; Y_1 | W_2 Q) \\
R_2 \leq I(X_2; Y_2 | W_1 Q) \\
R_1 + R_2 \leq I(X_2, W_1; Y_2 | Q) + I(X_1; Y_1 | W_1 W_2, Q) \\
R_1 + R_2 \leq I(X_1, W_2; Y_1 | Q) + I(X_2; Y_2 | W_1 W_2, Q) \\
R_1 + R_2 \leq I(X_1, W_2; Y_1 | W_1, Q) + I(X_2, W_1; Y_2 | W_2, Q) \\
2R_1 + R_2 \leq I(X_1, W_2; Y_1 | Q) + I(X_1; Y_1 | W_1, W_2, Q) + I(X_2, W_1; Y_2 | W_2, Q) \\
R_1 + 2R_2 \leq I(X_2, W_1; Y_2 | Q) + I(X_2; Y_2 | W_1, W_2, Q) + I(X_1, W_2; Y_1 | W_1, Q) \left. \right\}.
\]

The region \(\mathcal{R}_{\text{HK,complete}}\) is difficult to evaluate because it requires an optimization with respect to the joint distribution \(P(Q, W_1, W_2, X_1, X_2)\), where \(Q\) is a time-sharing random variable and \((W_1, W_2)\) has the meaning of common information decoded at both receivers. In order to have a region that can be evaluated easily, it is customary to assume jointly Gaussian input \(P(W_1, W_2, X_1, X_2 | Q)\) without time sharing, that is, the random variable \(Q\) is a deterministic constant. We set \(W_u \sim \mathcal{N}(0, P_{u,\text{common}})\) independent of \(T_u \sim \mathcal{N}(0, P_{u,\text{private}})\) and let \(X_u = W_u + T_u\) such that the total power constraint is met with equality, i.e., \(P_u = P_{u,\text{private}} + P_{u,\text{common}}, for u \in \{1, 2\}\). We further parameterize the ratio of the average private power to the total average power for a given user \(u\) as

\[
\alpha_u = \frac{1}{1 + x^{b_u}} \in [0, 1], \quad b_u \in \mathbb{R}, \tag{6}
\]

so that the average receive SNR/INR’s on channel \(c \in \{1, 2\}\) are

\[
\mathbb{E}[|H_{cu}|^2]P_{u,\text{private}} = \alpha_u \mathbb{E}[|H_{cu}|^2]P_u = \frac{x^{\beta_{cu}}}{1 + x^{b_u}} \\
\mathbb{E}[|H_{cu}|^2]P_{u,\text{common}} = (1 - \alpha_u) \mathbb{E}[|H_{cu}|^2]P_u = \frac{x^{\beta_{cu} + b_u}}{1 + x^{b_u}}, \quad u \in \{1, 2\}.
\]

Moreover, following [4], we set \(b_1 = \beta_{21}\) and \(b_2 = \beta_{12}\) so that the average interfering private power is below the level of the noise, that is, \(\mathbb{E}[|H_{cu}|^2]P_{u,\text{private}} = \frac{x^{\beta_{cu}}}{1 + x^{b_u}} \leq 1\).
With these choices, \( R_{HK, complete} \) reduces to

\[
\mathcal{R}_{HK} = \left\{ (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbb{R}^4 : \right. \\
\log(1 + x^{r_1}) \leq \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}}}{1 + x^{\beta_{12} - \gamma_{12}}} \right) \\
\log(1 + x^{r_2}) \leq \log \left( 1 + \frac{x^{\beta_{22} - \gamma_{22}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
\log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}} + x^{\beta_{12} - \gamma_{12}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
+ \log \left( 1 + \frac{x^{\beta_{22} - \gamma_{22}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
\log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}} + x^{\beta_{12} - \gamma_{12}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
+ \log \left( 1 + \frac{x^{\beta_{22} - \gamma_{22}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
2 \log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}} + x^{\beta_{12} - \gamma_{12}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
+ \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}}}{1 + x^{\beta_{12} - \gamma_{12}}} \right) + \log \left( 1 + \frac{x^{\beta_{22} - \gamma_{22}} + x^{\beta_{21} - \gamma_{21}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
\log(1 + x^{r_1}) + 2 \log(1 + x^{r_2}) \leq \log \left( 1 + \frac{x^{\beta_{22} - \gamma_{22}} + x^{\beta_{21} - \gamma_{21}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \\
+ \log \left( 1 + \frac{x^{\beta_{22} - \gamma_{22}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) + \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}} + x^{\beta_{12} - \gamma_{12}}}{1 + x^{\beta_{21} - \gamma_{21}}} \right) \right\}.
\]

(7a) – (7g)

B. Diversity

The probability of outage \( P_{\text{out}}(r_1, r_2) \) is defined as the probability that the fading realization \( (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \) is such that the rate pair \( (r_1, r_2) \) cannot be decoded. By using the outer bound region \( R_{ETW} \) in (5) and
the inner bound region $\mathcal{R}_{\text{HK}}$ in (7) we can bound the outage probability as

$$1 - \mathbb{P}[(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathcal{R}_{\text{ETW}}] \leq \mathbb{P}_{\text{out}}(r_1, r_2) \leq 1 - \mathbb{P}[(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathcal{R}_{\text{HK}}].$$

The diversity, or the high-SNR exponent of the outage probability, is defined as

$$d(r_1, r_2) = \lim_{x \to +\infty} -\frac{\log(\mathbb{P}_{\text{out}}(r_1, r_2))}{\log(x)},$$

and it is bounded by

$$d_{\text{HK}}(r_1, r_2) \leq d(r_1, r_2) \leq d_{\text{ETW}}(r_1, r_2), \quad (8)$$

where $d_{\text{ETW}}(r_1, r_2)$ and $d_{\text{HK}}(r_1, r_2)$ are defined similarly to $d(r_1, r_2)$.

The rest of the paper is devoted to the evaluation of $d_{\text{ETW}}(r_1, r_2)$ and $d_{\text{HK}}(r_1, r_2)$.

### III. Diversity Upper Bound

By using the Laplace’s integration method as in [17] we obtain

$$d_{\text{ETW}}(r_1, r_2) = \min_{\gamma \in (\bar{\mathcal{R}}_{\text{ETW}})^c} \{\gamma_{11} + \gamma_{12} + \gamma_{21} + \gamma_{22}\} \quad (9)$$

where $\bar{\mathcal{R}}_{\text{ETW}}$ is the large-$x$ approximation of $\mathcal{R}_{\text{ETW}}$ in (5) and is given by

$$\bar{\mathcal{R}}_{\text{ETW}} = \left\{(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbb{R}_+^4 : X_{ij} \overset{\Delta}{=} [\beta_{ij} - \gamma_{ij}]^+, \quad r_1 \leq X_{11}, \quad (10a) \right. \\

\left. \quad r_2 \leq X_{22}, \quad (10b) \right. \\

\left. \quad r_s \overset{\Delta}{=} r_1 + r_2 \leq [X_{11} - X_{21}]^+ + \max\{X_{21}, X_{22}\} \quad (10c) \right.

\left. \quad = \max\{X_{11}, X_{21}\} + \max\{X_{22}, X_{21}\} - X_{21} \right.

\left. \quad r_s \overset{\Delta}{=} r_1 + r_2 \leq [X_{22} - X_{12}]^+ + \max\{X_{12}, X_{11}\} \quad (10d) \right.

\left. \quad = \max\{X_{11}, X_{12}\} + \max\{X_{22}, X_{12}\} - X_{12} \right.

\left. \quad r_s \overset{\Delta}{=} r_1 + r_2 \leq \max\{X_{12}, X_{11} - X_{21}\} + \max\{X_{21}, X_{22} - X_{12}\} \quad (10e) \right.

\left. \quad = \max\{X_{11}, X_{21} + X_{12}\} + \max\{X_{22}, X_{21} + X_{12}\} - (X_{21} + X_{12}) \right.

\left. \quad r_f \overset{\Delta}{=} 2r_1 + r_2 \leq [X_{11} - X_{21}]^+ + \max\{X_{11}, X_{12}\} + \max\{X_{21}, X_{22} - X_{12}\} \quad (10f) \right.

\left. \quad = \max\{X_{11}, X_{21}\} + \max\{X_{11}, X_{12}\} + \max\{X_{22}, X_{21} + X_{12}\} - (X_{21} - X_{12}) \right.

\left. \quad r_g \overset{\Delta}{=} r_1 + 2r_2 \leq [X_{22} - X_{12}]^+ + \max\{X_{22}, X_{21}\} + \max\{X_{12}, X_{11} - X_{21}\} \quad (10g) \right.

\left. \quad = \max\{X_{22}, X_{21}\} + \max\{X_{22}, X_{12}\} + \max\{X_{11}, X_{21} + X_{12}\} - (X_{21} - X_{12}) \right\}.$$
where \([x]^+ \triangleq \max\{0, x\}\).

The optimization problem in (9) can be solved as follows: since the complement of \(\tilde{R}_{ETW}\) is the union of the complement of the conditions (10a) through (10g), by applying the union bound as in [21] it can be shown that the diversity in (9) evaluates to

\[
\begin{align*}
    d_{ETW}(r_1, r_2) &= \min_{\ell=a...g} \{d(10)\}, \\
    d(10) &\triangleq \beta_{11} + \beta_{12} + \beta_{21} + \beta_{22} - \max_{X's \text{ do NOT satisfy equation (10)}} \{X_{11} + X_{12} + X_{21} + X_{22}\}. 
\end{align*}
\]

We have:

- The diversity \(d_{10a}\) (corresponding to the constraint (10a)) is:

\[
\begin{align*}
    d_{10a} &= \beta_{11} - \max\{X_{11}\} \\
    &\quad \text{subj. to } 0 \leq X_{11} \leq \beta_{11}, \quad X_{11} \leq r_1, \\
    &= \beta_{11} - \min\{\beta_{11}, r_1\} = -\min\{0, r_1 - \beta_{11}\} = \max\{0, \beta_{11} - r_1\} \\
    &= [\beta_{11} - r_1]^+.
\end{align*}
\]

- Similarly to \(d_{10a}\), the diversity \(d_{10b}\) (corresponding to the constraint (10b)) is:

\[
    d_{10b} = [\beta_{22} - r_2]^+.
\]

- The diversity \(d_{10c}\) (corresponding to the constraint (10c)) is:

\[
\begin{align*}
    d_{10c} &= \beta_{11} + \beta_{21} + \beta_{22} - \max\{X_{11} + X_{21} + X_{22}\} \\
    &\quad \text{subj. to } 0 \leq X_{11} \leq \beta_{11}, \quad 0 \leq X_{21} \leq \beta_{21}, \quad 0 \leq X_{22} \leq \beta_{22}, \\
    &\quad \text{and to } \max\{X_{11}, X_{21}\} + \max\{X_{22}, X_{21}\} - X_{21} \leq r_s \triangleq r_1 + r_2.
\end{align*}
\]

We start by re-writing the last constraint as follows:

\[
\begin{align*}
    \max\{X, Y\} + \max\{Z, Y\} &\leq r_s + Y \iff \begin{cases} 
        X + Z \leq r_s + Y \\
        Y + Z \leq r_s + Y \\
        X + Y \leq r_s + Y \\
        Y + Y \leq r_s + Y
    \end{cases} \iff \begin{cases} 
        X + Z \leq r_s + Y \\
        \max\{X, Y, Z\} \leq r_s
    \end{cases}.
\end{align*}
\]
which implies
\[
\max \{X_{21} + (X_{11} + X_{22})\}
\]
subj. to \(0 \leq X_{21} \leq \min\{r_s, \beta_{21}\}, 0 \leq X_{11} \leq \min\{r_s, \beta_{11}\}, 0 \leq X_{22} \leq \min\{r_s, \beta_{22}\},\)
and to \(X_{11} + X_{22} \leq r_s + X_{21},\)

\[
= \max \left\{ X_{21} + \min \left\{ \min\{r_s, \beta_{11}\} + \min\{r_s, \beta_{22}\}, r_s + X_{21} \right\} \right\}
\]
subj. to \(0 \leq X_{21} \leq \min\{r_s, \beta_{21}\},\)

\[
= \min \left\{ \min\{r_s, \beta_{11}\} + \min\{r_s, \beta_{22}\} + \min\{r_s, \beta_{21}\}, r_s + 2 \min\{r_s, \beta_{21}\} \right\}.
\]

Hence we obtain:

\[
d_{10c} = \beta_{11} + \beta_{21} + \beta_{22} - \min \left\{ \min\{r_s, \beta_{11}\} + \min\{r_s, \beta_{22}\} + \min\{r_s, \beta_{21}\}, r_s + 2 \min\{r_s, \beta_{21}\} \right\}
\]

\[
= \max \{ [\beta_{11} - r_s]^+ + [\beta_{22} - r_s]^+ + [\beta_{21} - r_s]^+ , \beta_{11} + \beta_{22} - 2r_s + |\beta_{21} - r_s| \}.
\]

Remark: In the symmetric case, with \(\beta_{11} = \beta_{22} = 1, \beta_{12} = \beta_{21} = \alpha\) and \(r_1 = r_2 = r\) (i.e., \(r_s = 2r\)), \(d_{10c}\) reduces to

\[
d_{10c, sym} = 2 \max \left\{ [1 - 2r]^+ + \left[ \frac{\alpha}{2} - r \right]^+, (1 - 2r) + \left[ \frac{\alpha}{2} - r \right] \right\}
\]

\[
= 2 \left( [A]^+ + [B]^+ + \left[ B^- - [A^-]^+ \right]^+ \right) |_{A=1-2r, B=\frac{\alpha}{2} - r},
\]

where

\[
x = [x]^+ - [x]^-, \quad [x]^+ \overset{\Delta}{=} \max \{0, x\} \geq 0, \quad [x]^+ \overset{\Delta}{=} -\min \{0, x\} \geq 0. \quad \forall x \in \mathbb{R}.
\]

The corresponding bound in [11] is

\[
d_b = 2[1 - r - \min(r, \frac{\alpha}{2})]^+ + [\alpha - 2r]^+
\]

\[
= 2 \left( [A]^+ + [B]^+ + \left[ A - B^+ - [A]^+ + [B]^+ \right]^+ \right) |_{A=1-2r, B=\frac{\alpha}{2} - r},
\]

which can be easily shown to be equivalent to \(d_{10c, sym}\) since for \(A \geq B : [A - B]^+ - [A]^+ + [B]^+ = A - B - [A]^+ + [B]^+ = -[A^-] + [B^-], \) and \(A < B : [A - B]^+ - [A]^+ + [B]^+ = -[A]^+ + [B]^+ \leq 0.\)

- The diversity \(d_{10d}\) (corresponding to the constraint (10d)) is as \(d_{10c}\) but with \(\beta_{21}\) replaced by \(\beta_{12}\), i.e., with the role of the users swapped.
Remark: In the symmetric case, $d_{10d_, sym} = d_{10c_, sym}$.

- The diversity $d_{10c}$ (corresponding to the constraint (10c)) is as $d_{10c}$ but with $\beta_2 + \beta_1$ instead of $\beta_2$.

Remark: In the symmetric case, $d_{10e_, sym}$ coincides with $d_c$ in [11].

- The diversity $d_{10f}$ (corresponding to the constraint (10f)) is:

$$d_{10f} = \beta_1 + \beta_2 + \beta_2 + \beta - \max\{X + Y + Z + W\}$$

subject to $0 \leq X \leq \beta_1$, $0 \leq W \leq \beta_2$, $0 \leq Y \leq \beta_2$, $0 \leq Z \leq \beta_2$,

and to $\max\{X, Y\} + \max\{X, W\} + \max\{Z, Y + W\} - (Y + W) \leq r_f \Delta = 2r_1 + r_2$.

The last constraint can be rewritten as

$$\begin{cases}
(X - Y) + (X - W) \leq r_f - \max\{Z, Y + W\} \\
X - Y \leq r_f - \max\{Z, Y + W\} \\
X - W \leq r_f - \max\{Z, Y + W\} \\
0 \leq r_f - \max\{Z, Y + W\}
\end{cases} \iff \begin{cases}
2X \leq r_f - [Z - (Y + W)]^+ \\
\max\{Z, Y + W\} \leq r_f
\end{cases}$$

that together with $X \leq \beta_1$ gives

$$\max\{X\} = \min\left\{\beta_1, \frac{r_f - [Z - (Y + W)]^+}{2}\right\}.$$

Next, the function

$$\max\{X\} + (Y + W) = \min\left\{\beta_1, \frac{r_f - [Z - (Y + W)]^+}{2}\right\} + (Y + W),$$

is increasing in $(Y + W)$, and since the constraints

$$W \leq \beta_2, \ Y \leq \beta_2, \ Z \leq \beta_2, \ \max\{Z, Y + W\} \leq r_f,$$

are equivalent to

$$Y + W \leq \min\{\beta_2 + \beta_1, r_f\}, \ Z \leq \min\{\beta_2, r_f\},$$

we have

$$\max\{X + Y + W\} = \min\left\{\beta_1, \frac{r_f - [Z - \min\{r_f, \beta_2 + \beta_1\}]^+}{2}\right\} + \min\{r_f, \beta_2 + \beta_1\}.$$

Finally, the function

$$Z + \max\{X + Y + W\} = \min\left\{\beta_1, \frac{r_f - [Z - \min\{r_f, \beta_2 + \beta_1\}]^+}{2}\right\} + \min\{r_f, \beta_2 + \beta_1\} + Z,$$
is also increasing in $Z$, hence, subject to $Z \leq \min\{r_f, \beta_{22}\}$, we finally have

$$d_{[10]} = -\max\{X + Y + W + Z\} + (\beta_{11} + \beta_{12} + \beta_{21} + \beta_{22})$$

$$= -\min\left\{\beta_{11}, \frac{r_f - \min\{r_f, \beta_{22}\} - \min\{r_f, \beta_{21} + \beta_{12}\}}{2}\right\} - \min\{r_f, \beta_{21} + \beta_{12}\} - \min\{r_f, \beta_{22}\}$$

$$+ (\beta_{11} + \beta_{12} + \beta_{21} + \beta_{22})$$

$$= \left[\beta_{11} - \frac{r_f - \beta_{22} - (\beta_{21} + \beta_{12}) + a - b}{2}\right]^+ + a + b, \quad a \triangleq [(\beta_{21} + \beta_{12}) - r_f]^+, b \triangleq [\beta_{22} - r_f]^+.$$ 

Remark: In the symmetric case, $d_{[10]}$ reduces to

$$d_{[10], sym} = \frac{1 + b + \|[a] - [b]\|^+}{2} + [a]^+ + [b]^+, \quad a = 2\alpha - 3r, b = 1 - 3r$$

which is not equivalent to $d_d$ in [11]. In fact, it turns out that $d_d$ in [11] is not correct. Consider the following numerical example: let $r_1 = r_2 = 0.4$, $\beta_{11} = \beta_{22} = 1$, $\beta_{12} = \beta_{21} = 0.5$, which corresponds to $r_f = 1.2, \alpha = 0.5$ in [11]. The optimization problem for $d_d$ is

$$d_d = \min\{\gamma_{11} + \gamma_{12} + \gamma_{21} + \gamma_{22}\}$$

subj.to $[[1 - \gamma_{11}]^+ + [\alpha - \gamma_{12}]^+]^+ + \max([1 - \gamma_{11}]^+, [\alpha - \gamma_{21}]^+)$

$$+ \max([\alpha - \gamma_{12}]^+, [1 - \gamma_{22}]^+, [\alpha - \gamma_{21}]^+) \leq r_f$$

It can be easily verified that $\gamma_{11} = 0.4$, $\gamma_{12} = \gamma_{21} = \gamma_{22} = 0$ is a feasible solution that gives $(\gamma_{11} + \gamma_{12} + \gamma_{21}, \gamma_{22}) = 0.4$. However, according

$$d_d = \max\{(1 - \frac{3r}{2})^+ + (1 - 3r)^+ + (2\alpha - 3r)^+, \min\{3 - 3r - \min(3r, 2\alpha)\}^+, \max(1, 2 - 3r - \min(3r, 2\alpha))\}$$

we have $d_d = 0.8$.

• Finally $d_{[10g]}$ (corresponding to the constraint [10g]) is as $d_{[10]}$ but with $\beta_{22}$ instead of $\beta_{11}$ and $r_g$ instead of $r_f$, i.e., the role of the users is swapped.

### IV. Diversity Lower Bound

The evaluation of the diversity lower bound $d_{HK}$ in (8) can be carried out similarly to the evaluation of the diversity upper bound $d_{ETW}$ in the previous section. We will consider both the case of no rate splitting and the particular choice of the power split among common and private messages inspired by [4] which led to (7).
A. Diversity lower bound without rate splitting

Without rate splitting in the HK region, a user either sends all private information or all common information. These two modes of operation correspond to either treating the interference as noise at the receiver, or performing joint decoding as in a MAC channel.

Consider first the case where the interference is treated as noise. User 1 can be successfully decoded at receiver 1 by treating user 2 as noise if

$\{ (\gamma_{11}, \gamma_{12}) \in \mathbb{R}^2 : \log(1 + x^{r_1}) \leq \log \left( 1 + \frac{x^{\beta_{11} - \gamma_{11}}}{1 + x^{\beta_{12} - \gamma_{12}}} \right) \}$.

By following the same approach used in the derivation of the diversity upper bound, we have that the exponent of the probability that user 1 cannot be decoded successfully at receiver 1 by treating user 2 as noise is given by:

$$d_{NI1} = \beta_{11} + \beta_{12} - \max\{X_{11} + X_{12}\}$$

subj. to $0 \leq X_{11} \leq \beta_{11}, \ 0 \leq X_{12} \leq \beta_{12}, \ [X_{11} - X_{12}]^+ \leq r_1$

$= \beta_{11} + \beta_{12} - \max\{X_{11} + X_{12}\}$

subj. to $0 \leq X_{11} \leq \beta_{11}, \ 0 \leq X_{12} \leq \beta_{12}, \ X_{11} \leq X_{12} + r_1$

$= \beta_{11} + \beta_{12} - \max\{\min\{\beta_{11}, X_{12} + r_1\} + X_{12}\}$

subj. to $0 \leq X_{12} \leq \beta_{12}$,

$= \beta_{11} + \beta_{12} - \max\{\min\{\beta_{11}, \beta_{12} + r_1\} + \beta_{12}\}$

$= [\beta_{11} - r_1 - \beta_{12}]^+$,

Similarly, the exponent of the probability that user 2 cannot be successfully decoded at receiver 2 by treating user 1 as noise is

$$d_{NI2} = [\beta_{22} - r_2 - \beta_{21}]^+.$$  

Consider now the case where the users are jointly decoded. User 1 and user 2 can be successfully jointly decoded at receiver 2 as in a MAC if

$\{ (\gamma_{21}, \gamma_{22}) \in \mathbb{R}^4 :$

$\log(1 + x^{r_1}) \leq \log(1 + x^{\beta_{21} - \gamma_{21}}) \}$

$\log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{22} - \gamma_{22}}) \}$

$\log(1 + x^{r_1}) + \log(1 + x^{r_2}) \leq \log(1 + x^{\beta_{21} - \gamma_{21}} + x^{\beta_{22} - \gamma_{22}}) \}.$
The exponent of the probability that both users cannot be jointly decoded is given by
\[ d_{\text{MAC2}} = \min\{[\beta_{21} - r_1]^+, [\beta_{22} - r_2]^+, [\beta_{22} - r_s]^+ + [\beta_{21} - r_s]^+\}, \quad r_s \overset{\Delta}{=} r_1 + r_2, \]
where the last argument of the minimum in \( d_{\text{MAC2}} \) can be derived as follows:
\[ \beta_{22} + \beta_{21} - \max\{X_{21} + X_{22}\} \]
\[ \text{subj. to } 0 \leq X_{22} \leq \beta_{22}, \quad 0 \leq X_{21} \leq \beta_{21}, \quad \max\{X_{22}, X_{21}\} \leq r_s, \]
\[ = \beta_{22} + \beta_{21} - \min\{\beta_{21}, r_s\} - \min\{\beta_{22}, r_s\} \]
\[ = [\beta_{22} - r_s]^+ + [\beta_{21} - r_s]^+. \]

Similarly, the exponent of the probability that user 1 and user 2 cannot be successfully jointly decoded at receiver 1 is
\[ d_{\text{MAC1}} = \min\{[\beta_{11} - r_1]^+, [\beta_{12} - r_2]^+, [\beta_{12} - r_s]^+ + [\beta_{11} - r_s]^+\}, \quad r_s \overset{\Delta}{=} r_1 + r_2. \]

Hence, without rate splitting, we have
\[ d_{\text{HK-wors}} = \max\{d_{00}, d_{01}, d_{10}, d_{11}\}, \quad (11) \]
where “wors” stands for “without rate splitting” and where

- \( d_{11} \) is the diversity when both sources send only private information (which is sum-rate optimal for very weak interference unfaded GIFCs [5]) given by
\[ d_{11} = \min\{d_{\text{NI1}}, d_{\text{NI2}}\}. \]

- \( d_{10} \) (and similarly for \( d_{01} \) but the role of the users swapped) is the diversity when user 1 sends only private information and the user 2 sends only common information (which is sum-rate optimal for mixed interference unfaded GIFCs [22]) given by
\[ d_{10} = \min\{d_{\text{NI1}}, d_{\text{MAC2}}\}. \]

- \( d_{00} \) is the diversity when both sources send common information (which is optimal for strong interference unfaded GIFCs [?]) given by
\[ d_{00} = \min\{d_{\text{MAC1}}, d_{\text{MAC2}}\}. \]
Since \( \max\{\min\{a, b_1\}, \min\{a, b_2\}\} = \min\{a, \max\{b_1, b_2\}\} \), we further rewrite \( d_{\text{HK-wors}} \) in (11) as
\[
d_{\text{HK-wors}} = \max \left\{ \min\{d_{\text{NI1}, d_{\text{NI2}}}, \min\{d_{\text{NI1}, d_{\text{MAC2}}}, \min\{d_{\text{MAC1}, d_{\text{NI2}}}, \min\{d_{\text{MAC1}, d_{\text{MAC2}}}\}\}\} \right\}
\]
\[
= \max \left\{ \min\{d_{\text{NI1}, \max\{d_{\text{NI2}, d_{\text{MAC2}}}, \min\{d_{\text{MAC1}, \max\{d_{\text{NI2}, d_{\text{MAC2}}}}\}\}\}\right\}
\]
\[
= \min \left\{ \max\{d_{\text{NI1}, d_{\text{MAC1}}}, \max\{d_{\text{NI2}, d_{\text{MAC2}}})\} \right\}
\]
\[
= \min \{d_{\text{warp}_1}, d_{\text{warp}_2}\}, \quad d_{\text{warp}_u} \triangleq \max\{d_{\text{NI} u}, d_{\text{MAC} u}\},
\]
that is, the diversity \( d_{\text{HK-wors}} \) has the following intuitive explanation: each user \( u \in \{1, 2\} \) chooses the best strategy between treating the interference as noise \( (d_{\text{NI} u}) \) and joint decoding \( (d_{\text{MAC} u}) \), which gives diversity \( d_{\text{warp}_u} \), and the overall diversity is dominated by the the worst user.

B. Diversity lower bound with rate splitting

In the general case of rate splitting, with the power split indicated in Section III we have:
\[
d_{\text{HK}}(r_1, r_2) = \min_{\gamma \in (\tilde{R}_{\text{HK}})^*} \{\gamma_{11} + \gamma_{12} + \gamma_{21} + \gamma_{22}\}
\]
(12)
where \( \tilde{R}_{\text{HK}} \) is the large-\( x \) approximation of \( R_{\text{HK}} \) in (7) and is given by:
\[
\tilde{R}_{\text{HK}} = \left\{ \left(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}\right) \in \mathbb{R}_+^4 : \quad X_{ij} \triangleq \beta_{ij} - \gamma_{ij},
\right. \]
\[
r_1 \leq [X_{11}]^+
\]
\[
r_2 \leq [X_{22}]^+
\]
\[
r_s \triangleq r_1 + r_2 \leq \max\{X_{22}, X_{21}\}^+ + [X_{11} - \beta_{21}]^+
\]
\[
r_s \triangleq r_1 + r_2 \leq \max\{X_{11}, X_{12}\}^+ + [X_{22} - \beta_{12}]^+
\]
\[
r_s \triangleq r_1 + r_2 \leq \max\{X_{11} - \beta_{21}, X_{12}\}^+ + \max\{X_{22} - \beta_{12}, X_{21}\}^+
\]
\[
r_f \triangleq 2r_1 + r_2 \leq [\max\{X_{11}, X_{12}\}]^+ + [X_{11} - \beta_{21}]^+
\]
\[
\quad + [\max\{X_{22} - \beta_{12}, X_{21}\}]^+
\]
\[
r_g \triangleq r_1 + 2r_2 \leq [\max\{X_{22}, X_{21}\}]^+ + [X_{22} - \beta_{12}]^+
\]
\[
\quad + [\max\{X_{11} - \beta_{21}, X_{12}\}]^+.
\]
(13)
We finally have that the diversity is given by:

\[ d_{HK} = \min_{\ell=a \ldots g} \{ d_{13\ell} \}, \]

\[ d_{13\ell} \overset{\Delta}{=} \beta_{11} + \beta_{12} + \beta_{21} + \beta_{22} - \max_{X's \ do \ NOT \ satisfy \ equation \ (13f)} \{ X_{11} + X_{12} + X_{21} + X_{22} \}. \]

where

- The diversity \( d_{13a} \) and \( d_{13b} \) (corresponding to the constraint (13a) and (13b), respectively) are:

\[ d_{13a} = d_{10a} = [\beta_{11} - r_1]^+, \]

and

\[ d_{13b} = d_{10b} = [\beta_{22} - r_2]^+, \]

as for the upper bound.

- The diversity \( d_{13c} \) (corresponding to the constraint (13c)) is:

\[ d_{13c} = \beta_{22} + \beta_{21} + \beta_{11} - \max \{ X_{22} + X_{21} + X_{11} \} \]

subj. to \( X_{22} \leq \beta_{22}, \ X_{21} \leq \beta_{21}, \ X_{11} \leq \beta_{11}, \)

and to \( [\max \{ X_{22}, X_{21} \}]^+ + [X_{11} - \beta_{21}]^+ \leq r_s \Delta = r_1 + r_2. \)

If rewrite the optimization domain as

\[ X_{22} \leq \beta_{22}, \ X_{21} \leq \beta_{21}, \ \max \{ X_{22}, X_{21} \} \leq \alpha r_s, \]

\[ X_{11} \leq \beta_{11}, \ X_{11} - \beta_{21} \leq (1 - \alpha)r_s, \]

\( \alpha \in [0, 1], \)

we immediately obtain

\[ d_{13c} = \beta_{22} + \beta_{21} + \beta_{11} - \max_{\alpha \in [0, 1]} \{ \min \{ \beta_{22}, \alpha r_s \} + \min \{ \beta_{21}, \alpha r_s \} + \min \{ \beta_{11}, (1 - \alpha)r_s + \beta_{21} \} \} \]

\[ = \min_{\alpha \in [0, 1]} \{ [\beta_{22} - \alpha r_s]^+ + [\beta_{21} - \alpha r_s]^+ + [\beta_{11} - \beta_{21} - (1 - \alpha)r_s]^+ \}, \]

and the optimal \( \alpha \) (see Appendix A) is

\[ \alpha_{13c} \overset{\Delta}{=} \min \{ 1, \max \{ \beta_{22}, \beta_{21} \}/r_s \}. \]

Remark: In general, it is difficult to compare the lower bound \( d_{13c} \) with the upper bound \( d_{10c} \). By inspection, if \( r_s \geq \beta_{21} \geq \max \{ \beta_{11}, \beta_{22} \} \) the two bounds meet. It is possible that this condition we can be relaxed.
• The diversity $d_{13d}$ (corresponding to the constraint (13d)) is as $d_{13c}$ but with the role of the users reversed.

• The diversity $d_{13e}$ (corresponding to the constraint (13e)) is:

$$d_{13e} = \beta_{22} + \beta_{21} + \beta_{11} + \beta_{12} - \max\{X_{22} + X_{21} + X_{11} + X_{12}\}$$

subj. to $X_{22} \leq \beta_{22}$, $X_{21} \leq \beta_{21}$, $X_{11} \leq \beta_{11}$, $X_{12} \leq \beta_{12}$,

and to $[\max\{X_{11} - \beta_{21}, X_{12}\}]^+ + [\max\{X_{22} - \beta_{12}, X_{21}\}]^+ \leq r_s$.

If rewrite the optimization domain as

$$X_{22} \leq \beta_{22}, \ X_{21} \leq \beta_{21}, \ \max\{X_{22} - \beta_{12}, X_{21}\} \leq \alpha r_s,$$

$$X_{11} \leq \beta_{11}, \ X_{12} \leq \beta_{12}, \ \max\{X_{11} - \beta_{21}, X_{12}\} \leq (1 - \alpha)r_s,$$

$\alpha \in [0, 1]$,

we immediately obtain

$$d_{13e} = \beta_{22} + \beta_{21} + \beta_{11} + \beta_{12}$$

$$- \max_{\alpha \in [0,1]} \left\{ \min\{\beta_{22}, \beta_{12} + \alpha r_s\} + \min\{\beta_{21}, \alpha r_s\} + \min\{\beta_{11}, \beta_{21} + (1 - \alpha)r_s\} + \min\{\beta_{12}, (1 - \alpha)r_s\} \right\}$$

$$= \min_{\alpha \in [0,1]} \left\{ [\beta_{22} - \beta_{12} - \alpha r_s]^+ + [\beta_{21} - \alpha r_s]^+ + [\beta_{11} - \beta_{21} - (1 - \alpha)r_s]^+ + [\beta_{12} - (1 - \alpha)r_s]^+ \right\},$$

and the optimal $\alpha$ (see Appendix B) is

$$\alpha_{13c} \triangleq \min \left\{ 1, \frac{\max\{\beta_{22} - \beta_{21}, \beta_{21}\}}{r_s} \right\}.$$

• The diversity $d_{13f}$ (corresponding to the constraint (13f)) is:

$$d_{13f} = \beta_{22} + \beta_{21} + \beta_{11} + \beta_{12} - \max\{X_{22} + X_{21} + X_{11} + X_{12}\}$$

subj. to $X_{22} \leq \beta_{22}$, $X_{21} \leq \beta_{21}$, $X_{11} \leq \beta_{11}$, $X_{12} \leq \beta_{12}$,

and to $[\max\{X_{11}, X_{12}\}]^+ + [X_{11} - \beta_{21}]^+ + [\max\{X_{22} - \beta_{12}, X_{21}\}]^+ \leq r_f$.

If rewrite the optimization domain as

$$X_{22} \leq \beta_{22}, \ X_{21} \leq \beta_{21}, \ \max\{X_{22} - \beta_{12}, X_{21}\} \leq \alpha r_f,$$

$$X_{11} \leq \beta_{11}, \ X_{12} \leq \beta_{12}, \ \max\{X_{11}, X_{12}\} + [X_{11} - \beta_{21}]^+ \leq (1 - \alpha)r_f,$$

$\alpha \in [0, 1]$,
and \( \max\{X_{11}, X_{12}\} + [X_{11} - \beta_{21}]^+ \leq (1 - \alpha)r_f \) as

\[
X_{11} + X_{11} \leq \beta_{21} + (1 - \alpha)r_f \\
X_{11} + \beta_{21} \leq \beta_{21} + (1 - \alpha)r_f \\
X_{12} + X_{11} \leq \beta_{21} + (1 - \alpha)r_f \\
X_{12} + \beta_{21} \leq \beta_{21} + (1 - \alpha)r_f
\]

we obtain that the optimization domain is

\[
X_{22} \leq \min\{\beta_{22}, \beta_{12} + \alpha r_f\}, \ X_{21} \leq \min\{\beta_{21}, \alpha r_f\}, \\
X_{11} \leq \min\{\beta_{11}, (1 - \alpha)r_f, \frac{\beta_{21} + (1 - \alpha)r_f}{2}\}, \ X_{12} \leq \min\{\beta_{12}, (1 - \alpha)r_f\}, \\
X_{11} + X_{12} \leq \beta_{21} + (1 - \alpha)r_f, \ \alpha \in [0, 1],
\]

and we immediately obtain

\[
d_{13f} = \beta_{22} + \beta_{21} + \beta_{11} + \beta_{12} - \max_{\alpha \in [0, 1]} \left\{ \min\{\beta_{22}, \beta_{12} + \alpha r_f\} + \min\{\beta_{21}, \alpha r_f\} \\
+ \min\{\beta_{21} + (1 - \alpha)r_f, \ \min\{\beta_{11}, (1 - \alpha)r_f, \frac{\beta_{21} + (1 - \alpha)r_f}{2}\} + \min\{\beta_{12}, (1 - \alpha)r_f\} \right\},
\]

and the optimal \( \alpha \) can be found in Appendix C.

- The diversity \( d_{13g} \) (corresponding to the constraint (13g)) is as \( d_{13f} \) but with the role of the users reversed.

Remark: In this section we derived the achievable diversity for the case of power split such that the average power of the private message at the non-intended receiver is below the noise floor. In Appendix C, we derive the achievable diversity for a generic power split. The expression is quite complex and not very insightful. By numerical optimization we found that the particular power split we chose in Section II is optimal, or very close to optimal, for a very large set of channel parameters \((\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})\).

V. NUMERICAL RESULTS

In this section we present numerical evaluations of the diversity upper bound \( d_{ETW} \) in (9) and the diversity lower bound without rate splitting \( d_{HK-\text{wors}} \) in (11) and the diversity lower bound with rate splitting \( d_{HK} \) in (12) for different values of the channel parameters \((\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})\).
A. Symmetric channels

We first consider symmetric channels. We set the average received power of the direct links to $\beta_{11} = \beta_{22} = 1$ and the average received power of the cross links to $\beta_{12} = \beta_{21} = \beta \geq 0$. In Figs. 1, 2, 3, 4, 5, and 6 we plot the diversity vs. the common multiplexing gain $r_1 = r_2 = r$ for $\beta = 0.2, 0.5, 0.6, 2/3, 1.1,$ and 1.5, respectively.

In Figs. 1 and 2 (weak interference) $d_{ETW} = d_{HK}$ and the dominant constraint at low rate is the “single user diversity” $d_{10a} = d_{10b}$, while at high rate is the “sum-rate diversity” $d_{10c}$.

In Figs. 3 and 4 (weak interference) $d_{ETW} = d_{HK}$ and the dominant constraint at low rate is the “single user diversity” $d_{10a} = d_{10b}$, at medium rate is $d_{10f} = d_{10g}$ and $d_{10c} = d_{10d}$, while at high rate is $d_{10a} = d_{10b}$ again. These figures show that the expression given for $d_{ETW}$ in [11] given by $d_{AL}$ is not correct, as pointed out in a remark earlier.

In Fig. 5 (strong interference) $d_{ETW} \neq d_{HK}$ and $d_{HK-wors} = d_{HK}$. The dominant constraint at low rate is the “single user diversity” $d_{10a} = d_{10b}$, at medium rate is the “sum-rate diversity” $d_{10c}$ while at high rate is again “single user diversity” $d_{10a} = d_{10b}$.

In Fig. 6 (strong interference) $d_{ETW} = d_{HK-wors} = d_{HK}$ and the dominant constraint at low rate is the
“single user diversity” \( d_{10a} = d_{10b} \), while at high rate is the “sum-rate diversity” \( d_{10c} \). In this case no-rate splitting is optimal. In [12] it was show rate splitting is not needed in very strong interference, that is, for \( \beta_{12} = \beta_{21} = \beta \geq 2 \). Here we show numerically that the threshold of 2 for the average interference power can be lowered. We found by simulation that \( d_{ETW} \) is not achievable for symmetric channels for \( \beta \in [0.680, 1.500] \).

B. Asymmetric channels

In Figs. 7, 8 and 9 we plot the diversity vs. the common multiplexing gain \( r_1 = r_2 = r \) for asymmetric channels with \( \beta_{11} = \beta_{22} = 1 \) and \( \beta_{12} \neq \beta_{21} \).

In Fig. 7 (weak interference) \( d_{ETW} = d_{HK} \) and the dominant constraint at low rate is the “single user diversity” \( d_{10a} = d_{10b} \) since \( \beta_{11} = \beta_{22} \), at medium rate is \( d_{10d} \) and \( d_{10c} \), while at high rate is \( d_{10a} = d_{10b} \) again.

In Fig. 8 (mixed interference) \( d_{ETW} = d_{HK} \) and the dominant constraint at low rate is the “single user diversity” \( d_{10a} = d_{10b} \).
Fig. 3. Symmetric channel in weak interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1$, $\beta_{12} = \beta_{21} = 0.6$.

diversity” $d_{10a} = d_{10b}$, at medium rate is $d_{10c}$, and $d_{10c}$, while at high rate is $d_{10a} = d_{10b}$ again.

In Fig. 9 (very strong interference) $d_{ETW} = d_{HK-wors}$ and the dominant constraint is $d_{10a} = d_{10b}$. In very strong interference, i.e., $\min\{\beta_{12}, \beta_{21}\} \geq \beta_{11} + \beta_{22}$, the interference is so strong that each user can completely remove the unintended signal before decoding its own signal. In this case the capacity region, and hence the diversity, is the cartesian product of the single user capacities without interference.

VI. CONCLUSION

In this paper, we analyzed the diversity-multiplexing trade-off of two-source block-fading Gaussian interference channels without channel state information at the transmitters. As opposed to previous works, we considered generic asymmetric networks. We found that, a simple inner bound based on the HK scheme with fixed power split achieves the outer bound based on perfect channel state information at the transmitter for wide range of channel parameters.
Fig. 4. Symmetric channel in weak interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1, \beta_{12} = \beta_{21} = 2/3$. 
Fig. 5. Symmetric channel in strong interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1, \beta_{12} = \beta_{21} = 1.1$.

Fig. 6. Symmetric channel in strong interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1, \beta_{12} = \beta_{21} = 1.5$. 
Fig. 7. Asymmetric channel in weak interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1, \beta_{12} = 0.9, \beta_{21} = 0.2$. 
Fig. 8. Asymmetric channel with mixed interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1, \beta_{12} = 1.2, \beta_{21} = 0.5$. 
Fig. 9. Asymmetric channel in strong interference: diversity vs. $r_1 = r_2 = r$ for $\beta_{11} = \beta_{22} = 1, \beta_{12} = 3, \beta_{21} = 5.$
A. Optimization of $d^{(13c)}$

The optimization problem with respect to $\alpha$ in $d^{(13c)}$ involves a function of the type

$$d^{(13c)}(\alpha) = [m - \alpha]^+ + [M - \alpha]^+ + [-m_2 + \alpha]^+, \quad m \leq M.$$ 

For $\alpha \leq m$

$$d^{(13c)}(\alpha) = m + M - 2\alpha + [-m_2 + \alpha]^+$$

which is decreasing in $\alpha$ since, depending on the value of $m_2$, $d^{(13c)}(\alpha)$ is a straight line of slope -2 or -1. For $m < \alpha \leq M$

$$d^{(13c)}(\alpha) = M - \alpha + [-m_2 + \alpha]^+$$

which is non-increasing in $\alpha$ since, depending on the value of $m_2$, $f_c(\alpha)$ is a straight line of slope -1 or 0. For $\alpha > M$

$$d^{(13c)}(\alpha) = [-m_2 + \alpha]^+$$

which is non-decreasing in $\alpha$ since, depending on the value of $m_2$, $d^{(13c)}(\alpha)$ is a straight line of slope 0 or +1. Hence the function has a minimum (maybe not unique) at $\alpha = M$.

In general, it can be easily shown that the function $d^{(13c)}(\alpha)$ is flat in the interval with extreme points $\max\{m, m_2\}$ and $M$. This means that the minimum of $d^{(13c)}(\alpha)$ over $\alpha \in [0, 1]$ is achieved by any $\alpha$ in the interval with extreme points $\min\{1, \max\{m, m_2\}\}$ and $\min\{1, M\}$. In particular, we can choose $\alpha = \min\{1, M\}$.

B. Optimization of $d^{(13e)}$

The optimization problem with respect to $\alpha$ in $d^{(13e)}$ involves a function of the type

$$d^{(13e)}(\alpha) = [m_1 - \alpha]^+ + [M_1 - \alpha]^+ + [-m_2 + \alpha]^+ + [-M_2 + \alpha]^+$$

with $m_1 \leq M_1$ and with $m_2 \leq M_2$. With similar reasoning as in Appendix A it can be shown that the function is flat in the interval with extreme points $\max\{m_1, m_2\}$ and $\min\{M_1, M_2\}$. This means that the minimum of $d^{(13e)}(\alpha)$ over $\alpha \in [0, 1]$ is achieved by any $\alpha$ in the interval with extreme points $\min\{1, \max\{m_1, m_2\}\}$ and $\min\{1, M_1, M_2\}$. In particular, we can choose $\alpha = \min\{1, M_1, M_2\}$. Notice that $d^{(13e)} = d^{(13c)}$ for $M_2 = +\infty$. 
C. General rate splitting in the HK achievable region

The large-$x$ approximation of the HK region for a general power split

\[
\begin{align*}
E[|H_{cu}|^2] P_{u,\text{private}} &= \alpha_u E[|H_{cu}|^2] P_u = \frac{x^{\beta_u}}{1 + x^{b_u}} \\
E[|H_{cu}|^2] P_{u,\text{common}} &= (1 - \alpha_u) E[|H_{cu}|^2] P_u = \frac{x^{\beta_u + b_u}}{1 + x^{b_u}}, \quad u \in \{1, 2\},
\end{align*}
\]

is

\[
\tilde{R}_{\text{HK}} = \left\{ (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbb{R}^4 : \quad X_{ij} \overset{\Delta}{=} \beta_{ij} - \gamma_{ij},
\right. \\
r_1 \leq [X_{11}]^+ - [X_{12} - [-b_2]^+]^+ \\
r_2 \leq [X_{22}]^+ - [X_{21} - [-b_1]^+]^+ \\
r_1 + r_2 \leq \max\{[X_{22}]^+, [X_{21} - [+b_1]^+]^+\} - [X_{21} - [-b_1]^+]^+ \\
+ [X_{11} - [+b_1]^+] - [X_{12} - [+b_2]^+]^+ \\
r_1 + r_2 \leq \max\{[X_{11}]^+, [X_{12} - [+b_2]^+]^+\} - [X_{12} - [-b_2]^+]^+ \\
+ [X_{22} - [+b_2]^+] - [X_{21} - [+b_1]^+]^+ \\
r_1 + r_2 \leq \max\{X_{11} - [-b_1]^+, X_{12} - [+b_2]^+\} - [X_{12} - [-b_2]^+]^+ \\
+ [\max\{X_{22} - [-b_2]^+, X_{21} - [+b_1]^+\} - [X_{21} - [-b_1]^+]^+ \\
2r_1 + r_2 \leq \max\{X_{11}, X_{12} - [+b_2]^+\} - [X_{12} - [-b_2]^+]^+ \\
+ [X_{11} - [-b_1]^+] - [X_{12} - [-b_2]^+]^+ \\
+ [\max\{X_{22} - [-b_2]^+, X_{21} - [+b_1]^+\} - [X_{21} - [-b_1]^+]^+ \right. \\
r_1 + 2r_2 \leq \max\{X_{22}, X_{21} - [+b_1]^+\} - [X_{21} - [-b_1]^+]^+ \\
+ [X_{22} - [-b_2]^+] - [X_{21} - [-b_1]^+]^+ \\
+ [\max\{X_{11} - [-b_1]^+, X_{12} - [+b_2]^+\} - [X_{12} - [-b_2]^+]^+ \right.
\]

The evaluation of the diversity can be carried out similarly to the evaluation of the diversity upper and lower bounds as done previously. In particular:

For (14a) (and similarly for (14b)): is as “treating the interference as noise” but with interference level with $[\beta_{12} - [+b_2]^+]^+$ instead of $\beta_{12}$, that is,

\[
d_{14a} = [\beta_{11} - [\beta_{12} - [+b_2]^+]^+ - r_1]^+ \in [[\beta_{11} - \beta_{12} - r_1]^+, [\beta_{11} - r_1]^+] \]
The minimum value of $d_{(14a)}$ is attained for $b_2 \leq 0$, while the maximum value is attained for $b_2 \geq \beta_{12}$. Recall, $b_2 = \beta_{12}$ is the power split we chose in the main section of this paper.

Before the derivation of $d_{(14c)} \triangleq d_c$ (and similarly for $d_{(14d)}$, $d_{(14e)} \triangleq d_e$ and $d_{(14f)} \triangleq d_f$) corresponding to the constraint in (14c), (14e), (14f) let

\[
A = [\beta_{11}]^+, \quad B = [\beta_{11} - [b_1]^+]^+, \\
C = [\beta_{22}]^+, \quad D = [\beta_{22} - [b_2]^+]^+, \\
E = [\beta_{12} - [b_2]^+]^+, \quad F = [\beta_{12} - [b_2]^+]^+, \\
G = [\beta_{21} - [b_1]^+]^+, \quad H = [\beta_{21} - [b_1]^+]^+, \\
r_s = [r_1]^+ + [r_2]^+, \quad r_f = 2[r_1]^+ + [r_2]^+, \quad r_g = [r_1] + 2[r_2]^+.
\]

For (14c) (and similarly for (14d)) we need to solve:

\[
d_c = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \} \\
\text{subj. to } \max \{ C - \gamma_{22} - [H - \gamma_{21}]^+, [G - \gamma_{21}]^+ - [H - \gamma_{21}]^+ \}^+ + [B - \gamma_{11} - [F - \gamma_{12}]^+]^+ \leq r_s.
\]

We divide the optimization into two steps. First we solve

\[
d_{c1} = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \} \\
\text{subj. to } \max \{ C - \gamma_{22} - [H - \gamma_{21}]^+, [G - \gamma_{21}]^+ - [H - \gamma_{21}]^+ \}^+ \leq r_{s1},
\]

and

\[
d_{c2} = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \} \\
\text{subj. to } [B - \gamma_{11} - [F - \gamma_{12}]^+]^+ \leq r_{s2}.
\]

Then we solve

\[
d_c = \min \{ d_{c1} + d_{c2} \} \\
\text{subj. to } r_{s1} + r_{s2} = r_s \triangleq r_1 + r_2.
\]

The optimization problem has the following three forms:

1) **CASE 1:** $G \geq H$ and $r_s > G - [H]^+$.

   a) If $r_{s1} \leq G - [H]^+$, then:

   \[
d_c = [C - r_{s1}]^+ + [G - r_{s1}]^+ + [B - [F]^+ - r_{s2}]^+.
   \]
b) If \( \min\{G - [H]^+, r_s\} \leq \min\{C, G\} \), then:

\[
\min d_c = \min\{(C - r_{s1}) + (G - r_{s1}) + [B - [F]^+] - r_{s2}\}^+.
\]

Increasing one unit of \( r_{s1} \) results in two units decrease of the object function, while increase one unit of \( r_{s2} \) results in one unit (or less) decrease of the object function. Thus \( r_{s1} = \min\{G - [H]^+, r_s\} \), and \( r_{s2} = 0 \) and

\[
d_c = [C - G]^+ + [B - [F]^+].
\]

c) If \( \min\{G - [H]^+, r_s\} > \min\{C, G\} \), then:

\[
d_{c1} = \min\{[C - r_{s1}]^+ + [G - r_{s1}]^+ + [B - [F]^+] - r_{s2}\}^+.
\]

Reasoning as before, \( r_{s1} \) should be as large as \( \min\{C, G\} \), thus let’s assume \( r_{s1} = \min\{C, G\} + r'_{s1} \), then the object function turns out to be:

\[
d_{c1} = \min\{[\max\{C, G\} - \min\{C, G\} - r'_{s1}]^+ + [B - [F]^+] - r_{s2}\}^+.
\]

trivially solved as:

\[
d_{c1} = [\max\{C, G\} - \min\{C, G\} + [B - [F]^+] - r_{s}']^+.
\]

Hence, by defining \( T_1 = \min\{C, G, G - [H]^+, r_s\}^+ \), and \( T_2 = r_s - T_1 \) we obtain

\[
d_{c1} = [C + G - 2T_1 + [B - [F]^+] - T_2]^+.
\]

d) If \( r_{s1} \geq G - H \), then

\[
d_{c2} = \min[C - H - r_{s1}]^+ + [B - F - r_{s2}]^+
\]

and hence

\[
d_c = \min\{d_{c1}, d_{c2}\}.
\]

2) CASE 2: \( G \geq H \) and \( r_s < G - [H]^+ \).

If \( r_{s1} \leq G - H \), by applying the reasoning in the previous subsection and by define \( T_1 = \min\{C, G, G - H, r_s\}^+ \), and \( T_2 = r_s - T_1 \) we obtain

\[
\]

3) CASE 3: \( G < H \).
In this case
\[ d_c = [C - H - r_{s1}]^+ + [B - F - r_{s2}]^+ \]
and we trivially have
\[ \min d_c = [C - H + [B - F - r_s]]. \]

For \((14e)\) we need to solve:
\[
d_e = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \}
\]
subj. to \( \max \{ B - \gamma_{11} - [F - \gamma_{12}]^+, E - \gamma_{12} - [F - \gamma_{12}]^+ \}^+ + \max \{ D - \gamma_{22} - [H - \gamma_{21}]^+, G - \gamma_{21} - [H - \gamma_{21}]^+ \}^+ \leq r_s. \)

We divide the optimization into two steps. First we solve
\[
d_{e1} = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \}
\]
subj. to \( \max \{ D - \gamma_{22} - [H - \gamma_{21}]^+, G - \gamma_{21} - [H - \gamma_{21}]^+ \}^+ \leq r_{s1}, \)
and
\[
d_{e2} = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \}
\]
subj. to \( \max \{ B - \gamma_{11} - [F - \gamma_{12}]^+, E - \gamma_{12} - [F - \gamma_{12}]^+ \}^+ \leq r_{s2}. \)

Then we solve
\[
d_e = \min \{ d_{e1} + d_{e2} \}
\]
subj. to \( r_{s1} + r_{s2} = r_s. \)

The optimization problem has the following four forms:

1) CASE 1: \( G \geq H \) and \( E \geq F. \)
   a) If \( r_{s1} \geq G - H \) and \( r_{s2} \geq E - F, \) which requires
   \[ r_s \geq [G - H]^+ + [E - F]^+, \]
   then
   \[ d_{e1} = [(D - H)^+ + [B - F]^+ - r_s]. \]
   b) If \( r_{s1} \geq G - H \) and \( r_{s2} < E - F, \) which require \( r_s \geq [G - H]^+, \) then
   \[ d_{e2} = [D - H - r_{s1}]^+ + [B - r_{s2}]^+ + [E - r_{s2}]^+ \]
and
\[ \min d_{e2} = [B + E - 2T_1 + [D - H]^+ - T_2]^+ \]
where
\[ T_1 = \min\{B, E, E - F, r_s\} \]
\[ T_2 = r_s - T_1 \]

c) If \( r_{s1} < G - H \) and \( r_{s2} \geq E - F \), which requires \( r_s \geq [E - F]^+ \), then
\[ d_{e3} = [D - r_{s1}]^+ + [G - r_{s1}]^+ + [B - F - r_{s2}]^+ \]
and
\[ d_{e3} = [D + G - 2T_1 + [B - F]^+ - T_2] \]
where
\[ T_1 = \min\{D, G, [G - H]^+, r_s\} \]
\[ T_2 = r_s - T_1 \]

d) If \( r_{s1} < G - H \) and \( r_{s2} < E - F \), which requires \( r_s < [G - H]^+ + [E - F]^+ \), then
\[ d_{e4} = [D + G - 2T_1 + B + E - 2T_2 - T3]^+ \]
where
\[ T_1 = \min\{D, G, r_s, G - H\} \]
\[ T_2 = \min\{B, E, r_s - T_1, E - F\} \]
\[ T_3 = r_s - T_1 - T_2 \]

For the four cases above, we have
\[ d_e = \min\{d_{e1}, d_{e2}, d_{e3}, d_{e4}\}. \]

2) CASE 2: \( G \geq H \) and \( E < F \).

a) If \( r_{s1} \geq G - H \), which requires \( r_s \geq G - H \), then
\[ d_{e1} = [D - H - r_{s1}]^+ + [B - F - r_{s2}]^+ \]
and

\[
\min d_{e1} = [(D - H)^+ + [B - F]^+ - r_s]^+
\]

b) If \( r_{s1} < G - H \) than

\[
d_{e2} = [D - r_{s1}]^+ + [G - r_{s1}]^+ + [B - F - r_{s2}]^+
\]

and

\[
d_{e2} = [D + G - 2T_1 + [B - F]^+ - T_2]^+
\]

where

\[
T_1 = \min\{D, G, G - H, r_s\}
\]

\[
T_2 = r_f - T_1
\]

For these two cases we have

\[
d_e = \max\{d_{e1}, d_{e2}\}
\]

3) CASE 3 \( G < H \) and \( E \geq F \):

a) If \( r_{s2} \geq E - F \), which requires \( r_s \geq E - F \), then

\[
d_{e1} = [D - H - r_{s1}]^+ + [B - F - r_{s2}]^+
\]

and

\[
\min d_{e1} = [(D - H)^+ + [B - F]^+ - r_s]^+
\]

b) If \( r_{s2} < E - F \) than

\[
d_{e2} = [D - H - r_{s1}]^+ + [B - r_{s2}]^+ + [E - r_{s2}]^+
\]

and

\[
d_{e2} = [B + E - 2T_1 + [D - H]^+ - T_2]^+
\]

where

\[
T_1 = \min\{B, E, E - F, r_s\}
\]

\[
T_2 = r_f - T_1
\]

For these two cases we have

\[
d_e = \max\{d_{e1}, d_{e2}\}\]
4) CASE 4 $G < H$ and $E < F$.

We have simply:

$$d_e = [(D - H)^+ + B - F]^- + r_s^+.$$

For (14f):

We need to solve:

$$d_f = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \}$$

subj. to

$$\max \{ A - \gamma_{11} - [F - \gamma_{12}]^+, E - \gamma_{12} - [F - \gamma_{12}]^+ \}^+ + [B - \gamma_{22} - [F - \gamma_{12}]^+]^+$$

$$+ \max \{ D - \gamma_{22} - [H - \gamma_{21}]^+, G - \gamma_{21} - [H - \gamma_{21}]^+ \}^+ \leq r_f.$$

We divide the optimization into two steps. First we solve

$$d_{f1} = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \}$$

subj. to

$$\max \{ A - \gamma_{11} - [F - \gamma_{12}]^+, E - \gamma_{12} - [F - \gamma_{12}]^+ \}^+ + [B - \gamma_{22} - [F - \gamma_{12}]^+]^+ \leq r_{f1},$$

and

$$d_{f2} = \min \{ \gamma_{11} + \gamma_{21} + \gamma_{12} + \gamma_{22} \}$$

subj. to

$$\max \{ D - \gamma_{22} - [H - \gamma_{21}]^+, G - \gamma_{21} - [H - \gamma_{21}]^+ \}^+ \leq r_{f2}.$$

Then we solve

$$d_f = \min \{ d_{f1} + d_{f2} \}$$

subj. to

$$r_{f1} + r_{f2} = r_f.$$

The optimization problem has the following four forms:

1) CASE 1: $E \geq F$ and $G \geq H$:

a) $r_{f1} \geq E - F$ and $r_{f2} \geq G - H$, which requires $r_f \geq [E - F]^+ - [G - H]^+$.

If $r'_{f1} \leq \max(A', B') - \min(A', B')$, then

$$d_f = d_{f1} + d_{f2} = [\max \{ A', B' \} - r'_{f1}] + [D - H - r_{f2}]^+$$

and

$$\min d_{f1} = [\max(A', B') + [D - H]^+ - r_f]^+$$

with

$$A' = [A - F - [E - F]^+]^+, \quad B' = [B - F]^+$$
If \( r'_{f1} > \max(A', B') - \min(A', B') \), then

\[
\min d_{f2} = \frac{[A' + B' - r'_{f1}]^+}{2} + [D - H - r_f] \\
\]

where

\[
r_f = [D - H - R_f]^+ \\
r_{f1} = r_f - r_f^2 \\
r'_{f1} = [r_{f1} - [B - F]^+]^+ \\
\]

and

\[
d_f = \max\{d_{f1}, d_{f2}\} \\
\]

b) If \( r_{f1} \geq E - F \) and \( r_f^2 < G - H \), which requires \( r_f \geq E - F \).

If \( r'_{f1} \leq \max(A', B') - \min(A', B') \), then

\[
d_f = \max\{A', B'\} - R_{f1}^+ + [D - r_f^2]^+ + [G - r_{f2}]^+ \\
\]

and

\[
\min d_{f1} = \max\{A', B'\} + D + G - 2T_1 - T_2^+ \\
\]

where

\[
T_1 = \min(D, G, G - H, r_s) \\
T_2 = r_s - T_1 \\
\]

If \( r'_{f1} \geq \max(A', B') - \min(A', B') \), then

\[
\min d_{f2} = \frac{[A' + B' - r'_{f1}]^+}{2} + [D - H - r_f] \\
\]

where

\[
r_f = \min\{\max\{D, G\}, r_f\}^+ \\
r_{f1} = r_f - r_f^2 \\
r'_{f1} = [r_{f1} - [B - F]^+]^+ \\
\]

and

\[
d_f = \max\{d_{f1}, d_{f2}\} \\
\]
c) If \( r_{f1} < E - F \) and \( r_{f2} \geq G - H \), which requires \( r_f \geq G - H \), then

\[
d_f = [E - r_{f1}]^+ + [A - r_{f1}]^+ + [B - F]^+ + [D - H - r_{f2}]^+
\]

thus

\[
d_f = [E + A - 2T_1 + [D - H]^+ - T_2 + [B - F]^+]^+
\]

where

\[
T_1 = \min\{E, A, E - F, [r_f - [B - F]^+]^+\}
\]

\[
T_2 = r_f - [B - F]^+ - T_1
\]

\[
d_f = [E - r_{f1}]^+ + [A - r_{f1}]^+ + [B - F]^+ + [D - r_{f2}]^+ + [G - r_{f2}]^+
\]

Thus

\[
\min d_f = [E + A - 2T_1 + D + G - 2T_2 + [B - F]^+]^+ - T_3]^+
\]

where

\[
T_1 = \min\{E, A, E - F, [r_f - [B - F]^+]^+\}
\]

\[
T_2 = \min\{D, G, G - H, [r_f - T_1 - [B - F]^+]^+\}
\]

\[
T_3 = [r_f - T_1 - T_2]^+
\]

and hence

\[
d_f = \min\{d_{f1}, d_{f2}, d_{f3}, d_{f4}\}
\]

2) CASE 2: \( E \geq F \) and \( G < H \):

a) \( r_{f1} \geq E - F \) which requires \( r_{f1} \geq E - F \).

\[
d_{f1} = [[[A - F]^+]^+ + [B - F]^+ - r_{f1}]^+ + [D - H - r_{f2}]^+
\]

and

\[
\min d_f = [[[A - F]^+]^+ + [B - F]^+ + [D - H]^+ - r_f]^+
\]

b) \( r_{f1} < E - F \).

\[
d_{f2} = [E - r_{f1}]^+ + [A - r_{f1}]^+ + [B - F]^+ + [D - H - r_{f2}]^+
\]
and
\[ \min d_{f2} = \left[ (E + A - 2T_1 + [D - H]^+) - T_2 + [B - F]^+ \right]^+ \]

\[ d_f = \max\{d_{f1}, d_{f2}\} \]

3) CASE 3: $E < F$ and $G \geq H$.

a) If $r_{f2} \geq G - H$, which requires $r_{f2} \geq G - H$, then
\[ d_{f1} = \left[ ([A - F]^+) + [B - F]^+ - r_{f1} \right]^+ + [D - H - r_{f2}]^+ \]

and
\[ \min d_f = \left[ ([A - F]^+) + [B - F]^+ + [D - H]^+ - r_f \right]^+ \]

b) If $r_{f2} < G - H$ then
\[ d_{f2} = \left[ [A - F]^+ + [B - F]^+ - r_{f1} \right]^+ + [D - r_{f2}]^+ + [G - r_{f2}]^+ \]

and
\[ \min d_{f2} = \left[ [D + G - 2T_1 + [A - F]^+ + [B - F]^+ - T_2]^+ \right]^+ \]

where
\[ T_1 = \min\{D, G, G - H, r_s\} T_2 = r_s - T_1 \]

and hence
\[ d_f = \max\{d_{f1}, d_{f2}\} \]

4) CASE 4: $E < F$ and $G < H$.

We simply have
\[ d_f = \left[ ([A - F]^+) + [B - F]^+ - r_{f1} \right]^+ + [D - H - r_{f2}]^+ \]

and thus
\[ \min d_f = \left[ ([A - F]^+) + [B - F]^+ + [D - H]^+ - r_f \right]^+. \]
REFERENCES


