1. (8.1, 10 points) Let $M$ be the Turing machine defined by

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>B</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_0$</td>
<td>$q_1, B, R$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q_1$</td>
<td>$q_2, B, L$</td>
<td>$q_1, a, R$</td>
<td>$q_1, c, R$</td>
<td>$q_1, c, R$</td>
</tr>
<tr>
<td>$q_2$</td>
<td></td>
<td>$q_2, c, L$</td>
<td></td>
<td>$q_2, b, L$</td>
</tr>
</tbody>
</table>

a) Trace the computation for the input string $aabca$.

b) Trace the computation for the input string $bcbc$.

c) Give the state diagram of $M$.

d) Describe the result of a computation in $M$.

Solution:

a)

\[
\begin{align*}
q_0 & \rightarrow Bq_1aabcaB \\
& \rightarrow Bq_1abcB \\
& \rightarrow Baq_1bcaB \\
& \rightarrow Baacq_1caB \\
& \rightarrow Baaccq_1aB \\
& \rightarrow Baaccaq_1B \\
& \rightarrow Baacq_2aB \\
& \rightarrow Baacq_2ccB \\
& \rightarrow Baaq_2cbcB \\
& \rightarrow Baaq_2abbcB \\
& \rightarrow Bq_2bcbccB \\
& \rightarrow q_2 BccbccB
\end{align*}
\]

b)

\[
\begin{align*}
q_0 & \rightarrow BbebcB \\
& \rightarrow Bq_1bcbB \\
& \rightarrow Bcq_1cbB \\
& \rightarrow Bccq_1bcB \\
& \rightarrow Bccccq_1cB \\
& \rightarrow Bccccq_1B \\
& \rightarrow Bccq_2cbB \\
& \rightarrow Bccq_2bbB \\
& \rightarrow Bq_2ccbbB \\
& \rightarrow q_2 BcccbB
\end{align*}
\]

c) The state diagram of $M$ is

\[
\begin{align*}
a/aR, b/cR, c/eR \quad a/eL, c/bL
\end{align*}
\]

d) The result of a computation is to replace the $a$’s in the input string with $c$’s and the $c$’s with $b$’s.

2. (8.2, 10 points) Let $M$ be the Turing machine defined by
δ | B  | a  | b  | c  
---|----|----|----|----
q₀ | q₁,B,R | | | |
q₁ | q₁,B,R | q₁,a,R | q₁,b,R | q₂,c,L |
q₂ | q₂,b,L | q₂,a,L |

a) Trace the computation for the input string abcab.

b) Trace the first six transitions of the computation for the input string abab.

c) Give the state diagram of M.

d) Describe the result of a computation in M.

Solution:

a) 

\[ \begin{align*}
q₀ & \rightarrow BabcabB \\
B & \rightarrow Bq₁abcabB \\
B & \rightarrow Baq₁abcabB \\
B & \rightarrow Babq₁abcabB \\
B & \rightarrow Baq₂bcabB \\
B & \rightarrow Bq₂aacabB \\
q₂ & \rightarrow Bq₂BbacabB
\end{align*} \]

b) 

\[ \begin{align*}
q₀ & \rightarrow BababB \\
B & \rightarrow Bq₁ababB \\
B & \rightarrow Baq₁ababB \\
B & \rightarrow Babq₁ababB \\
B & \rightarrow Babq₁ababB \\
B & \rightarrow Bababq₁B \\
B & \rightarrow BababBq₁
\end{align*} \]

3. (8.3 (a,b,d), 20 points) Construct a Turing machine with input alphabet \( \{a, b\} \) to perform each of the following operations. Note that the tape head is scanning position zero in state \( q_f \) whenever a computation terminates.

a) Move the input one space to the right. Input configuration \( q₀BuB \), result \( q_fBBuB \).

b) Concatenate a copy of the reversed input string to the input. Input configuration \( q₀BuB \), result \( q_fBuuB \).

d) Erase the \( b \)'s from the input. Input configuration \( q₀BabaababB \), result \( q_fBaaaaB \).

Solution:
a) Move the input one space to the right.

First the head moves to the first blank following the input by transitions to states $q_0$ and $q_1$, and then moves to the left. At the state $q_2$, starting with the rightmost symbol in the input and working in a right-to-left manner, the machine moves each symbol one position to the right.

If the input is the null string, the computation halts in the final state $q_2$ with the tape head in position zero as desired. Otherwise, an $a$ is moved to the right by transitions to states $q_3$ and $q_4$. Similarly, states $q_5$ and $q_4$ shift a $b$. This process is repeated until the entire string has been shifted.

b) Concatenate a copy of the reversed input string to the input.

First the head moves to the rightmost symbol in the input. At state $q_2$, the machine works in a right-to-left manner, and copy a symbol to the end of the current string.

d) Erase the $b$’s from the input.

The machine works in a left-to-right manner. At state $q_1$, it skips $a$’s and whenever it reads a $b$ or a previously deleted position, denoted by symbol $X$, the machine move the first $a$ in the following string. If no further $a$ is find, it goes to the final state $q_4$ and change all the remaining symbols $b$, $X$ and $Y$ into blanks.
4. (8.5, 10 points) Construct a Turing machine with input alphabet \( \{a, b\} \) to accept each of the following languages by final state.

a) \( \{a^i b^j | i \geq 0, j \geq i \} \)

b) \( \{a^i b^i a^i b^i | i, j > 0 \} \)

c) Strings with the same number of a’s and b’s

d) \( \{u u^R | u \in \{a, b\}^* \} \)

e) \( \{u u | u \in \{a, b\}^* \} \)

Solution:

a) \( \{a^i b^j | i \geq 0, j \geq i \} \)

b) \( \{a^i b^i a^i b^i | i, j > 0 \} \)

c) Strings with the same number of a’s and b’s

A computation of the machine begins by finding the first a on the tape and replacing it with an X (state \( q_1 \)). The tape head is then returned to position zero and a search is initiated for a corresponding b. If a b is encountered in state \( q_3 \), an X is written and the tape head is repositioned to repeat the cycle \( q_2, q_3, q_4, q_5 \). If no matching b is found, the computation halts in state \( q_3 \) rejecting the input. After all the a’s have been processed, the entire string
is read in $q_5$ and $q_6$ is entered upon reading the trailing blank. The computation halts in the accepting state $q_f$ if no $b$'s remain on the tape.

d) $\{uu^R \mid u \in \{a, b\}^*\}$

e) $\{uu \mid u \in \{a, b\}^*\}$
5. (8.6, 10 points) Modify your solution to Exercise 5(a) to obtain a Turing machine that accepts the language \( \{ a^i b^j \mid i \geq 0, j \geq i \} \) by halting.

Solution:

6. (8.16, 20 points) Construct a two-tape Turing machine with input alphabet \( \{a, b, c\} \) that accepts the language \( \{a^i b^j c^k \mid i \geq 0\} \).

Solution:

7. (8.18, 20 points) Construct a two-tape Turing machine that accepts strings in which each \( a \) is followed by an increasing number of \( b \)'s; that is, the strings are of the form

\[ ab^{n_1} ab^{n_2} \cdots ab^{n_k}, k > 0, \]

where \( n_1 < n_2 < \cdots < n_k \).

Solution: