31 March 2011

NAME | Ima Sample
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ASU ID | 

You have 75 minutes to complete the exam.
Do not open the exam until you are instructed to do so.
*Notes and texts are permitted, provided that you do not disrupt your neighbour or encroach on their space.*
*Computers, calculators, or communication devices are not permitted.*
Write all answers on the examination paper itself.
**BUDGET YOUR TIME WELL!**
**SHOW ALL WORK!**

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Bonus question [1 mark]: What is $\emptyset(\emptyset^*)$?
It is $\emptyset$

[10 in total] Question 1: Give examples (no explanation is needed) of languages A and B so that
[2 marks] (a) $A \subseteq B$, A is regular, and B is context-free but not regular. $A = \{01\}$, $B = \{0^n1^n : n \geq 1\}$
[2 marks] (b) $A \cap B$ is regular, A is regular, and B is context-free but not regular. $A = \{01\}$, $B = \{0^n1^n : n \geq 1\}$
[2 marks] (c) A ⊆ B, A is context-free but not regular, and B is regular.
A = \{0^n1^n : n ≥ 1\}, B = \{0,1\}^*

[2 marks] (d) A ∪ B is regular, A is regular, and B is context-free but not regular.
A = \{0,1\}^*, B = \{0^n1^n : n ≥ 1\}

[2 marks] (e) A ⊆ B, A is context-free but not regular, and B is context-free but not regular.
A = \{0^n1^n : n ≥ 2\}, B = \{0^n1^n : n ≥ 1\}

[10 marks in total] Question 2. Consider the language L = \{0^a1^{a-b}0^b : a ≥ b ≥ 0\}.

[5] a) Is L regular? Answer yes or no, and then provide a clear justification for your answer.

L is not regular. Suppose to the contrary that it is, so that the pumping lemma for regular languages applies. Let p be the pumping constant. Consider the string s = 0^{2p}1^p0^p. Then s is in L (take a=2p and b=p).
In order to write s = xyz with |xy| ≤ p, y can contain only 0s; and because |y| > 0, y must have some number t of 0s with 1 ≤ t ≤ p. Now consider xy^pz = 0^{2p-t}1^p0^p. Because t > 0, 2p-t ≠ 2p, and so xy^pz is not in L. This contradicts the assumption that L is regular, so L is NOT regular.

[5] b) Is L context-free? Answer yes or no, and then provide a clear justification for your answer.

L is context-free. I give a CFG for L.
S → 0S0 | T
T → 0T1 | ε

[10 marks in total] Question 3. Consider the language L = \{0^a1^{a+1}0^{b+1} : a,b ≥ 0\}.

[5] a) Is L context-free? Answer yes or no, and then provide a clear justification for your answer.

L is not context-free. Suppose (to the contrary) that it were. Then the pumping lemma for CFLs applies, so let p be the pumping constant. Let s = 0^p1^{p+1}0^{p+1}. Then s = uvwxy for |vx| > 0, |vwx| ≤ p, and uv^iwx^iy is in L for all i ≥ 0. Now if v or x contains the substring 01, then uv^2wx^2y contains
the substring 01 at least twice, so is not in \( L \). The same happens if \( v \) or \( x \) contains the substring 10. So we need only treat cases in which \( v \) contains one type of symbol or is empty, and in which \( x \) contains one type of symbol or is empty. Not both are empty because \( |vx| > 0 \). Suppose that \( vx \) contains \( m \) 0s from the \( 0^p \) block, \( n \) 1s from the \( 1^{p+1} \) block, and \( q \) 0s from the \( 0^{q+1} \) block. Now of the numbers \( \{m,n,q\} \), at least one is 0, and at least one is not 0.

Now \( uv^2wx^2y = 0^{pp+m}1^{p+n+1}0^{p+q+1} \). For this to be in the language \( pp+m = (p+n)(p+q) = pp + (n+q)p + nq \). So if \( m=0 \), we would have \( 0 = (n+q)p + nq \), which is impossible because \( p \) and \( n+q \) are both positive. So \( m \neq 0 \).

But then we must have \( m = (n+q)p + nq \). If \( n=q=0 \), this is impossible, so \( n+q \neq 0 \). Because \( |vwx| \leq p \), \( m+n+q \leq p \); but then \( m < p \), and \( (n+q)p + nq \geq p \), a contradiction. So \( L \) cannot be context-free.

[5] b) Is \( L \) regular? Answer yes or no, and then provide a clear justification for your answer.

By the (a) part, \( L \) is not context-free. But every regular language is context-free, so \( L \) is not regular.

[10 marks] Question 4. Let \( G \) be the grammar with rules

\[
\begin{align*}
S & \rightarrow aABb \mid A \mid B \\
A & \rightarrow aSSb \mid C \mid D \mid \varepsilon \\
B & \rightarrow ab \mid BC \mid ECE \\
C & \rightarrow aBAb \mid CD \mid DC \mid \varepsilon \\
D & \rightarrow E \mid DE \mid FE \\
E & \rightarrow DD \\
F & \rightarrow BB \mid \varepsilon
\end{align*}
\]

Give an equivalent grammar in Chomsky normal form. Show your steps in producing the new grammar. (Hint: Can you simplify the grammar before converting it to Chomsky normal form?)

I am going to use the hint first. The main thing to notice is that once any derivation has used either of \( D \) or \( E \), we can never get rid of all the variables – every \( D \) makes an \( E \), and every \( E \) makes a \( D \). So the rules for these variables are useless in making any strings that only have terminals,
and we can delete variables D and E. But once D is deleted, there is no way to make an F, so we can delete variable F as well.
The rule A → C is not needed, because if A ⊢ C ⊢ ε, use A ⊢ ε. And if A ⊢ C ⊢ aBAb, use A ⊢ aSSb ⊢ aBSb ⊢ aBAb.
The rule S → aABb is not needed, because we can use S ⊢ A ⊢ aSSb ⊢ aASb ⊢ aABb.
Now C only appears once on the RHS of a rule, so substitute it into the RHS.
This simplifies the grammar to
\[ S \rightarrow A \mid B \]
\[ A \rightarrow aSSb \mid \varepsilon \]
\[ B \rightarrow ab \mid BaBAb \]
Now the start variable cannot appear on the RHS of any rule (and it does), so I introduce a new start variable S₀, and the grammar is
\[ S₀ \rightarrow S \]
\[ S \rightarrow A \mid B \]
\[ A \rightarrow aSSb \mid aSb \mid ab \]
\[ B \rightarrow ab \mid BaBAb \mid BaBb \]
Now I remove ε-rules (allowing S₀ → ε if needed)
\[ S₀ \rightarrow S \mid \varepsilon \]
\[ S \rightarrow A \mid B \]
\[ A \rightarrow aSSb \mid aSb \mid ab \]
\[ B \rightarrow ab \mid BaBAb \mid BaBb \]
Now I remove unit rules
\[ S₀ \rightarrow aSSb \mid aSb \mid ab \mid BaBAb \mid BaBb \mid \varepsilon \]
\[ S \rightarrow aSSb \mid aSb \mid ab \mid BaBAb \mid BaBb \]
\[ A \rightarrow aSSb \mid aSb \mid ab \]
\[ B \rightarrow ab \mid BaBAb \mid BaBb \]
Now I make every rule have either variables or terminals on RHS but not both
\[ S₀ \rightarrow X_aSSX_b \mid X_aSX_b \mid X_aX_b \mid BX_aBAX_b \mid BX_aBX_b \mid \varepsilon \]
\[ S \rightarrow X_aSSX_b \mid X_aSX_b \mid X_aX_b \mid BX_aBAX_b \mid BX_aBX_b \]
\[ A \rightarrow X_aSSX_b \mid X_aSX_b \mid X_aX_b \]
\[ B \rightarrow X_aX_b \mid BX_aBAX_b \mid BX_aBX_b \]
\[ X_a \rightarrow a \]
\[ X_b \rightarrow b \]
Now I break up the rules that have three or more variables on the RHS.

\[
S_0 \rightarrow X_a T_5 \mid X_a T_2 \mid X_a X_b \mid BT_7 \mid BT_4 \mid \epsilon \\
S \rightarrow X_a T_5 \mid X_a T_2 \mid X_a X_b \mid BT_7 \mid BT_4 \\
A \rightarrow X_a T_5 \mid X_a T_2 \mid X_a X_b \\
B \rightarrow X_a X_b \mid BT_7 \mid BT_4 \\
X_a \rightarrow a \\
X_b \rightarrow b \\
T_1 \rightarrow BX_b \\
T_2 \rightarrow SX_b \\
T_3 \rightarrow AX_b \\
T_4 \rightarrow X_a T_1 \\
T_5 \rightarrow ST_2 \\
T_6 \rightarrow BT_3 \\
T_7 \rightarrow X_a T_6
\]

Done, at long last. I did not expect anyone, in the time available, to get all of these details written down correctly!

[10 marks] Question 5. Draw a state diagram for a pushdown automaton to accept \( \{a^i b^j c^i d^k : i, j, k \geq 0 \} \). The PDA must accept by final state and empty stack.

![State diagram for pushdown automaton]