1. Question 1.

Let $G = (V, E)$ be a directed graph. For two vertices $x, y \in V$, the distance from $x$ to $y$, $\text{dist}(x, y)$, is the length of a shortest directed path starting at $x$ and ending at $y$. (Recall that paths cannot repeat vertices or edges.) The diameter of $G$, $\text{diam}(G)$, is $\max_{x,y \in V} \text{dist}(x, y)$. A spanning subdigraph of $G$ is a directed graph $H = (W, F)$ with $V = W$ and $E \supseteq F$.

- Show that $\text{Fat} = \{ \langle G, k \rangle : G$ has a spanning subdigraph $H$ with $\text{diam}(H) \leq k \}$ is NL-complete.
- Show that $\text{Obese} = \{ \langle G \rangle : G$ has a spanning subdigraph $H$ with $\text{diam}(H) \leq 2 \}$ is in $L = \text{SPACE}(\log n)$.

2. Question 2.

Let $G = (V, E)$ be an undirected graph. For two vertices $x, y \in V$, the distance from $x$ to $y$, $\text{dist}(x, y)$, is the length of a shortest path starting at $x$ and ending at $y$. (Recall that paths cannot repeat vertices or edges.) The diameter of $G$, $\text{diam}(G)$, is $\max_{x,y \in V} \text{dist}(x, y)$.

- Show that $\text{Thin} = \{ \langle G, k \rangle : G$ has a spanning connected subgraph $H$ with $\text{diam}(H) \geq k \}$ is NP-complete.
- Show that when $G$ is an $n$-vertex graph, we can decide whether $G$ has a spanning connected subgraph $H$ with $\text{diam}(H) \geq k$ in time $n^{O(k)}$. Explain why this does not contradict your conclusion about the NP-completeness.
3. Question 3.

Let’s play a game, Friendship. There is a graph $G$ in which the vertices have been partitioned into $4\ell$ disjoint sets, $(X_i, Y_i : 1 \leq i \leq 2\ell)$. Two players, Constructo and Disrupto, alternate taking turns; Constructo starts. At turn $i$ for $1 \leq i \leq \ell$, Constructo deletes either all vertices in $X_{2i-1}$ or all vertices in $Y_{2i-1}$. Then Disrupto deletes either all vertices in $X_{2i}$ or all vertices in $Y_{2i}$. Constructo wins at the end if there remains a clique of size $2\ell$; otherwise Disrupto wins. Show that Friendship $=$ \{$(G, (X_i, Y_i : 1 \leq i \leq 2\ell)) : \text{Constructo wins}$\} is PSPACE-complete.


- Using the pattern that we have seen for defining NL-, NP-, PSPACE- and EXPSPACE-completeness, carefully define EXPTIME-completeness.
- Show that $ATM,lim = \{\langle M, w, k \rangle : M \text{ is a TM that accepts } w \text{ within at most } k \text{ steps} \}$ is EXPTIME-complete. (Note that $k$ is input in binary.)

5. Question 5.

A narcissistic Turing machine is a TM $M$ for which $L(M) = \{\langle M \rangle\}$.

(a) Show that a narcissistic TM exists.
(b) Determine whether or not the language of (encodings of) narcissistic Turing-machines is Turing-recognizable. Also determine whether or not the language of (encodings of) narcissistic Turing-machines is co-Turing-recognizable.

Explain carefully.


Let $U = \{0, 1\}^*$. For a string $w \in U$, write $w = w_1 \cdots w_\ell$ with $w_i \in \{0, 1\}$ for $1 \leq i \leq \ell = |w|$. Define the relation $\text{AltDen} = \{(x, y, w) : |x| = |y| = |w| \text{ and } x_i \uparrow y_i = w_i \text{ when } 1 \leq i \leq |w|\}$.\ (Here $0 \uparrow 0 = 0 \uparrow 1 = 1 \uparrow 0 = 1 \uparrow 1 = 0$.) Write $(x, y, w) \in \text{AltDen}$ as $x \uparrow y = w$. We consider the theory of $U$ with $\uparrow$.

(a) Show that $x = y; x \lor y = z$ ("or"), and $x \land y = z$ ("and"), are definable in this theory.
(b) Show that the theory of $U$ with $\uparrow$ is decidable.
(c) Using the decidability of the theory, show in detail whether the statement $\forall x \forall y \forall z[(x \lor y) \land z = (x \land z) \lor (y \land z)]$ holds.

7. Question 7.

Let $\text{MINOR}_{NFA} = \{\langle M \rangle : M \text{ is an NFA with } L(M) \subseteq \{0, 1\}^* \text{ for which } L(M) \text{ contains fewer than half the strings in } \{0, 1\}^n \text{ for every } n \geq 0\}$. Show that $\text{MINOR}_{NFA}$ is NP-hard.
8. Question 8.

In this question, $L = SPACE(\log n)$.

(a) Suppose that $A$ is a regular language. Must $A^* \in L$? Explain.
(b) Suppose that $B \in L$. Must $B$ be context-free? Explain.
(c) Suppose that $C \in L$. Must $C^* \in NL$? Explain.