In a CFG $G$, a variable $A$ is

- **unreachable** if there is no derivation $S \Rightarrow^* w$ with $w$ containing $A$;
- **unproductive** if there is no derivation $A \Rightarrow^* w$ with $w$ containing only terminals;
- **redundant** if whenever there is a derivation $S \Rightarrow^* w$ with $w$ containing only terminals, there is a derivation that does not use the variable $A$ (in other words, variable $A$ and all rules containing it can be deleted without changing the language).

A CFG is a **regular grammar** if every rule is of the form $A \rightarrow bC$, $A \rightarrow C$, $A \rightarrow b$, or $A \rightarrow \varepsilon$.

1. **Regular languages:**
   - (a) Prove or disprove: $\{\langle G, A \rangle : G$ is a regular grammar and $A$ is an unreachable variable in $G\}$ is decidable.
   - (b) Prove or disprove: $\{\langle G, A \rangle : G$ is a regular grammar and $A$ is an unproductive variable in $G\}$ is decidable.
   - (c) Prove or disprove: $\{\langle G, A \rangle : G$ is a regular grammar and $A$ is a redundant variable in $G\}$ is decidable.

2. **Context-free languages:**
   - (a) Prove or disprove: $\{\langle G, A \rangle : G$ is a CFG and $A$ is an unreachable variable in $G\}$ is decidable.
   - (b) Prove or disprove: $\{\langle G, A \rangle : G$ is a CFG and $A$ is an unproductive variable in $G\}$ is decidable.
   - (c) Prove or disprove: $\{\langle G, A \rangle : G$ is a CFG and $A$ is a redundant variable in $G\}$ is decidable.

3. **An instance of the Post Correspondence Problem consists of a finite set of tiles $\left\{ t_i, b_i : 1 \leq i \leq \ell, t_i, b_i \in \Sigma^+ \right\}$. An instance is periodic if there is a finite sequence $i_1, \ldots, i_k$ for which $t_{i_1} \cdots t_{i_k} = b_{i_1} \cdots b_{i_k}$. It is aperiodic if it is not periodic, but there is an infinite sequence $i_1, i_2, \ldots$, so that for $t_{i_1} t_{i_2} \cdots = b_{i_1} b_{i_2} \cdots$. For example, $\left\{ \begin{bmatrix} a \\ aa \end{bmatrix}, \begin{bmatrix} aa \\ a \end{bmatrix} \right\}$ is periodic, $\left\{ \begin{bmatrix} a \\ aa \end{bmatrix} \right\}$ is aperiodic, and $\left\{ \begin{bmatrix} a \\ aab \end{bmatrix} \right\}$ is neither.
   - (a) Prove or disprove: $\{\langle P \rangle : P$ is an aperiodic instance of the PCP\}$ is Turing recognizable.
   - (b) Prove or disprove: $\{\langle P \rangle : P$ is an aperiodic instance of the PCP\}$ is co-recognizable.
(4) For a Turing machine $M$, let $\nu(\langle M \rangle)$ be 0 if $M$ runs forever on the empty string as input; otherwise let $\nu(\langle M \rangle)$ be the number of nonblank tape cells when $M$ terminates. Then define $\rho(n) = \max\{\nu(\langle M \rangle) : M \text{ is a TM with exactly } n \text{ states}\}$. Prove or disprove: The function $\rho(n)$ is Turing computable.

(5) We established Rice’s theorem by a reduction from $A_{TM}$. Instead prove Rice’s theorem as a consequence of the recursion theorem.