CSE 555 HOMEWORK THREE

DUE 06 MARCH 2014, 1:30 P.M. MST BY EMAIL TO THE PROFESSOR AND THE TA

(1) Relation \( R \subseteq \mathbb{N}^k \) is definable in \( T_h(\mathbb{N}, +) \) if there is a formula \( \phi \) with \( k \) free variables \( a_1, \ldots, a_k \) so that \( \phi \) uses only the relation PLUS, and whenever \( a_1, \ldots, a_k \in \mathbb{N} \),
\[
\phi(a_1, \ldots, a_k) \text{ is true } \iff (a_1, \ldots, a_k) \in R.
\]
Consider the logical formula \((x \times y = z)\), and rewrite it as
\[
((y = 0) \rightarrow (z = 0)) \land ((y \neq 0) \rightarrow (x + (x \times (y - 1)) = z)).
\]
Repeating this until \( y = 0 \), we can eliminate all occurrences of \( \times \), so it appears that we have defined \( \times \) in terms of \(+\). Explain carefully why this appearance is not correct, and that this does not make \( \times \) definable in \( T_h(\mathbb{N}, +) \). Then explain why \( \times \) cannot be definable in \( T_h(\mathbb{N}, +) \) by any method.

(2) In thinking about \( T_h(\mathbb{N}, +) \), we have built our DFA for \(+\) using big endian order (most significant bit first, so that decimal 11 is written as 1011 or 01011 or 001011 or ...). Unfortunately our friend has built her DFA so that it uses little endian order (least significant bit first, so that decimal 11 is written as 1101 or 11010 or 110100 or ...). Insisting as we do on being big endian, when we use her DFA we get unusual results: adding 011010 and 000110 with her machine we get 011101 (because 22 + 24 = 46), but in big endian order, the result is 100000 or 32 (because 26 + 6 = 32). To avoid confusion, we refer to the operation of her machine as \( \oplus \), when it is applied to numbers in big endian order; we call the operation of the original machine \( + \).

(a) Show that \( T_h(\mathbb{N}, +, \oplus) \) is decidable. Explain carefully. (Remember, the inputs are given as big endian, as both machines now expect!)

(b) We are wondering whether her machine does the addition “correctly” infinitely often. To do this, explain how to use your general method to decide the truth of the sentence
\[
\forall y \exists a \exists b ((a \geq y) \land (b \geq y) \land (a + b = a \oplus b))
\]

First rewrite the sentence using only \(+, \oplus\), and logical operations by defining \( \geq \) and \( = \) using the basic relations. Note: There is no credit for giving a proof or disproof that does not use the general method developed in the first part.

(3) Establish the recursion theorem for oracle Turing machines. That is, establish that a TM with an oracle for \( L \) can implement “obtain own description”.

(4) Establish that \( E'_{TM} = \{(M) : M \text{ is a TM with an oracle for } A_{TM} \text{ and } L(M) = \emptyset\} \) is undecidable relative to \( A_{TM} \).

(5) Show (in general) that \( L \) is recognizable if and only if \( L \) is mapping reducible to \( A_{TM} \). Is it possible that \( L \) can be decidable relative to \( A_{TM} \), yet \( L \) is not recognizable? neither recognizable nor co-recognizable? Explain.