(1) Design a DFA to verify addition in binary for the integers $\mathbb{Z}$ rather than just the natural numbers $\mathbb{N}$. (Your DFA have an alphabet of characters representing 3-tuples of bits.) You will need a method to represent and handle negative integers to do this. Whatever representation you choose, your DFA should accept if and only if $a + b = c$ where $a, b, c$ are the integers represented in the first, second, third bits, respectively.

(2) We examine $Th(\mathbb{Z}, +)$.
   (a) Using your DFA for Question 1, show that $Th(\mathbb{Z}, +)$ is decidable.
   (b) Show that every statement in the model $(\mathbb{N}, +)$ can be written as a statement in the model $(\mathbb{Z}, +)$ so that the truth value is unchanged. (Be careful: $\exists x PLUS(4, x, 3)$ is false in $Th(\mathbb{N}, +)$ but true in $Th(\mathbb{Z}, +)$.)

(3) A Turing machine running on empty input may halt or run forever. We are interested in how long it can run but eventually halt. Let $S(n, k)$ be the largest number of transitions applied by any $n$-state Turing machine with tape alphabet having $k$ symbols running on empty input, which halts.
   (a) Show that, for every fixed $n$ and $k$, the number of Turing machines with $n$ states having tape alphabet of size $k$ is finite. Can you conclude that $S(n, k)$ is finite for every choice of $n$ and $k$?
   (b) Prove or disprove: $S(n, k)$ is a Turing computable function.

(4) Let $S(n, k)$ be as in Question 3.
   (a) Prove or disprove: $S(n, k)$ is a Turing computable function given an oracle for $A_{TM}$.
   (b) Let $S = \{(n, k, \ell) : S(n, k) \leq \ell$ for $n, k, \ell \in \mathbb{N}\}$. Prove or disprove: $A_{TM}$ is decidable relative to $S$.

(5) An instance of the Post Correspondence Problem consists of a finite set of tiles $\left\{ \begin{bmatrix} t_i \\ b_i \end{bmatrix} : 1 \leq i \leq \ell, t_i, b_i \in \Sigma^+ \right\}$. An instance is periodic if there is a finite sequence $i_1, \ldots, i_k$ for which $t_{i_1} \cdots t_{i_k} = b_{i_1} \cdots b_{i_k}$. It is aperiodic if it is not periodic, but there is an infinite sequence $i_1, i_2, \ldots$, so that for $t_{i_1} t_{i_2} \cdots = b_{i_1} b_{i_2} \cdots$.
   (a) Prove or disprove: $\{\langle P \rangle : P$ is an aperiodic instance of the PCP$\}$ is decidable relative to $HALT_{TM}$.
   (b) Prove or disprove: $\{\langle P \rangle : P$ is a periodic instance of the PCP$\}$ is decidable relative to $HALT_{TM}$.