When proving NP-completeness in this homework, use only the known NP-completeness results in Sipser’s Chapter 7. Do not simply cite NP-completeness results from other sources.

1) **Short PCP** is the decision problem: Given a set $T$ of $n$ tiles $[t_i] = [b_i]_1$, $1 \leq i \leq n$, and an integer $k$ written in unary, is there a PCP solution $t_{i_1} \cdots t_{i_\ell} = b_{i_1} \cdots b_{i_\ell}$ in which $\ell \leq k$? Show that short PCP is NP-complete.

2) Suppose that we are provided an oracle to answer the question: Given $\langle T, k \rangle$, does the set $T$ of tiles have a PCP solution with $k$ or fewer tiles (with $k$ written in unary)? The oracle takes zero time to compute, but we must write the question to it, and read its answer, and these do take time.
   
   (a) We want to find a solution (solve the construction problem, not just the decision problem). Show how to use the oracle (repeatedly) to find a PCP solution of size at most $k$, if one exists, in polynomial time.
   
   (b) We want to find a smallest solution (solve the optimization problem, not just the construction problem). Show how to use the oracle (repeatedly) to find a PCP solution with the fewest tiles, if one exists having at most $k$ tiles, in polynomial time.

3) An $n$-fan is a graph $G = (V, E)$ where $V = \{v_0, \ldots, v_{n-1}\}$ and $E = \{\{v_i, v_{i+1}\} : 0 \leq i < n-1\} \cup \{\{v_i, v_{n-1}\} : 0 \leq i < n-2\}$. We say that $H = (W, F)$ is a spanning subgraph of $G = (V, E)$ if $W = V$ and $F \subseteq E$. Show that the question “Given input $\langle G \rangle$, does $G$ have a spanning fan?” is NP-complete.

4) A maximal path is a set $\{x_0, x_1, \ldots, x_k\}$ in a graph so that $\{x_i, x_{i+1}\}$ is an edge for $0 \leq i < k$, and there is no edge $\{x_0, y\}$ or $\{x_k, y\}$ unless $y \in \{x_0, x_1, \ldots, x_k\}$. The value of $k$ is the length of the path. Show that the question “Given graph $G$ on $n$ vertices, does $G$ have a maximal path of length equal to $\lfloor \sqrt{n} \rfloor$?” is NP-complete.

5) Earlier we saw that $\text{ALL}_{\text{DFA}} = \{\langle M \rangle : L(M) = \Sigma^* \}$ for the DFA $M$ whose input alphabet is $\Sigma$ and $\text{ALL}_{\text{NFA}} = \{\langle M \rangle : L(M) = \Sigma^* \}$ for the NFA $M$ whose input alphabet is $\Sigma$ are both decidable.
   
   (a) Show that $\text{ALL}_{\text{DFA}}$ is in $P$.
   
   (b) Show that $\text{ALL}_{\text{NFA}}$ is NP-hard.