Acceleration for General Circular Motion

The position of an object in circular motion of radius $r$ can be expressed as

$$\mathbf{r}(t) = r(\cos(\theta(t))\mathbf{i} + \sin(\theta(t))\mathbf{j}).$$

The angle $\theta$ is in general an unknown function of time $t$. If $\theta$ is measured in radians then the arclength $s(t)$ at any time $t$ is given by

$$s(t) = r\theta(t).$$

The speed of the particle as it goes around the circle is the time derivative of the arclength.

$$v(t) = \frac{d}{dt}s(t).$$

We will need the time derivative of the angle, which we can now find.

$$\frac{d}{dt}\theta(t) = \frac{d}{dt}\left(\frac{s(t)}{r}\right) = \frac{v(t)}{r}.$$

In the work below, we will use this frequently.

Now we set out to find the acceleration, by taking two derivatives of the position.

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = v(t)(-\sin(\theta(t))\mathbf{i} + \cos(\theta(t))\mathbf{j})$$

To find the derivative of the velocity we must use the product rule for derivatives.

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = -\frac{v^2(t)}{r}(\cos(\theta(t))\mathbf{i} + \sin(\theta(t))\mathbf{j}) + \frac{d}{dt}v(t)(-\sin(\theta(t))\mathbf{i} + \cos(\theta(t))\mathbf{j}).$$

In terms of radial and tangential unit vectors, this is

$$\mathbf{a}(t) = -\frac{v^2(t)}{r}\mathbf{r} + \frac{d}{dt}v(t)\mathbf{t}.$$