Comments

**Prob 1** is straightforward. Note that there are two solutions. See attached sample solution.

**Prob 2**
There are many ways to rewrite the original equation into the form, \( x = g(x) \), for the iterative procedure. Just to name three: 

(A) \( g(x) = (10 \exp(-x))^{1/3} \) 
(B) \( g(x) = x - 0.1 \, x^3 + \exp(-x) \) 
(C) \( g(x) = -\ln(0.1 \, x^3) \)

As indicated by Eq. (3.30) in the textbook, the process is expected to converge only if \( |g'(x)| < 1 \) in the neighborhood of the solution. Some of you first noted, by inspecting the plot of \( f(x) = 0.1 \, x^3 - \exp(-x) \), that the solution is located between 1 and 2. You then showed that \( |g'(1)| < 1 \) and \( |g'(2)| < 1 \) for your choice of \( g(x) \) before performing the iteration. That was a sensible practice, but one can make it even more convincing by plotting \( g'(x) \) and \( f(x) \) together to show that \( |g'(x)| < 1 \) right where the solution is located. In the following figure, the black curve is \( f(x) \), and red, green, and blue curves are \( g'(x) \) for (A), (B), and (C). The two dashed lines indicate the bounds of 1 and -1. Clearly, (A) and (B) are viable choices while (C) is not.

Almost everyone who got it right chose form (A) - see attached sample solution. Only two persons chose to use form (B). Note that by using (B), we have \( f(x) = x - g(x) \Rightarrow f(x_i) = x_i - g(x_i) = x_i - x_{i+1} \). (The last step used the relation, \( x_{i+1} = g(x_i) \), from the fixed point iterative procedure.) Thus, in this case \( |f(x_i)| \) is immediately known as a by-product of the iterative process.
D Solve the equation \(0.8x^2 - \sin(x) + 0.1 = 0\)

Find all solutions using Newton's Method.

Solve to accuracy \(|f(x_n)| < 0.0001\).

Using my TI-84 Plus calculator, I graphed the function
\[0.8x^2 - \sin(x) + 0.1 = 0\]

By the graph, the solution for zeros should be around \(x = 0\) \(\pm\) \(x = 1\).

Knowing this, my initial guess for each solution will be \(x = -1\) and \(x = 2\).

\[x_1 = -1\]

\[x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = (-1) - \frac{(0.8(-1)^2) - \sin(-1) + 0.1}{(-1) - \cos(-1)} = -0.641163867\]

\[x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = (-0.641163867) - \frac{0.8x_2^2 - \sin(x_2) + 1}{(x_2) - \cos(x_2)} = 0.922973333\]

\[x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = (0.922973333) - \frac{0.8x_3^2 - \sin(x_3) + 1}{(x_3) - \cos(x_3)} = 1.056828036\]

So, \(f(1.056828036) = 9.824005369 \times 10^{-5}\)

This solution is well within our accuracy tolerance of \(|f(x_n)| < 0.0001\). By choosing \(x = -1\) to start, this allowed the method to converge.

\[x_n = 1.056828036\]
$x_1 = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = (2) - \frac{3x_1^2 - \sin(x_1) + 1}{x_1 - \cos(x_1)} = 1.807169482$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = (1.807169482) - \frac{3x_2^2 - \sin(x_2) + 1}{x_2 - \cos(x_2)} = 1.342442948$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = (1.342442948) - \frac{3x_3^2 - \sin(x_3) + 1}{x_3 - \cos(x_3)} = 1.31821868$

$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = (1.31821868) - \frac{3x_4^2 - \sin(x_4) + 1}{x_4 - \cos(x_4)} = 1.317576931$

$f(1.317576931) = 2.8872321 \times 10^{-7}$

So, for the 2nd zero (right side)

$x_n = 1.317576931$
2. Solve \( 0.1x^3 - e^{-x} = 0 \) using the fixed point iteration method

\[ f(x) = 0.1x^3 - e^{-x} \]

\[ x = 1 \quad f(x) = -0.267879 \quad \rightarrow \text{sign change} \]

\[ x = 2 \quad f(x) = 0.664665 \]

Case #1

\[ 0 = 0.1x^3 - e^{-x} \]

\[ 0.1x^3 = e^{-x} \]

\[ \ln(0.1x^3) = \ln(e^{-x}) \]

\[ \ln(0.1x^3) = -x \]

\[ x = g(x) = -\ln(0.1x^3) \]

\[ g'(x) = -\frac{3}{x} \]

\[ g'(1) = -3 \]

\[ g'(2) = -\frac{3}{2} = -1.5 \]

\[ |g'(x)| > 1 \]

Case #2

\[ 0.1x^3 = e^{-x} \]

\[ x^3 = \frac{e^{-x}}{0.1} \]

\[ x = \left( \frac{e^{-x}}{0.1} \right)^{1/3} \]

\[ g(x) = \left( \frac{e^{-x}}{0.1} \right)^{1/3} = (10e^{-x})^{1/3} \]

\[ g'(x) = \frac{1}{3} (10e^{-x})^{-2/3} \]

\[ g'(1) = -0.514576 \]

\[ g'(2) = -0.368708 \]

\[ |g'(x)| < 1 \]

\[ x_1 = 1 \]

\[ x_2 = g(x_1) = (10e^{-1})^{1/3} = 1.54387 \]

\[ x_3 = g(x_2) = (10e^{-1.54372})^{1/3} = 1.28783 \]

\[ x_4 = g(x_3) = (10e^{-1.28783})^{1/3} = 1.1025 \]

\[ x_5 = g(x_4) = (10e^{-1.1025})^{1/3} = 1.3499 \]

\[ x_6 = g(x_5) = (10e^{-1.3499})^{1/3} = 1.37377 \]

\[ x_n = 1.373774268464 \]

Solution: \( x_n = 1.37377 \)