An example for the method of characteristics

Example 1

For \( u(x,t) \) defined on the infinite domain, \(-\infty < x < \infty \) and \( t \geq 0 \), solve the PDE

\[
\frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} = 0
\]

with the boundary condition,

\[
u(x,0) = P(x) ,
\]

where

\[
P(x) = \begin{cases} 
1 & \text{if } x < 0 \\
1 + x^2 & \text{if } 0 \leq x \leq 1 \\
2 & \text{if } x > 1 
\end{cases}
\]

Solution:

Along the characteristics defined as

\[
dx/dt = 2 \ u ,
\]

the PDE is reduced to an ODE,
\[ \frac{du}{dt} = 0. \quad (2) \]

Solving Eq. (2) we have

\[ u(t) = u(0), \quad \text{or} \quad u(x(t), t) = u(x(0), 0) = P(x(0)). \]

Since \( u(t) = u(0) \), Eq. (1) becomes \( \frac{dx}{dt} = 2u(0) \) which leads to

\[ x(t) = x(0) + 2u(0) \cdot t, \]
\[ = x(0) + 2P(x(0)) \cdot t. \quad (3) \]

This is the equation for the characteristics which can be used to trace any given pair of \((x, t)\) back to the corresponding \(x(0)\). Once we obtain \(x(0)\), the solution for the PDE is immediately known as \(u(x, t) = P(x(0))\). For the convenience of later discussions, we will write \(x(0)\) as \(x_0\).

**Case 1: \(x_0 < 0\)**

In this case, \(P(x_0) = 1\) such that we immediately obtain the solution as \(u(x, t) = 1\). Moreover, by Eq. (3), \(x_0 < 0\) corresponds to \(x - 2 P(x_0) \cdot t < 0\), or \(x < 2t\). We obtain the 1st part of the solution as

\[ u(x, t) = 1, \quad \text{if} \quad x < 2t. \quad (S1) \]

**Case 2: \(0 \leq x_0 \leq 1\)**

In this case, \(P(x_0) = 1 + (x_0)^2\), which means the solution is

\[ u(x, t) = 1 + (x_0)^2. \quad (4) \]
What remains to be done, however, is to represent \( x_0 \) in Eq. (4) in terms of \( x \) and \( t \). Using Eq. (3), we have

\[
x_0 = x - 2 \left[ 1 + (x_0)^2 \right] t,
\]

which is a quadratic equation for \( x_0 \) for any given \((x, t)\). Solving it and discarding the solution that contradicts the assumption of \( 0 \leq x_0 \leq 1 \), we obtain

\[
x_0 = \left( \frac{1}{4t} \right) \left[ -1 + \left( 1 - 16t^2 + 8xt \right)^{1/2} \right].
\]

The solution for the PDE, \( u(x, t) \), can be obtained by plugging this expression back to Eq. (4). It is left to you as an exercise to show that the condition, \( 0 \leq x_0 \leq 1 \), corresponds to \( 2t \leq x \leq 4t + 1 \). Thus, the 2nd part of the solution is

\[
u(x, t) = 1 + \left\{ \left( \frac{1}{4t} \right) \left[ -1 + \left( 1 - 16t^2 + 8xt \right)^{1/2} \right] \right\}^2, \text{ if } 2t \leq x \leq 4t + 1. \quad (S2)
\]

**Case 3: \( x_0 > 1 \)**

In this case, \( P(x_0) = 2 \) and the solution is simply \( u(x, t) = 2 \). Moreover, by Eq. (3), \( x_0 > 1 \) corresponds to \( x > 4t + 1 \). The 3rd part of the solution is

\[
u(x, t) = 2, \text{ if } x > 4t + 1. \quad (S3)
\]

Combining (S1)-(S3), we have the complete solution for \( u(x, t) \). See plots in the next 2 pages for the solution at selected values of \( t \) and examples of the characteristics in the \( x-t \) plane.
Solution at $t = 0, 0.1, \text{ and } 0.2$
Characteristics